Ramanujan's Lost Notebook

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Ramanujan's Lost Notebook

Part III

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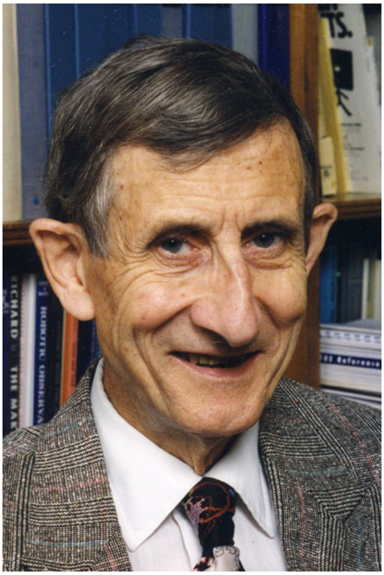
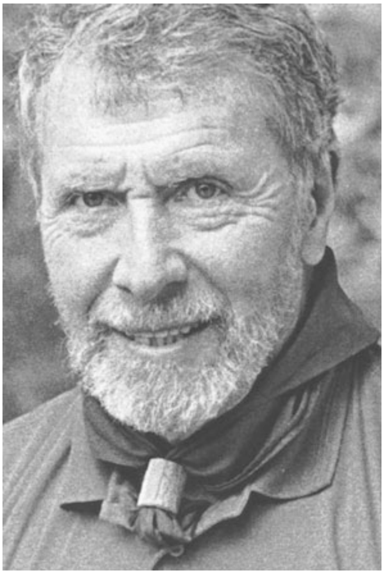
The CEO's of Ranks and Cranks


Freeman Dyson


Frank Garvan


Oliver Atkin


H.P.F. Swinnerton-Dyer

I felt the joy of an explorer who suddenly discovers the key to the language lying hidden in the hieroglyphs which are beautiful in themselves.

–Rabindranath Tagore, *The Religion of Man*

Preface

This is the third of five volumes that the authors plan to write in their examination of all the claims made by S. Ramanujan in *The Lost Notebook and Other Unpublished Papers*, which was published by Narosa in 1988. This publication contains the "Lost Notebook," which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included therein are other partial manuscripts, fragments, and letters that Ramanujan wrote to G.H. Hardy from nursing homes during 1917–1919. Our third volume contains ten chapters and focuses on some of the most important and influential material in *The Lost Notebook and Other Unpublished Papers.* At center stage is the partition function $p(n)$. In particular, three chapters are devoted to ranks and cranks of partitions. Ramanujan's handwritten manuscript on the partition and tau functions is also examined.

Contents

1

Introduction

This is the third volume devoted to Ramanujan's lost notebook and to partial manuscripts, fragments, and letters published with the lost notebook [283]. The centerpiece of this volume is the partition function $p(n)$. Featured in this book are congruences for $p(n)$, ranks and cranks of partitions, the Ramanujan τ-function, the Rogers–Ramanujan functions, and the unpublished portion of Ramanujan's paper on highly composite numbers [274].

The first three chapters are devoted to ranks and cranks of partitions. In 1944, F. Dyson [127] defined the rank of a partition to be the largest part minus the number of parts. If $N(m,t,n)$ denotes the number of partitions of n with rank congruent to m modulo t, then Dyson conjectured that

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \qquad 0 \leq k \leq 4, \tag{1.0.1}$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \qquad 0 \leq k \leq 6. \tag{1.0.2}$$

Thus, if (1.0.1) and (1.0.2) were true, the partitions counted by $p(5n+4)$ and $p(7n+5)$ would fall into five and seven equinumerous classes, respectively, thus providing combinatorial explanations and proofs for Ramanujan's famous congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$. Dyson's conjectures were first proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [28] in 1954.

Dyson observed that the corresponding analogue to (1.0.1) and (1.0.2) does not hold for the third famous Ramanujan congruence $p(11n+6) \equiv 0 \pmod{11}$, and so he conjectured the existence of a statistic that he called the *crank* that would combinatorially explain this congruence. In his doctoral dissertation [144], F.G. Garvan defined a crank for vector partitions, which became the forerunner of the *true crank*, which was discovered by Andrews and Garvan [17] during the afternoon of June 6, 1987, at Illinois Street Residence Hall, a student dormitory at the University of Illinois, following a meeting on June 1–5 to commemorate the centenary of Ramanujan's birth.

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Although Ramanujan did not record any written text about ranks and cranks in his lost notebook [283], he did record theorems about their generating functions. Chapter 2 is devoted to the five and seven-dissections of each of these two generating functions. Cranks are the exclusive topic of Chapter 3, where dissections for the generating function for cranks are studied, but now in the context of congruences. A particular formula found in the lost notebook and proved in Chapter 4 is employed in our proofs in Chapter 3. As we argue in the following two paragraphs, it is likely that Ramanujan was working on cranks up to four days before his death on April 26, 1920.

In January 1984, the second author, Berndt, was privileged to have a very pleasant and exceptionally informative conversation with Ramanujan's widow, Janaki. In particular, this author asked her about the extent of papers that her late husband possessed at his death, and remarked that the only papers that have been passed down to us are those constituting the lost notebook of 138 pages. She claimed that Ramanujan had many more than 138 pages in his possession at his death. She related that as her husband "did his sums," he would deposit his papers in a large leather trunk beneath his bed, and that the number of pages in this trunk certainly exceeded 138. She told Berndt that during her husband's funeral, certain people, whom she named but whom we do not name here, came to her home and stole most of Ramanujan's papers and never returned them. She later donated those papers that were not stolen to the University of Madras. These papers certainly contain, or possibly are identical to, the lost notebook.

It is our contention that Ramanujan kept at least two stacks of papers while doing mathematics in his last year. In one pile, he put primarily those pages containing the statements of his theorems, and in another stack or stacks he put papers containing his calculations and proofs. The one stack of papers containing the lost notebook was likely in a different place and missed by those taking his other papers. (Of course, it is certainly possible that more than one pile of papers contained statements of results that Ramanujan wanted to save.) Undoubtedly, Ramanujan produced scores of pages containing calculations, scratch work, and proofs, but the approximately twenty pages of scratch work in the lost notebook apparently pertain more to cranks than to any other topic. Our guess is that when Ramanujan ceased research four days prior to his death, he was thinking about cranks. His power series expansions, factorizations, preliminary tables, and scratch work were part of his deliberations and had not yet been put in a secondary pile of papers. Thus, these sheets were found with the papers that had been set aside for special keeping and so unofficially became part of his lost notebook. In particular, pages 58–89 in the lost notebook likely include some pages that Ramanujan intended to keep in his principal stack, but most of this work probably would have been relegated to a secondary pile if Ramanujan had lived longer. Further remarks can be found in [64].

Ramanujan's famous manuscript on the partition and tau functions is examined in Chapter 5. This chapter is a substantially revised and extended

version of the original publication by the second author and K. Ono [67] appearing in a volume honoring the first author on his 60th birthday. Difficult decisions in the presentation of this manuscript were necessary. As readers peruse the manuscript, it will become immediately clear that Ramanujan left out many details, and that the frequency of omitted details increases as the manuscript progresses. Often, especially in beginning sections, it is not difficult to insert missing details. Thus, to augment readability, we have inserted such details in square brackets, so that readers can easily separate Ramanujan's exposition from that of the authors. However, other claims require considerably more amplification or are completely lacking in details. It was decided that such claims should be either proved or discussed in an appendix. Thus, further decisions needed to be made: Should all of the necessary arguments be presented, or should readers be referred to papers where complete proofs can be given. If details for all of Ramanujan's claims were to be supplied, because of the increased number of pages, this volume might necessarily be devoted *only* to this manuscript.

G.H. Hardy [280] extracted a portion of Ramanujan's manuscript and added several details in giving proofs of his aforementioned famous congruences for the partition function, namely,

$$p(5n+4) \equiv 0 \pmod 5, \; p(7n+5) \equiv 0 \pmod 7, \; p(11n+6) \equiv 0 \pmod{11}. \tag{1.0.3}$$

Thus, we feel that it is unnecessary to give any further commentary on these passages here; readers can proceed to [280] or [281, 232–238] for complete proofs. From the remarkable recent work of S. Ahlgren and M. Boylan [5], we now know that (1.0.3) are the only congruences for $p(n)$ in which the prime moduli of the congruences match the moduli of the arithmetic progressions in the arguments. We remark that we are also following the practice of Hardy, who placed additional details in square brackets, so that readers could see precisely what Ramanujan had recorded and what he had not.

These congruences (1.0.3) are the first cases of the infinite families of congruences

$$p(5^k n + \delta_{5,k}) \equiv 0 \pmod{5^k}, \tag{1.0.4}$$

$$p(7^k n + \delta_{7,k}) \equiv 0 \pmod{7^{[k/2]+1}}, \tag{1.0.5}$$

$$p(11^k n + \delta_{11,k}) \equiv 0 \pmod{11^k},$$

where $\delta_{p,k} :\equiv 1/24 \pmod{p^k}$. In Ramanujan's manuscript, he actually gives a complete proof of (1.0.4), but many of the details are omitted. These details were supplied by G.N. Watson [336], who unfortunately did not mention that his proof had its genesis in Ramanujan's unpublished manuscript. Ramanujan also began a proof of (1.0.5), but he did not finish it. If he had done so, then he would have seen that his original conjecture was incorrect and needed to be corrected as given in (1.0.5). Since proofs of (1.0.4) and (1.0.5) can now be found in several sources (which we relate in Chapter 5), there is no need to give proofs here.

It was surprising for us to learn that Ramanujan had also found congruences for $p(n)$ for the moduli 13, 17, 19, and 23 and had formulated a general conjecture about congruences for any prime modulus. However, unlike (1.0.3), these congruences do not give divisibility of $p(n)$ in any arithmetic progressions. In his doctoral dissertation, J.M. Rushforth [305] supplied all of the missing details for Ramanujan's congruences modulo 13, 17, 19, and 23. Since Rushforth's work has never been published and since his proofs are motivated by those found by Ramanujan, we have decided to publish them here for the first time. In fact, almost all of Rushforth's thesis is devoted to Ramanujan's unpublished manuscript on $p(n)$ and $\tau(n)$, and so we have extracted from it further proofs of results claimed by Ramanujan in this famous manuscript. Ramanujan's general conjecture on congruences for prime moduli was independently corrected, proved, and generalized in two distinct directions by H.H. Chan and J.-P. Serre and by Ahlgren and Boylan [5]. The proof by Chan and Serre is given here for the first time.

Many of the results in Ramanujan's manuscript are now more efficiently proved using the theory of modular forms. Indeed, much of this manuscript has given impetus for further work not only on $p(n)$ but also on the Fourier coefficients of other modular forms. Some of this work is briefly described in Chapter 5, but except for the proof by Chan and Serre, we have not employed the theory of modular forms in proofs within our commentary on Ramanujan's manuscript.

A series of Ramanujan's claims in the $p(n)/\tau(n)$ manuscript are wrong. Rushforth first noted and examined these mistakes in his thesis [305]. However, P. Moree has made a thorough examination of all these erroneous claims and corrected them in a particularly illuminating paper [228].

Lastly, we remark that the $p(n)/\tau(n)$ manuscript is found on pages 133–177, 238–243 of [283], with the latter portion, designated as Part II, in the handwriting of Watson. In fact, the original version of Part II in Ramanujan's own handwriting can be found in the library at Trinity College. One might therefore ask why Narosa published a facsimile of Watson's handwritten copy instead of Ramanujan's own version. There are two possible explanations. First, Watson's copy is closely written, while Ramanujan's more sprawling version would have required more pages in the published edition [283]. Second, the editors might not have been aware of Ramanujan's original manuscript in his *own* handwriting.

Having given an extensive account on our approach to the $p(n)/\tau(n)$ manuscript in Chapter 5, we turn to other chapters.

Chapter 6 is devoted to six entries on page 189 of the lost notebook [283], all of which are related to the content of Chapter 5, and to entries on page 182, which are related to Ramanujan's paper on congruences for $p(n)$ [276] and of course also to Chapter 5. In particular, we give proofs of two of Ramanujan's most famous identities, immediately yielding the first two congruences in (1.0.3). On page 182, we also see that Ramanujan briefly examined congruences for $p_r(n)$, where $p_r(n)$ is defined by

$$(q;q)_\infty^r = \sum_{n=0}^{\infty} p_r(n) q^n, \qquad |q| < 1.$$

Apparently, page 182 is page 5 from a manuscript, but unfortunately all of the remaining pages of this manuscript are likely lost forever. We have decided also to discuss in Chapter 6 various scattered, miscellaneous entries on $p(n)$. Most of this mélange can be found in Ramanujan's famous paper with Hardy establishing their asymptotic series for $p(n)$ [167].

In Chapter 7, we examine nine congruences that make up page 178 in the lost notebook. These congruences are on generalized tau functions and are in the spirit of Ramanujan's famous congruences for $\tau(n)$ discussed in Chapter 5.

The Rogers–Ramanujan functions are the focus of Chapter 8, wherein Ramanujan's 40 famous identities for these functions are examined. Having been sent some, or possibly all, of the 40 identities in a letter from Ramanujan, L.J. Rogers [304] proved eight of them, with Watson [333] later providing proofs for six further identities as well as giving different proofs of two of the identities proved by Rogers. For several years after Ramanujan's death, the list of 40 identities was in the hands of Watson, who made a handwritten copy for himself, and it is this copy that is published in [283]. Fortunately, he did not discard the list in Ramanujan's handwriting, which now resides in the library at Trinity College, Cambridge. Approximately ten years after Watson's death, B.J. Birch [75] found Watson's copy in the library at Oxford University and published it in 1975, thus bringing it to the mathematical public for the first time. D. Bressoud [81] and A.J.F. Biagioli [74] subsequently proved several further identities from the list.

Our account of the 40 identities in Chapter 8 is primarily taken from a *Memoir* [65] by Berndt, G. Choi, Y.-S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi. The goal of these authors was to provide proofs for as many of these identities as possible that were in the spirit of Ramanujan's mathematics. In doing so, they borrowed some proofs from Rogers, Watson, and Bressoud, while supplying many new proofs as well. After the publication of [65] in which proofs of 35 of the 40 identities were given in the spirit of Ramanujan, Yesilyurt [347], [348] devised ingenious and difficult proofs of the remaining five identities, and so these papers [347], [348] are the second primary source on which Chapter 8 is constructed.

Chapter 9 is devoted to one general theorem on certain sums of positive integral powers of theta functions, and five examples in illustration. Many offered original ideas about the entries in this chapter; in particular, Heng Huat Chan and Hamza Yesilyurt deserve special thanks. Ramanujan's primary theorem has inspired several generalizations, but it seems likely that Ramanujan's approach has not yet been discovered.

In 1915, the London Mathematical Society published Ramanujan's paper on highly composite numbers [274], [281, 78–128]. However, this is only part of the paper that Ramanujan submitted. The London Mathematical Society was in poor financial condition at that time, and to diminish expenses,

they did not publish all of Ramanujan's paper. Fortunately, the remainder of the paper has not been lost and resides in the library at Trinity College, Cambridge. In its original handwritten form, it was photocopied along with Ramanujan's lost notebook in 1988 [283]. J.-L. Nicolas and G. Robin prepared an annotated version of the paper for the first volume of the *Ramanujan Journal* in 1997 [284]. In particular, they inserted text where gaps occurred, and at the end of the paper, they provided extensive commentary on research in the field of highly composite numbers accomplished since the publication of Ramanujan's original paper [274]. Chapter 10 contains this previously unpublished manuscript of Ramanujan on highly composite numbers, as completed by Nicolas and Robin, and a moderately revised and extended version of the commentary originally written by Nicolas and Robin.

The first author is grateful to Frank Garvan, whose ideas and insights permeate Chapter 2. The second author thanks Heng Huat Chan, Song Heng Chan, and Wen-Chin Liaw for their collaboration on the papers [62] and [63], from which Chapters 3 and 4 were prepared. The last section of the former paper, which corresponds to Section 3.8 of Chapter 3, is due to Garvan, whom we thank for the many valuable remarks and suggestions on ranks and cranks that he made to the authors of [62] and [63]. Atul Dixit read Chapters 2 and 9 in detail and offered several corrections and suggestions.

We thank Paul Bateman, Heng Huat Chan, Frank Garvan, Michael Hirschhorn, Pieter Moree, Robert A. Rankin, and Jean-Pierre Serre for helpful comments on Chapter 5. We are particularly grateful to Hirschhorn for reading several versions of Chapter 5 and providing insights that we would not have otherwise observed. In particular, the argument given in square brackets near the beginning of Section 5.21 is his. He showed us that Ramanujan's conjecture on the value of c_λ at the beginning of Section 5.23 is correct. He also provided the meaning of the four mysterious numbers that Ramanujan recorded at the end of Section 5.21, but which we moved to a more proper place at the end of Section 5.24. Lastly, he provided references that Ono and the second author had overlooked in our earlier version [67] of the $p(n)/\tau(n)$ manuscript.

We are grateful to the late Professor W.N. Everitt, the School of Mathematics, and the Library at the University of Birmingham for supplying us with a copy of Rushforth's dissertation and for permission to use material from it in this volume.

Our account of Chapter 6 originates primarily from two papers by the second author that he coauthored, the first with Ae Ja Yee and Jinhee Yi, and the second with Chadwick Gugg and Sun Kim. We thank all of them for their kind collaboration. One particular entry on page 331 that we discuss in Chapter 6 was particularly puzzling, and we are grateful to L. Bruce Richmond for helpful correspondence.

The authors thank Heng Huat Chan for informing us that the results on page 189 of the lost notebook were briefly discussed by K.G. Ramanathan [273, pp. 154–155], and for discussion on one of the incorrect entries on page

189. We are also pleased to thank Scott Ahlgren for his proof of another entry on page 189.

Chapter 7 is entirely due to Dennis Eichhorn, who completed this work as part of a research assistantship under the second author at the University of Illinois.

The second author is greatly indebted to his coauthors, G. Choi, Y.-S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi, of the *Memoir* [65], which has been revised for Chapter 8 in this volume.

Chapter 9 has been significantly enhanced by correspondence that the second author had with Hamza Yesilyurt, who provided material from his forthcoming paper with A. Berkovich and Garvan [53].

It is our great pleasure to thank J.-L. Nicolas and G. Robin for their initial preparation of Chapter 10 and in particular for their insightful comments accompanying it. We thank K.S. Williams for providing several references for our commentary on Chapter 10.

We are indebted to J.P. Massias for calculating largely composite numbers and finding the meaning of the table appearing in [283, p. 280].

Heng Huat Chan, Atul Dixit, Byungchan Kim, Pieter Moree, Jaebum Sohn, and Michael Somos read in detail large portions of the manuscript for this volume and provided many useful comments and corrections.

David Kramer, who is likely the most careful and knowledgeable copy editor in the mathematics publishing realm, uncovered many errors and inconsistencies, and we thank him for his usually superb copy editing of our book. We are also indebted to Springer's TEXexpert, Rajiv Monsurate, for considerable advice.

We thank the librarian and staff at Trinity College Library, Cambridge, for providing the authors with copies of the lost notebook and several other manuscripts by Ramanujan.

The first author thanks the National Science Foundation for support while this book was being prepared. The second author thanks the National Security Agency for summer support. The second author is particularly grateful to the Sloan Foundation for a grant that relieved him of teaching duties for the 2011–2012 academic year and enabled him to complete the writing of this volume with his coauthor.

2

Ranks and Cranks, Part I

2.1 Introduction

This somewhat lengthy chapter concerns some of the most important formulas from the lost notebook [283], which are contained in only a few lines. We first introduce some standard notation that will be used throughout this chapter (and most of this book). Secondly, we record the two formulas listed at the top of page 20 (one of which is repeated in the middle of page 18). After stating these formulas, we provide history demonstrating that these entries are the genesis of some of the most important developments in the theory of partitions during the twentieth and twenty-first centuries. Next, we offer two further claims found in the lost notebook. Lastly, we provide proofs for all four claims.

For each nonnegative integer n, set

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a)_\infty := (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad |q|<1.$$

Also, set

$$(a_1,\ldots,a_m;q)_n := (a_1;q)_n\cdots(a_m;q)_n$$

and

$$(a_1,\ldots,a_m;q)_\infty := (a_1;q)_\infty\cdots(a_m;q)_\infty. \tag{2.1.1}$$

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1. \tag{2.1.2}$$

It satisfies the well-known Jacobi triple product identity [60, p. 10, Theorem 1.3.3], [12, p. 21, Theorem 2.8]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1.3}$$

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Also recall that [55, p. 34, Entry 18(iv)] for any integer n,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{2.1.4}$$

We now state the first of the two aforementioned remarkable entries from the lost notebook.

Entry 2.1.1 (pp. 18, 20). *Let* ζ_5 *be a primitive fifth root of unity, and let*

$$F_5(q) := \frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty (\zeta_5^{-1} q;q)_\infty}. \tag{2.1.5}$$

Then

$$\begin{aligned} F_5(q) = A(q^5) &- (\zeta_5 + \zeta_5^{-1})^2 q B(q^5) \\ &+ (\zeta_5^2 + \zeta_5^{-2}) q^2 C(q^5) - (\zeta_5 + \zeta_5^{-1}) q^3 D(q^5), \end{aligned} \tag{2.1.6}$$

where

$$A(q) := \frac{(q^5;q^5)_\infty G^2(q)}{H(q)}, \tag{2.1.7}$$

$$B(q) := (q^5;q^5)_\infty G(q), \tag{2.1.8}$$

$$C(q) := (q^5;q^5)_\infty H(q), \tag{2.1.9}$$

$$D(q) := \frac{(q^5;q^5)_\infty H^2(q)}{G(q)}, \tag{2.1.10}$$

with

$$G(q) := \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{2.1.11}$$

and

$$H(q) := \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{2.1.12}$$

We remark that by the famous Rogers–Ramanujan identities [15, Chapter 10],

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}.$$

The identity (2.1.6) is an example of a *dissection*. Since this and the following chapter are devoted to dissections, we offer below their definition.

Definition 2.1.1. *Let* $P(q)$ *denote any power series in* q*. Then the* t*-dissection of* P *is given by*

$$P(q) =: \sum_{k=0}^{t-1} q^k P_k(q^t). \tag{2.1.13}$$

Note that (2.1.6) provides a 5-dissection for $F_5(q)$, i.e., (2.1.6) separates $F_5(q)$ into power series according to the residue classes modulo 5 of their powers. In analogy with (2.1.6), we see that (2.1.17) in the next entry provides a 5-dissection for $f_5(q)$.

Of the dissections offered by Ramanujan in his lost notebook, some, such as (2.1.6), are given as equalities in terms of roots of unity; others are given as congruences in terms of a variable a. In Chapter 3, we establish Ramanujan's dissections in terms of congruences, while in this chapter we prove 5- and 7-dissections in the form of equalities for each of the rank and crank generating functions, whose representations are given, respectively, in (2.1.24) and (2.1.27) below. The precise definitions of the *rank* and *crank* of a partition will be given after we record the second of the two aforementioned fundamental identities.

In order to explicate our remark about congruences in the preceding paragraph, following Ramanujan in his lost notebook, we define the more general function

$$F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{2.1.14}$$

(Note that the notation (2.1.14) conflicts with that of (2.1.5); the right-hand side of (2.1.5) would be $F_{\zeta_5}(q)$ in the notation (2.1.14).) Set

$$A_n := a^n + a^{-n} \qquad \text{and} \qquad S_n := \sum_{k=-n}^{n} a^k.$$

Then [62, p. 105, Theorem 5.1]

$$F_a(q) \equiv A(q^5) + (A_1 - 1)qB(q^5) + A_2 q^2 C(q^5) - A_1 q^3 D(q^5) \pmod{S_2}. \tag{2.1.15}$$

Thus, we have replaced the primitive root ζ_5 by the general variable a. The congruence (2.1.15) is then a generalization of (2.1.6), because if we set $a = \zeta_5$ in (2.1.15), the congruence is transformed into an identity. An advantage of (2.1.15) over (2.1.6) is that we can put $a = 1$ in (2.1.15) and so immediately deduce the Ramanujan congruence

$$p(5n+4) \equiv 0 \pmod{5},$$

where $p(n)$ is the number of partitions of n. Although (2.1.15) appears to be more general than (2.1.6), in fact, *it is not*. It is shown in [62, pp. 118–119] that (2.1.15) can be derived from (2.1.6). In Section 3.8 of the following chapter we reproduce that argument, which is due to F.G. Garvan.

Entry 2.1.2 (p. 20). *Let ζ_5 be a primitive fifth root of unity, and let*

$$f_5(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n \left(\zeta_5^{-1} q;q\right)_n}. \tag{2.1.16}$$

Then

$$f_5(q) = A(q^5) + \left(\zeta_5 + \zeta_5^{-1} - 2\right)\phi(q^5) + qB(q^5) + \left(\zeta_5 + \zeta_5^{-1}\right) q^2 C(q^5) - \left(\zeta_5 + \zeta_5^{-1}\right) q^3 \left\{ D(q^5) - \left(\zeta_5^2 + \zeta_5^{-2} - 2\right) \frac{\psi(q^5)}{q^5} \right\}, \tag{2.1.17}$$

where $A(q)$*,* $B(q)$*,* $C(q)$*, and* $D(q)$ *are given in* (2.1.7)–(2.1.10)*, and where*

$$\phi(q) := \sum_{n=0}^{\infty} \phi_n q^n := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \tag{2.1.18}$$

and

$$\frac{\psi(q)}{q} := -\frac{1}{q} + \sum_{n=0}^{\infty} \psi_n q^n := \sum_{n=0}^{\infty} \frac{q^{5n^2-1}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}. \tag{2.1.19}$$

Corollaries of the preceding entry appear in the middle of page 184 in the lost notebook. Since their proofs are immediate consequences of Entry 2.1.2, we offer them here.

Entry 2.1.3 (p. 184). *Write*

$$\sum_{n=0}^{\infty} \lambda_n q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 + \frac{\sqrt{5}+1}{2} q + q^2) \cdots (1 + \frac{\sqrt{5}+1}{2} q^n + q^{2n})}. \tag{2.1.20}$$

Then,

$$\sum_{n=0}^{\infty} \lambda_{5n+1} q^n = \frac{(q^5;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = (q^5;q^5)_\infty G(q), \tag{2.1.21}$$

$$\sum_{n=0}^{\infty} \lambda_{5n+2} q^n = -\frac{\sqrt{5}+1}{2} \frac{(q^5;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = -\frac{\sqrt{5}+1}{2} (q^5;q^5)_\infty H(q), \tag{2.1.22}$$

$$\lambda_{5n-1} \text{ is identically zero.} \tag{2.1.23}$$

Proof. In the definition (2.1.16), set $\zeta_5 = e^{4\pi i/5}$; therefore, $\zeta_5 + \zeta_5^{-1} = -\frac{\sqrt{5}+1}{2}$. Using then the notation (2.1.20), equate coefficients of q^{5n+1} on both sides of (2.1.17). Divide both sides by q and lastly replace q^5 by q in the resulting identity to establish (2.1.21). Similarly, to prove (2.1.22), equate coefficients of q^{5n+2} on both sides of (2.1.17). Divide both sides by q^2 and replace q^5 by q. Finally, we note that the dissection (2.1.17) does not have any powers of the form q^{5n-1}, and so (2.1.23) is immediate. □

Before presenting the third and fourth entries for this chapter, as remarked above, it is appropriate to say something about these results, which lay hidden

during one of the most interesting developments in the theory of partitions during the twentieth century.

In 1944, F. Dyson [127] published a paper filled with fascinating conjectures from the theory of partitions. Namely, Dyson began by defining the *rank* of a partition to be the largest part minus the number of parts. Dyson's objective was to provide a purely combinatorial description of Ramanujan's theorem that 5 divides $p(5n+4)$. In particular, Dyson conjectured that the partitions of $5n+4$ classified by their rank modulo 5 did, indeed, produce five sets of equal cardinality, namely $p(5n+4)/5$. He was also led to conjecture that the partitions of $7n+5$, classified by rank, split into seven sets each of cardinality $p(7n+5)/7$. This would prove the second Ramanujan congruence, namely, that 7 divides $p(7n+5)$. He also conjectured a generating function for ranks. If $N(m,n)$ denotes the number of partitions of n with rank m, then Dyson's observations make clear he knew that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n}. \tag{2.1.24}$$

Observe that if we take $z=1$ in (2.1.24), then (2.1.24) reduces to the well-known generating function for $p(n)$,

$$\sum_{n=0}^{\infty} p(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2},$$

which is due to Euler. If we set $z=-1$ in (2.1.24), we obtain Ramanujan's mock theta function $f(q)$.

Unfortunately, it turned out that the Ramanujan congruence

$$p(11n+6) \equiv 0 \pmod{11} \tag{2.1.25}$$

was *not* explicable in the same way that worked for $p(5n+4)$ and $p(7n+5)$. So Dyson conjectured the existence of an unknown parameter of partitions, which he whimsically called "the crank," to explain (2.1.25).

In 1954, A.O.L. Atkin and H.P.F. Swinnerton-Dyer [28] proved all of Dyson's conjectures; however, the crank remained undiscovered.

The real breakthrough in this study was made by Garvan in his Ph.D. thesis [146] at Pennsylvania State University in 1986. Garvan's thesis is primarily devoted to the Entries 2.1.1 and 2.1.2 given above. Observe that Entry 2.1.2 is devoted to a special case of the generating function (2.1.24) for ranks. Not only was Garvan able to prove these two entries, but he also deduced all of the Atkin and Swinnerton-Dyer results for the modulus 5 from Entry 2.1.2. As for Entry 2.1.1, Garvan defined a "vector crank," which did provide a combinatorial explanation for 11 dividing $p(11n+6)$, but did this via certain triples of partitions, i.e., vector partitions. Subsequently, Garvan and Andrews [17] found the actual crank. Namely, for any given partition π, let $\ell(\pi)$ denote

the largest part of π, $\omega(\pi)$ the number of ones appearing in π, and $\mu(\pi)$ the number of parts of π larger than $\omega(\pi)$. Then the crank, $c(\pi)$, is given by

$$c(\pi) = \begin{cases} l(\pi), & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0. \end{cases} \tag{2.1.26}$$

For $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n \leq 1$ we set

$$M(m,n) = \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

The generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n) a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{2.1.27}$$

As shown by Andrews and Garvan [17], the combinatorial equivalent of (2.1.27) is given by (2.1.26). Note that if we set $a = 1$ in (2.1.27), we obtain Euler's original generating function for $p(n)$,

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

Observe that Entry 2.1.1 provides an identity for a special instance of the generating function for cranks.

Thus, although Ramanujan did not record combinatorial definitions of the rank and crank in his lost notebook (in fact, there are hardly any words at all in the lost notebook), he had discovered their generating functions. From the entries on ranks and cranks in this and the following two chapters, it is clear that Ramanujan placed considerable importance on these ideas, and it is regrettable indeed that we do not know Ramanujan's motivations and thoughts on these two fundamental concepts in the theory of partitions.

We finally record the last two results to be included in this chapter. Actually in each entry below, Ramanujan gives only the left-hand side or hints at it. However, the analogies with Entries 2.1.1 and 2.1.2 are so clear that we have filled in what was clearly intended for the right-hand sides. For Entry 2.1.4, Garvan has supplied the right-hand side in [146, p. 62].

Entry 2.1.4 (p. 19). *Let ζ_7 be a primitive seventh root of unity, and let*

$$F_7(q) := \frac{(q;q)_\infty}{(\zeta_7 q;q)_\infty (\zeta_7^{-1} q;q)_\infty}. \tag{2.1.28}$$

Then

$$F_7(q) = (q^7;q^7)_\infty \Big\{ X^2(q^7) + \left(\zeta_7 + \zeta_7^{-1} - 1\right) qX(q^7)Y(q^7) \tag{2.1.29}$$
$$+ \left(\zeta_7^2 + \zeta_7^{-2}\right) q^2Y^2(q^7) + \left(\zeta_7^3 + \zeta_7^{-3} + 1\right) q^3X(q^7)Z(q^7)$$
$$- \left(\zeta_7 + \zeta_7^{-1}\right) q^4Y(q^7)Z(q^7) - \left(\zeta_7^2 + \zeta_7^{-2} + 1\right) q^6Z^2(q^7) \Big\},$$

where

$$X(q) := \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 3 \pmod 7}}^{\infty} (1-q^n)^{-1}, \tag{2.1.30}$$
$$Y(q) := \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 2 \pmod 7}}^{\infty} (1-q^n)^{-1}, \tag{2.1.31}$$
$$Z(q) := \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 1 \pmod 7}}^{\infty} (1-q^n)^{-1}. \tag{2.1.32}$$

There are series representations for $X(q)$, $Y(q)$, and $Z(q)$ that yield analogues of the Rogers-Ramanujan identities for $G(q)$ and $H(q)$ [12, p. 117, Exercise 10].

In order to state the last major entry of this chapter, we need considerable notation. First, introducing the notation of Atkin and Swinnerton-Dyer [28, p. 94], we let

$$\Sigma(z,\zeta,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n\zeta^n q^{3n(n+1)/2}}{1-zq^n}. \tag{2.1.33}$$

Furthermore, to simplify future considerations, in particular to state and prove Entry 2.1.5 below, we make the conventions

$$P_7(a) := \left(q^{7a}, q^{49-7a}; q^{49}\right)_\infty \quad (a \neq 0), \tag{2.1.34}$$
$$P_7(0) := \left(q^{49}; q^{49}\right)_\infty, \tag{2.1.35}$$
$$\Sigma_7(a,b) := \Sigma\left(q^{7a}, q^{7b}, q^{49}\right) \quad (a \neq 0), \tag{2.1.36}$$
$$\Sigma_7(0,b) := \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{147n(n+1)/2+7bn}}{1-q^{49n}}. \tag{2.1.37}$$

We note in passing that by (2.1.30)–(2.1.32),

$$P_7(1) = \frac{(q^7;q^7)_\infty Z(q^7)}{(q^{49};q^{49})_\infty}, \tag{2.1.38}$$
$$P_7(2) = \frac{(q^7;q^7)_\infty Y(q^7)}{(q^{49};q^{49})_\infty}, \tag{2.1.39}$$
$$P_7(3) = \frac{(q^7;q^7)_\infty X(q^7)}{(q^{49};q^{49})_\infty}. \tag{2.1.40}$$

Finally, we are ready to supply the right-hand side for the analogue of Entry 2.1.2 for the modulus 7.

Entry 2.1.5 (p. 19). *Let ζ_7 be a primitive seventh root of unity, and let*

$$f_7(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n}. \tag{2.1.41}$$

Then

$$\begin{aligned} f_7(q) &= \left(2-\zeta_7-\zeta_7^{-1}\right)\left(1-A_7(q^7)+q^7Q_1(q^7)\right)+A_7(q^7) \\ &\quad + qT_1(q^7)+q^2\left\{\left(\zeta_7+\zeta_7^{-1}\right)B_7(q^7)+q^{14}Q_3(q^7)(\zeta_7+\zeta_7^{-1}-\zeta_7^{-2}-\zeta_7^{2}\right\} \\ &\quad + q^3T_2(q^7)\left(1+\zeta_7^2+\zeta_7^{-2}\right)-q^4\left(\zeta_7^2+\zeta_7^{-2}\right)T_3(q^7) \\ &\quad + q^6\left\{q^7Q_2(q^7)\left(\zeta_7^2+\zeta_7^{-2}-\zeta_7^3-\zeta_7^{-3}\right)-C_7(q^7)\left(1+\zeta_7^3+\zeta_7^{-3}\right)\right\}, \end{aligned} \tag{2.1.42}$$

where

$$A_7(q) := \frac{(q^7,q^3,q^4;q^7)_\infty}{(q,q^2,q^5,q^6;q^7)_\infty}, \tag{2.1.43}$$

$$B_7(q) := \frac{(q^7,q^2,q^5;q^7)_\infty}{(q,q^3,q^4,q^6;q^7)_\infty}, \tag{2.1.44}$$

$$C_7(q) := \frac{(q^7,q,q^6;q^7)_\infty}{(q^2,q^3,q^4,q^5;q^7)_\infty}, \tag{2.1.45}$$

and for $m=1,2,3$,

$$Q_m(q^7) := \frac{\varSigma_7(m,0)}{P_7(0)} \tag{2.1.46}$$

and

$$T_m(q^7) := \frac{P_7(0)}{P_7(m)}. \tag{2.1.47}$$

We remark that the functions $Q_m(q^7)$ in (2.1.46) can be expressed in terms of the generating function for ranks. By a result of Garvan [146, p. 68, Lemma (7.9)], for $|q|<|z|<1/|q|$ and $z\neq 1$,

$$-1+\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n} = \frac{z}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{1-zq^n}.$$

Hence, after modest rearrangement, we find that

$$\varSigma_7(m,0) = \frac{(q^{49};q^{49})_\infty}{q^{7m}}\left\{-1+\sum_{n=0}^{\infty}\frac{q^{49n^2}}{(q^{7m};q^{49})_{n+1}(q^{49-7m};q^{49})_n}\right\}.$$

Throughout this chapter our work will follow closely the marvelous papers by Atkin and Swinnerton-Dyer [28] and Garvan [146].

2.2 Proof of Entry 2.1.1

Here we shall follow the elegant proof given by Garvan [146]. Throughout this section ζ_5 is a primitive fifth root of unity. We begin with the observation [146, p. 58, Lemma (3.9)]

$$\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty}=G(q^5)+q(\zeta_5+\zeta_5^{-1})H(q^5), \tag{2.2.1}$$

where $G(q)$ is defined in (2.1.11) and $H(q)$ is defined in (2.1.12). We prove the identity (2.2.1). Using the Jacobi triple product identity (2.1.3) twice, we find that

$$\begin{aligned}
\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty}&=\frac{(q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}{(q,\zeta_5 q,\zeta_5^{-1}q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}\\
&=\frac{(q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\\
&=\frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\zeta_5^{2n}q^{(n^2+n)/2}\\
&=\frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{\nu=-2}^{2}\sum_{m=-\infty}^{\infty}(-1)^{5m+\nu}\zeta_5^{10m+2\nu}q^{(5m+\nu)(5m+\nu+1)/2}\\
&=\frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{\nu=-2}^{2}(-1)^\nu\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\sum_{m=-\infty}^{\infty}(-1)^m q^{\left(25m^2+(10\nu+5)m\right)/2}\\
&=\frac{1}{(1-\zeta_5^{-2})}\sum_{\nu=-2}^{2}(-1)^\nu\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\frac{f(-q^{15+5\nu},-q^{10-5\nu})}{(q^5;q^5)_\infty}\\
&=\frac{1}{(1-\zeta_5^{-2})}\sum_{\nu=-2}^{2}(-1)^\nu\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\frac{\left(q^{15+5\nu},q^{10-5\nu},q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty}.
\end{aligned}$$

Now, by (2.1.11) and (2.1.12),

$$\frac{\left(q^{15+5\nu},q^{10-5\nu},q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty}=\begin{cases}G(q^5), & \text{if } \nu=0,-1,\\ H(q^5), & \text{if } \nu=1,-2,\\ 0, & \text{if } \nu=2.\end{cases}$$

Hence,

$$\begin{aligned}
\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty}&=\frac{1}{(1-\zeta_5^{-2})}G(q^5)\left(1-\zeta_5^{-2}\right)\\
&\quad+\frac{1}{(1-\zeta_5^{-2})}H(q^5)\left(-\zeta_5^2 q+\zeta_5^{-4}q\right)\\
&=G(q^5)+q\left(\zeta_5+\zeta_5^{-1}\right)H(q^5),
\end{aligned}$$

which is (2.2.1).

Next, we continue to follow Garvan in [146, p. 60, Lemma 3.18] and so employ the identity

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left(\frac{G(q^5)}{H(q^5)} - q - q^2 \frac{H(q^5)}{G(q^5)} \right), \tag{2.2.2}$$

which is one of the famous identities for the Rogers–Ramanujan continued fraction [15, p. 11, equation (1.1.10)]

$$\frac{1}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+}\cdots = \frac{H(q)}{G(q)}.$$

We now multiply together (2.2.1) and (2.2.2) to obtain

$$\begin{aligned}
&\frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty} = \left(G(q^5) + q(\zeta_5+\zeta_5^{-1})H(q^5)\right) \\
&\quad \times (q^{25};q^{25})_\infty \left(\frac{G(q^5)}{H(q^5)} - q - q^2 \frac{H(q^5)}{G(q^5)} \right) \\
&= (q^{25};q^{25})_\infty \left\{ \frac{G^2(q^5)}{H(q^5)} + q\left(-1+\zeta_5+\zeta_5^{-1}\right)G(q^5) \right. \\
&\qquad \left. + q^2\left(-1-(\zeta_5+\zeta_5^{-1})\right)H(q^5) + q^3\left(-(\zeta_5+\zeta_5^{-1})\right)\frac{H^2(q^5)}{G(q^5)} \right\} \\
&= A(q^5) - q(\zeta_5+\zeta_5^{-1})^2 B(q^5) \\
&\quad + q^2\left(\zeta_5^2+\zeta_5^{-2}\right)C(q^5) - \left(\zeta_5+\zeta_5^{-1}\right)q^3 D(q^5),
\end{aligned}$$

and Entry 2.1.1 is proved.

2.3 Background for Entries 2.1.2 and 2.1.4

As was mentioned in Section 2.1, Atkin and Swinnerton-Dyer [28] proved the conjectures of Dyson [127]. Garvan [146] proved that their work for the modulus 5 was in fact equivalent to Entry 2.1.2. Our proof here relies completely on Garvan's observation. We will modify the work of Atkin and Swinnerton-Dyer to the extent that we will eschew using their Lemma 2, which we state below.

Lemma 2.3.1. *Let $f(z)$ be a single-valued analytic function of z, except possibly for a finite number of poles, in every region $0 \le z_1 \le |z| \le z_2$; and suppose that for some constants A and w with $0 < |w| < 1$, and some (positive, zero, or negative) integer n, we have*

$$f(zw) = Az^n f(z)$$

identically in z. Then either $f(z)$ has exactly n more poles than zeros in

$$|w| \le |z| \le 1,$$

or $f(z)$ vanishes identically.

While this is a beautiful, powerful, and useful result, it is unlikely to have been the type of result that Ramanujan would have utilized.

The principal idea is to transform (2.1.16), (2.1.18), and (2.1.19) into certain bilateral series, which are called higher-level Appell series [355]. In particular, see Lemma 2.4.1 and the functions (2.1.33) and (2.3.11), which we define and develop in the next several pages.

The next identity does not appear in the lost notebook. However, it is effectively a generalization of Entries 12.4.4 (as restated in (12.4.15)) and 12.5.3 (as restated in (12.5.14)) in our first book [15, pp. 276, 283]. Consequently, it is a partial fraction decomposition of precisely the sort that Ramanujan often considered.

Lemma 2.3.2. [28, p. 94, Lemma 7] *For $\Sigma(z,\zeta,q)$ defined by* (2.1.33),

$$\zeta^3\Sigma(z\zeta,\zeta^3,q)+\Sigma(z\zeta^{-1},\zeta^{-3},q)-\zeta\frac{(\zeta^2,q/\zeta^2;q)_\infty}{(\zeta,q/\zeta;q)_\infty}\Sigma(z,1,q)$$
$$=\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}. \tag{2.3.1}$$

This formula was first proved by G.N. Watson [335], and we shall follow his proof. M. Jackson [185] has given a third proof from the theory of q-hypergeometric series, and S.H. Chan [105] has established a considerable generalization of Lemma 2.3.2.

Proof. Let us fix a positive integer N and consider the partial fraction decomposition with respect to z of the rational function

$$F_N(z):=\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_N}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_N}. \tag{2.3.2}$$

This function has simple poles at $z=\zeta q^m$, q^m, and $\zeta^{-1}q^m$ for $-(N-1)\le m\le N$. Hence, we see that

$$F_N(z):=\sum_{m=-N}^{N-1}\frac{A_m(N)}{1-z\zeta q^m}+\sum_{m=-N}^{N-1}\frac{B_m(N)}{1-zq^m/\zeta}+\sum_{m=-N}^{N-1}\frac{C_m(N)}{1-zq^m}. \tag{2.3.3}$$

Now for any integer m, algebraic simplification reveals that

$$(xq^{-m},q^{1+m}/x;q)_N=(-1)^mq^{-m(m+1)/2}x^m(q/x;q)_{N+m}(x;q)_{N-m}. \tag{2.3.4}$$

First, after three applications of (2.3.4), with $x=\zeta^{-2},\zeta^{-1},1$, respectively, we find that

$$A_m(N)=\lim_{z\to\zeta^{-1}q^{-m}}(1-z\zeta q^m)F_N(z)=\frac{(-1)^mq^{3m(m+1)/2}\zeta^{3m+3}}{(q/\zeta^2;q)_{N-m-1}(\zeta^2;q)_{N+m+1}}$$
$$\times\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_N}{(q/\zeta;q)_{N-m-1}(\zeta;q)_{N+m+1}(q;q)_{N-m-1}(q;q)_{N+m}}, \tag{2.3.5}$$

and

$$\lim_{N\to\infty} A_m(N) = (-1)^m q^{3m(m+1)/2}\zeta^{3m+3}. \tag{2.3.6}$$

Second, applying (2.3.4) three times once again, but now with $x = 1, \zeta, \zeta^2$, respectively, we find that

$$\begin{aligned} B_m(N) &= \lim_{z\to\zeta q^{-m}} (1 - z\zeta^{-1}q^m)F_N(z) \qquad (2.3.7)\\ &= \frac{(-1)^m q^{3m(m+1)/2}\zeta^{-3m}(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_N}{(\zeta;q)_{N-m}(q/\zeta;q)_{N+m}(\zeta^2;q)_{N-m}(q/\zeta^2;q)_{N+m}(q;q)_{N-m-1}(q;q)_{N+m}}, \end{aligned}$$

and

$$\lim_{N\to\infty} B_m(N) = (-1)^m q^{3m(m+1)/2}\zeta^{-3m}. \tag{2.3.8}$$

Third, applying (2.3.4) with $x = \zeta^{-1}, 1, \zeta$, respectively, we find that

$$\begin{aligned} C_m(N) &= \lim_{z\to q^{-m}} (1 - zq^m)F_N(z) \qquad (2.3.9)\\ &= \frac{-\zeta(-1)^m q^{3m(m+1)/2}(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_N}{(\zeta;q)_{N-m}(q/\zeta;q)_{N+m}(\zeta;q)_{N+m+1}(q/\zeta;q)_{N-m-1}(q;q)_{N-m-1}(q;q)_{N+m}}, \end{aligned}$$

and

$$\lim_{N\to\infty} C_m(N) = \frac{-\zeta(\zeta^2, q/\zeta^2;q)_\infty(-1)^m q^{3m(m+1)/2}}{(\zeta, q/\zeta;q)_\infty}. \tag{2.3.10}$$

We can now easily deduce (2.3.1). Clearly $F_N(z)$ converges uniformly to the right-hand side of (2.3.1) as $N \to \infty$.

Equations (2.3.6), (2.3.8), and (2.3.10) when applied to (2.3.3) yield the left-hand side of (2.3.1), provided we are allowed to take the limit $N \to \infty$ inside the summation signs, and indeed this interchange of limit and summation is legitimate because the convergence is uniformly independent of m, and the resulting series, after letting $N \to \infty$, is convergent as long as $|q| < 1$ and z is restricted away from the poles. Thus (2.3.1) is proved. □

Following Atkin and Swinnerton-Dyer [28, p. 96], we now define

$$\begin{aligned} g(z,q) := z\frac{(z^2, q/z^2;q)_\infty}{(z, q/z;q)_\infty}\Sigma(z,1,q) - z^3\Sigma(z^2,z^3,q) \qquad (2.3.11)\\ - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n z^{-3n}q^{3n(n+1)/2}}{1-q^n}. \qquad (2.3.12) \end{aligned}$$

Now the definition of $g(z,q)$ is motivated as follows. We would like to set $\zeta = z$ in (2.3.1); however, this would produce an undefined term at $n = 0$ in $\sum(1, z^{-3}, q)$ in (2.3.1). Note that $g(z,q)$ is the negative of the left-hand side of (2.3.1), with $\zeta = z$ and the one offending term at $n = 0$ in $\sum(1, z^{-3}, q)$ removed. Thus,

$$g(z,q) = \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right). \tag{2.3.13}$$

It is now a straightforward exercise to prove the next lemma, which is the second half of Lemma 8 in [28, p. 96].

Lemma 2.3.3. *We have*

$$g(z,q) + g(q/z,q) = 1. \tag{2.3.14}$$

Proof. We proceed as follows:

$$\begin{aligned}
g(z,q) + g(q/z,q) &= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right) \\
&\quad + \lim_{\zeta \to q/z} \left(\frac{1}{1 - q/(z\zeta)} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(q/(z\zeta), z\zeta, q/z, z, q\zeta/z, z/\zeta; q)_\infty} \right) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right. \\
&\quad \left. + \frac{1}{1 - \zeta/z} + \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} + \frac{1}{1 - \zeta/z} \right) \\
&= 1,
\end{aligned}$$

where in the antepenultimate line we replaced ζ by q/ζ in the second limit and algebraically simplified the second infinite product into the first product with opposite sign. This then completes the proof of (2.3.14). □

Our next objective is to establish a second component of Lemma 8 of Atkin and Swinnerton-Dyer [28, p. 96].

Lemma 2.3.4. *We have*

$$g(z,q) + g(z^{-1},q) = -2. \tag{2.3.15}$$

Proof. Replacing ζ by $1/\zeta$ in the second equality below, we find that

$$\begin{aligned}
&g(z,q) + g(z^{-1},q) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} + \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right) \\
&\quad + \lim_{\zeta \to 1/z} \left(\frac{1}{1 - 1/(z\zeta)} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(1/(z\zeta), q\zeta z, 1/z, qz, \zeta/z, qz/\zeta; q)_\infty} \right) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{1-\zeta/z} - \frac{(1/\zeta, \zeta q, 1/\zeta^2, \zeta^2 q, q, q; q)_\infty}{(\zeta/z, qz/\zeta, 1/z, qz, 1/(\zeta z), qz\zeta; q)_\infty}\Bigg) \\
&= \lim_{\zeta\to z}\left(1 - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty}\right. \\
&\left.+ \frac{z^3}{\zeta^3}\frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty}\right) \\
&= \lim_{\zeta\to z}\left(1 - \frac{(1-z^3/\zeta^3)}{(1-z/\zeta)}\frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(qz/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty}\right) \\
&= 1 - \lim_{\zeta\to z}\frac{(1-z^3/\zeta^3)}{(1-z/\zeta)} \\
&= 1 - \lim_{\zeta\to z}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2}\right) \\
&= 1 - 3 = -2,
\end{aligned}
$$

as desired to prove (2.3.15). ☐

As an immediate consequence of (2.3.15), we deduce the next corollary.

Corollary 2.3.1. *With $g(z,q)$ defined by (2.3.11),*

$$
g(z,q) - g(zq,q) = -3.
$$

Proof. By (2.3.15) and (2.3.14),

$$
\begin{aligned}
g(z,q) - g(zq,q) &= \big(g(z,q) + g(z^{-1},q)\big) - \big(g(z^{-1},q) + g(zq,q)\big) \\
&= -2 - 1 = -3, \qquad (2.3.16)
\end{aligned}
$$

as desired to prove (2.3.15). ☐

We shall now prove the other identity that occurs in Lemma 8 of [28].

Lemma 2.3.5. *If $g(z,q)$ is defined by (2.3.11), then*

$$
2g(z,q) - g(z^2,q) - \frac{\left(z^3, \dfrac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \dfrac{q}{z}; q\right)_\infty^2 \left(z^4, \dfrac{q}{z^4}; q\right)_\infty} + 1 = 0. \qquad (2.3.17)
$$

Proof. To prove (2.3.17), we denote the left-hand side of (2.3.17) by $f(z)$. Then, by Corollary 2.3.1,

$$
\begin{aligned}
f(z) - f(zq) &= 2\big(g(z,q) - g(zq,q)\big) \\
&\quad - \big\{\big(g(z^2,q) - g(z^2q,q)\big) + \big(g(z^2q,q) - g(z^2q^2,q)\big)\big\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\left(z^3, \frac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \frac{q}{z}; q\right)_\infty^2 \left(z^4, \frac{q}{z^4}; q\right)_\infty} + \frac{\left(z^3q^3, \frac{1}{z^3q^2}, q; q\right)_\infty^2}{\left(zq, \frac{1}{z}; q\right)_\infty^2 \left(z^4q^4, \frac{1}{z^4q^3}; q\right)_\infty} \\
&= 2(-3) - (-3-3) \\
&- \frac{\left(z^3, \frac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \frac{q}{z}; q\right)_\infty^2 \left(z^4, \frac{q}{z^4}; q\right)_\infty} + \frac{\left(z^3, \frac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \frac{q}{z}; q\right)_\infty^2 \left(z^4, \frac{q}{z^4}; q\right)_\infty} \\
&= 0. \qquad (2.3.18)
\end{aligned}
$$

Next, we show that if $\omega = e^{2\pi i/3}$, then

$$
g(\omega, q) = -1. \qquad (2.3.19)
$$

By (2.3.11), we see that

$$
\begin{aligned}
g(\omega, q) &= -\sum(\omega, 1, q) - \sum(\omega^2, 1, q) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-q^n} \\
&= -\frac{1}{1-\omega} - \frac{1}{1-\omega^2} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{3n(n+1)/2} \\
&\quad \times \left(\frac{1}{1-\omega q^n} + \frac{1}{1-\omega^2 q^n} + \frac{1}{1-q^n} \right) \\
&= -1 - 3\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-q^{3n}} \\
&= -1,
\end{aligned}
$$

because the nth and $(-n)th$ terms of the sum cancel. This proves (2.3.19).

Substituting $z = \omega$ on the left-hand side of (2.3.17), invoking (2.3.19), and observing that $g(\omega^2, q) = -1$ as well, we see that

$$
f(\omega) = -2 + 1 - 0 + 1 = 0. \qquad (2.3.20)
$$

Observe that (2.3.18) and (2.3.20) imply that

$$
f(\omega q^n) = 0 \quad \text{for} \quad n \geq 0.
$$

Therefore we need to prove that $f(z)$ is analytic except possibly at $z = 0, \infty$. However, the functional equation (2.3.18), namely

$$
f(z) = f(qz),
$$

means that we need to examine the possible poles only in the annulus $|q| < |z| \leq 1$. Potential poles occur at $z = \pm 1, \pm i, \pm q^{1/4}, \pm i q^{1/4}$ and are at worst simple poles. However, when we return to the definition of g in (2.3.11) to calculate the residue at each possible pole, we find that it is 0. Consequently, $f(z)$ is analytic except possibly at 0 and at ∞. However, $f(z)$ must, in fact, be analytic at $z = 0$ also, because all values of $f(z)$ in a deleted neighborhood of 0 are bounded by the maximum value of $|f(z)|$ in the annulus $|q| < |z| \leq 1$, owing to the functional equation above, and if $f(z)$ had a singularity at $z = 0$ (either a pole or an essential singularity), then it would have to be unbounded in a neighborhood of $z = 0$.

So we have shown that $f(z)$ is analytic for $|z| < 1$ and that $f(z)$ is identically 0 on a sequence of points ωq^n that converge in the interior of $|z| < 1$. We conclude that

$$f(z) \equiv 0,$$

and (2.3.17) is established. □

2.4 Proof of Entry 2.1.2

Let

$$S_5(b) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n q^{bn+n(3n+1)/2}}{1 - q^{5n}}. \tag{2.4.1}$$

Replacing n by $-n$, we see that

$$S_5(b) = -S_5(4 - b), \tag{2.4.2}$$

from which it readily follows that

$$S_5(2) = 0. \tag{2.4.3}$$

Furthermore, either applying the Jacobi triple product identity (2.1.3) and algebraic simplification or applying (2.1.4) with $n = b/3$, $(b-1)/3$, and $(b+1)/3$, respectively, we find that

$$\begin{aligned} S_5(b) - S_5(b+5) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{bn+n(3n+1)/2} - 1 = f(-q^{2+b}, -q^{1-b}) - 1 \\ &= \begin{cases} (-1)^b q^{-b(b+1)/6} (q;q)_\infty - 1, & \text{if } b \equiv 0 \pmod{3}, \\ -1, & \text{if } b \equiv 1 \pmod{3}, \\ (-1)^{b-1} q^{-b(b+1)/6} (q;q)_\infty - 1, & \text{if } b \equiv 2 \pmod{3}. \end{cases} \end{aligned} \tag{2.4.4}$$

We now establish the relationship between $S_5(b)$ and Entry 2.1.2.

Lemma 2.4.1. *We have*

$$(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(q/z)_n} = (1-z) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n}. \tag{2.4.5}$$

Proof. Recall that Entry 4.2.16 of [16, p. 89] is given by

$$\begin{aligned}(abq)_\infty &\sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(-aq)_n(-bq)_n} \\ &= 1 + (1+a)(1+b) \sum_{n=1}^{\infty} \frac{(-1)^n (abq)_{n-1}(1-abq^{2n}) a^n b^n q^{n(3n+1)/2}}{(q)_n(1+aq^n)(1+bq^n)}.\end{aligned}$$

Setting $a = 1/b = -z$ above, we find that

$$\begin{aligned}(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(q/z)_n} &= 1 + (1-z)(1-1/z) \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n) q^{n(3n+1)/2}}{(1-zq^n)(1-q^n/z)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}\{(1-z)(1-q^n/z) + (1-1/z)(1-zq^n)\}}{(1-zq^n)(1-q^n/z)} \\ &= 1 + (1-z) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n} - \frac{1-z}{z} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-q^n/z} \\ &= 1 + (1-z) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n} + (1-z) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n-1)/2}}{1-zq^{-n}} \\ &= (1-z) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n},\end{aligned}$$

which establishes (2.4.5). □

Lemma 2.4.1 gives a connection with a variant of the rank-generating function (2.1.41), namely

$$R_b(q) := \sum_{n=0}^{\infty} \frac{q^{7n^2}}{(q^b;q^7)_{n+1}(q^{7-b};q^7)_n}, \tag{2.4.6}$$

and the functions (2.1.33) and (2.1.36). By (2.4.5) and (2.4.6), with $z = q^{7b}$,

$$\begin{aligned}R_b(q^7) &= \sum_{n=0}^{\infty} \frac{q^{49n^2}}{(q^{7b};q^{49})_{n+1}(q^{49-7b};q^{49})_n} \\ &= \frac{1}{(q^{49};q^{49})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(3n+1)/2}}{1-q^{49n+7b}} \\ &= \frac{\Sigma_7(b,-7)}{P_7(0)},\end{aligned}$$

by (2.1.33) and (2.1.36).

Lemma 2.4.2. *Let ζ_5 be a primitive fifth root of unity. Then*

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n(\zeta_5^{-1}q;q)_n} \\ = 1 + \big(S_5(1) - 2S_5(4)\big) + (\zeta_5 + \zeta_5^{-1})\big(2S_5(1) + S_5(4)\big). \tag{2.4.7}$$

Proof. By (2.4.5), and then by (2.4.2) and (2.4.3),

$$\begin{aligned}
&(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n(\zeta_5^{-1}q;q)_n} \\
&= 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1-\zeta_5)(1-q^n)(1-\zeta_5^2 q^n)(1-\zeta_5^3 q^n)(1-\zeta_5^4 q^n)}{1-q^{5n}} \\
&= 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-q^{5n}} \Big((1-q^{4n}) + (q^n-1)\zeta_5 + (q^{2n}-q^n)\zeta_5^2 \\
&\quad + (q^{3n}-q^{2n})\zeta_5^3 + (q^{4n}-q^{3n})\zeta_5^4\Big) \\
&= 1 + S_5(0)(1-\zeta_5) + S_5(1)(\zeta_5-\zeta_5^2) + S_5(2)(\zeta_5^2-\zeta_5^3) + S_5(3)(\zeta_5^3-\zeta_5^4) \\
&\quad + S_5(4)(\zeta_5^4-1) \\
&= 1 + (\zeta_5+\zeta_5^{-1}-2)S_5(4) + (\zeta_5+\zeta_5^{-1}-\zeta_5^2-\zeta_5^{-2})S_5(1) \\
&= 1 - 2S_5(4) + (\zeta_5+\zeta_5^{-1})\big(S_5(4)+S_5(1)\big) - (\zeta_5^2+\zeta_5^{-2})S_5(1) \\
&= 1 + \big(S_5(1)-2S_5(4)\big) + (\zeta_5+\zeta_5^{-1})\big(2S_5(1)+S_5(4)\big).
\end{aligned}$$

∎

Lemma 2.4.3. *Recall that $\phi(q)$ and $\psi(q)$ are defined in (2.1.18) and (2.1.19), respectively, and that $\sum(z,\zeta,q)$ is defined in (2.1.33). Then*

$$\phi(q) = \frac{q}{(q^5;q^5)_\infty}\Sigma(q,1,q^5), \tag{2.4.8}$$

$$\frac{\psi(q)}{q} = \frac{q}{(q^5;q^5)_\infty}\Sigma(q^2,1,q^5). \tag{2.4.9}$$

Proof. To prove (2.4.8), we apply (2.4.5) with q replaced by q^5 and then $z=q$. Then dividing both sides by $1-q$ and using (2.1.18), we find that

$$(q^5;q^5)_\infty\{\phi(q)+1\} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(3n+1)/2}}{1-q^{5n+1}}.$$

Subtracting $(q^5;q^5)_\infty$ from both sides above and using the pentagonal number theorem, we find that

$$\begin{aligned}
(q^5;q^5)_\infty \phi(q) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(3n+1)/2}}{1-q^{5n+1}} - \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(3n+1)/2} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(3n+1)/2} \left(\frac{1}{1-q^{5n+1}} - 1 \right) \\
&= q \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{15n(n+1)/2}}{1-q^{5n+1}} \\
&= q\Sigma(q,1,q^5),
\end{aligned}$$

by (2.1.33).

To obtain (2.4.9), we apply (2.4.5) with q replaced by q^5 and then $z = q^2$. Now proceed with the same steps as in the foregoing proof, but now using (2.1.19) instead of (2.1.18), and we deduce (2.4.9). □

To simplify further considerations, we make the following conventions:

$$P_5(a) := (q^{5a}, q^{25-5a}; q^{25})_\infty \qquad (a \neq 0), \tag{2.4.10}$$

$$P_5(0) := (q^{25}; q^{25})_\infty, \tag{2.4.11}$$

$$\Sigma_5(a,b) := \Sigma(q^{5a}, q^{5b}, q^{25}) \qquad (a \neq 0), \tag{2.4.12}$$

$$\Sigma_5(0,b) := \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{75n(n+1)/2+5bn}}{1-q^{25n}}, \tag{2.4.13}$$

$$\begin{aligned}
g_5(a) &:= g(q^{5a}, q^{25}) \qquad (2.4.14)\\
&= \frac{q^{5a} P_5(2a)}{P_5(a)} \Sigma_5(a,0) - q^{15a}\Sigma_5(2a,3a) - \Sigma_5(0,-3a),
\end{aligned}$$

by (2.3.11). We note in passing that the Rogers–Ramanujan identities (2.1.11) and (2.1.12) can be written in the forms

$$P_5(1) = \frac{1}{G(q^5)}, \tag{2.4.15}$$

$$P_5(2) = \frac{1}{H(q^5)}. \tag{2.4.16}$$

Lemma 2.4.4. *If $S_5(b)$ is defined by (2.4.1), then*

$$S_5(1) = -g_5(2) - q^8 \frac{\Sigma_5(2,0)}{P_5(0)} (q;q)_\infty - q^3 \frac{P_5^2(0)}{P_5(2)}. \tag{2.4.17}$$

Proof. We begin by dissecting the series for $S_5(1)$ modulo 5. By (2.4.1),

$$S_5(1) = \sum_{b=0}^{4} \sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^{m+b} q^{(5m+b)(15m+3b+1)/2+5m+b}}{1-q^{25m+5b}} \tag{2.4.18}$$

$$= \sum_{b=0}^{4} (-1)^b q^{3b(b+1)/2} \sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^m q^{75m(m+1)/2+5m(3b-6)}}{1-q^{25m+5b}}$$
$$= \Sigma_5(0,-6) - q^3\Sigma_5(1,-3) + q^9\Sigma_5(2,0) - q^{18}\Sigma_5(3,3) + q^{30}\Sigma_5(4,6)$$
$$= q^9\Sigma_5(2,0) + \Sigma_5(0,-6) + q^{30}\Sigma_5(4,6) - q^3\big(q^{15}\Sigma_5(3,3) + \Sigma_5(1,-3)\big).$$

Now, by (2.4.14),

$$\begin{aligned} g_5(2) &= g_5(q^{10}, q^{25}) \\ &= \frac{q^{10}P_5(4)}{P_5(2)}\Sigma_5(2,0) - q^{30}\Sigma_5(4,6) - \Sigma_5(0,-6), \end{aligned}$$

and by Lemma 2.3.2 with q replaced by q^{25}, $\zeta = q^5$, and $z = q^{10}$, we find that

$$q^{15}\Sigma_5(3,3) + \Sigma_5(1,-3) = q^5\frac{P_5(2)}{P_5(1)}\Sigma_5(2,0) + \frac{P_5^2(0)P_5(1)P_5(2)}{P_5(3)P_5(2)P_5(1)}.$$

Therefore, by (2.4.18), the last two equalities, (2.4.15), and (2.4.16),

$$\begin{aligned} S_5(1) &= -g_5(2) + \frac{q^{10}P_5(4)}{P_5(2)}\Sigma_5(2,0) + q^9\Sigma_5(2,0) \\ &\quad - q^3\left(\frac{q^5P_5(2)}{P_5(1)}\Sigma_5(2,0) + \frac{P_5^2(0)}{P_5(3)}\right) \\ &= -g_5(2) - q^3\Sigma_5(2,0)\left(\frac{G(q^5)}{H(q^5)}q^5 - q^6 - q^7\frac{H(q^5)}{G(q^5)}\right) - q^3\frac{P_5^2(0)}{P_5(2)} \\ &= -g_5(2) - q^8\frac{\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - q^3\frac{P_5^2(0)}{P_5(2)}, \end{aligned}$$

by (2.2.2). ∎

Lemma 2.4.5. *For $S_5(b)$ defined by (2.4.1), we have*

$$S_5(4) = -g_5(1) + \frac{q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + q^2\frac{P_5^2(0)}{P_5(1)}. \tag{2.4.19}$$

Proof. By (2.4.1),

$$\begin{aligned} S_5(4) &= \sum_{b=0}^{4}\sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^{m+b}q^{(5m+b)(15m+3b+1)/2+(20m+4b)}}{1-q^{25m+5b}} \\ &= \sum_{b=0}^{4}(-1)^b q^{3b(b+3)/2}\sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^m q^{75m(m+1)/2+5(3b-3)m}}{1-q^{25m+5b}} \\ &= \Sigma_5(0,-3) - q^6\Sigma_5(1,0) + q^{15}\Sigma_5(2,3) - q^{27}\Sigma_5(3,6) + q^{42}\Sigma_5(4,9) \end{aligned}$$

$$= \Sigma_5(0,-3) + q^{15}\Sigma_5(2,3) - q^6\Sigma_5(1,0) \\ - q^{-3}(q^{30}\Sigma_5(3,6) + \Sigma_5(-1,-6)), \tag{2.4.20}$$

where we have replaced m by $m-1$ in the sum for $\Sigma_5(4,9)$. Now, by (2.4.14),

$$g_5(1) = \frac{q^5 P_5(2)}{P_5(1)}\Sigma_5(1,0) - q^{15}\Sigma_5(2,3) - \Sigma_5(0,-3), \tag{2.4.21}$$

and by Lemma 2.3.2 with q replaced by q^{25}, $\zeta = q^{10}$, and $z = q^5$,

$$q^{30}\Sigma_5(3,6) + \Sigma_5(-1,-6) - q^{10}\frac{P_5(4)}{P_5(2)}\Sigma_5(1,0) - \frac{P_5^2(0)P_5(2)P_5(4)}{P_5(3)P_5(1)P_5(-1)} = 0. \tag{2.4.22}$$

Therefore, by (2.4.20), (2.4.21), (2.4.22), (2.4.15), and (2.4.16),

$$\begin{aligned} S_5(4) &= -g_5(1) + q^5\frac{P_5(2)}{P_5(1)}\Sigma_5(1,0) - q^6\Sigma_5(1,0) \\ &\quad - q^{-3}\left(\frac{q^{10}P_5(4)}{P_5(2)}\Sigma_5(1,0) + \frac{P_5^2(0)P_5(4)}{P_5(1)P_5(-1)}\right) \\ &= -g_5(1) + q^5\Sigma_5(1,0)\left(\frac{G(q^5)}{H(q^5)} - q - q^2\frac{H(q^5)}{G(q^5)}\right) + q^2\frac{P_5^2(0)}{P_5(1)} \\ &= -g_5(1) + q^5\frac{\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + q^2\frac{P_5^2(0)}{P_5(1)}, \end{aligned}$$

by (2.2.2). □

Lemma 2.4.6. *Recall that $S_5(b)$ is defined by (2.4.1). Then*

$$\begin{aligned} &1 + S_5(1) - 2S_5(4) \\ &= \left\{\frac{P_5(0)P_5(2)}{P_5^2(1)} - 2q^5\frac{\Sigma_5(1,0)}{P_5(0)} + q\frac{P_5(0)}{P_5(1)} - q^8\frac{\Sigma_5(2,0)}{P_5(0)}\right\}(q;q)_\infty. \end{aligned} \tag{2.4.23}$$

Proof. By Lemmas 2.4.4 and 2.4.5,

$$\begin{aligned} 1 + S_5(1) - 2S_5(4) &= 1 - g_5(2) - \frac{q^8\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{q^3P_5^2(0)}{P_5(2)} \\ &\quad + 2g_5(1) - \frac{2q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} - \frac{2q^2P_5^2(0)}{P_5(1)} \\ &= \frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} - \frac{q^8\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{q^3P_5^2(0)}{P_5(2)} \\ &\quad - \frac{2q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} - \frac{2q^2P_5^2(0)}{P_5(1)}, \end{aligned}$$

by (2.3.17). Thus in order to establish (2.4.23), we need to show that

$$\frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} - \frac{q^3 P_5^2(0)}{P_5(2)} - \frac{2q^2 P_5^2(0)}{P_5(1)} = \left\{ \frac{P_5(0)P_5(2)}{P_5^2(1)} + q\frac{P_5(0)}{P_5(1)} \right\} (q;q)_\infty. \tag{2.4.24}$$

But by (2.4.15), (2.4.16), and (2.2.2), we see that

$$(q;q)_\infty = P_5(0)\left(\frac{P_5(2)}{P_5(1)} - q - \frac{q^2 P_5(1)}{P_5(2)}\right). \tag{2.4.25}$$

Consequently, the right-hand side of (2.4.24) is equal to

$$\begin{aligned}
\left\{ \frac{P_5(0)P_5(2)}{P_5^2(1)} + q\frac{P_5(0)}{P_5(1)} \right\} & P_5(0) \left\{ \frac{P_5(2)}{P_5(1)} - q - \frac{q^2 P_5(1)}{P_5(2)} \right\} \\
&= \frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} + \frac{qP_5^2(0)P_5(2)}{P_5^2(1)} - \frac{qP_5^2(0)P_5(2)}{P_5^2(1)} \\
&\quad - \frac{q^2 P_5^2(0)}{P_5(1)} - \frac{q^2 P_5^2(0)}{P_5(1)} - \frac{q^3 P_5^2(0)}{P_5(2)} \\
&= \frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} - \frac{q^3 P_5^2(0)}{P_5(2)} - \frac{2q^2 P_5^2(0)}{P_5(1)}.
\end{aligned}$$

Thus (2.4.24) has been proved, and therefore (2.4.23) has also been proved. □

Lemma 2.4.7. *With $S_5(b)$ as given in the previous lemmas,*

$$\begin{aligned}
&2S_5(1) + S_5(4) \\
&= \left\{ \frac{q^5 \Sigma_5(1,0)}{P_5(0)} + \frac{q^2 P_5(0)}{P_5(2)} - \frac{2q^8 \Sigma_5(2,0)}{P_5(0)} - \frac{q^3 P_5(0)P_5(1)}{P_5^2(2)} \right\} (q;q)_\infty.
\end{aligned} \tag{2.4.26}$$

Proof. By Lemmas 2.3.3 and 2.3.5, we find that

$$2g_5(2) + g_5(1) = 2g_5(2) - g_5(4) + 1 = \frac{P_5^2(6)P_5^2(0)}{P_5^2(2)P_5(8)} = -\frac{q^5 P_5^2(1)P_5^2(0)}{P_5^3(2)}. \tag{2.4.27}$$

Next, by Lemmas 2.4.4 and 2.4.5 and (2.4.27),

$$\begin{aligned}
2S_5(1) + S_5(4) &= -2g_5(2) - \frac{2q^8 \Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{2q^3 P_5^2(0)}{P_5(2)} \\
&\quad - g_5(1) + \frac{q^5 \Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + \frac{q^2 P_5^2(0)}{P_5(1)} \\
&= \frac{q^5 P_5^2(0)P_5^2(1)}{P_5^3(2)} - \frac{2q^8 \Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{2q^3 P_5^2(0)}{P_5(2)} \\
&\quad + \frac{q^5 \Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + \frac{q^2 P_5^2(0)}{P_5(1)}.
\end{aligned}$$

Thus, in order to establish (2.4.26), we need to show that

$$\frac{q^5 P_5^2(0) P_5^2(1)}{P_5^3(2)} - \frac{2q^3 P_5^2(0)}{P_5(2)} + \frac{q^2 P_5^2(0)}{P_5(1)} = \left\{ \frac{q^2 P_5(0)}{P_5(2)} - \frac{q^3 P_5(0) P_5(1)}{P_5^2(2)} \right\} (q;q)_\infty. \quad (2.4.28)$$

Invoking (2.4.25), which is a restatement of (2.2.2), we see that the right-hand side of (2.4.28) is equal to

$$\begin{aligned}
&\left\{ \frac{q^2 P_5(0)}{P_5(2)} - \frac{q^3 P_5(0) P_5(1)}{P_5^2(2)} \right\} P_5(0) \left\{ \frac{P_5(2)}{P_5(1)} - q - q^2 \frac{P_5(1)}{P_5(2)} \right\} \\
&\qquad = \frac{q^2 P_5^2(0)}{P_5(1)} - \frac{q^3 P_5^2(0)}{P_5(2)} - \frac{q^3 P_5^2(0)}{P_5(2)} + \frac{q^4 P_5^2(0) P_5(1)}{P_5^2(2)} \\
&\qquad\quad - \frac{q^4 P_5^2(0) P_5(1)}{P_5^2(2)} + \frac{q^5 P_5^2(0) P_5^2(1)}{P_5^3(2)} \\
&\qquad = \frac{q^2 P_5^2(0)}{P_5(1)} - \frac{2q^3 P_5^2(0)}{P_5(2)} + \frac{q^5 P_5^2(0) P_5^2(1)}{P_5^3(2)}.
\end{aligned}$$

Thus (2.4.28) and therefore (2.4.26) have been proved. □

We are finally ready to put all this together.

Proof of Entry 2.1.2. We first note that by Lemma 2.4.3,

$$\phi(q^5) = \frac{q^5}{P_5(0)} \Sigma_5(1,0) \quad (2.4.29)$$

and

$$\frac{\psi(q^5)}{q^5} = \frac{q^5}{P_5(0)} \Sigma_5(2,0). \quad (2.4.30)$$

Now by Lemma 2.4.2,

$$\begin{aligned}
&(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n (\zeta_5^{-1} q;q)_n} \\
&= 1 + S_5(1) - 2S_5(4) + (\zeta_5 + \zeta_5^{-1})\big(2S_5(1) + S_5(4)\big) \\
&= (q;q)_\infty \left\{ \frac{P_5(0)P_5(2)}{P_5^2(1)} - \frac{2q^5 \Sigma_5(1,0)}{P_5(0)} + \frac{qP_5(0)}{P_5(1)} - \frac{q^8 \Sigma_5(2,0)}{P_5(0)} \right. \\
&\quad \left. + (\zeta_5 + \zeta_5^{-1}) \left(\frac{q^5 \Sigma_5(1,0)}{P_5(0)} + \frac{q^2 P_5(0)}{P_5(2)} - \frac{2q^8 \Sigma_5(2,0)}{P_5(0)} - \frac{q^3 P_5(0) P_5(1)}{P_5^2(2)} \right) \right\},
\end{aligned}$$

by Lemmas 2.4.6 and 2.4.7. So, by (2.4.29), (2.4.30), (2.4.15), (2.4.16), and (2.1.7)–(2.1.10),

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n(\zeta_5^{-1}q;q)_n} \\
&= \frac{(q^{25};q^{25})_\infty G^2(q^5)}{H(q^5)} - 2\phi(q^5) + q\left(q^{25};q^{25}\right)_\infty G(q^5) - q^3\frac{\psi(q^5)}{q^5} + \left(\zeta_5+\zeta_5^{-1}\right) \\
&\quad \times \left(\phi(q^5) + q^2\left(q^{25};q^{25}\right)_\infty H(q^5) - 2q^3\frac{\psi(q^5)}{q^5} - \frac{q^3(q^{25};q^{25})_\infty H^2(q^5)}{G(q^5)}\right) \\
&= A(q^5) + \left(\zeta_5+\zeta_5^{-1}-2\right)\phi(q^5) + qB(q^5) + \left(\zeta_5+\zeta_5^{-1}\right)q^2C(q^5) \\
&\quad - q^3\left(\zeta_5+\zeta_5^{-1}\right)D(q^5) + \frac{\psi(q^5)}{q^5}q^3\Big(-1-2\left(\zeta_5+\zeta_5^{-1}\right)\Big) \\
&= A(q^5) + \left(\zeta_5+\zeta_5^{-1}-2\right)\phi(q^5) + qB(q^5) + \left(\zeta_5+\zeta_5^{-1}\right)q^2C(q^5) \\
&\quad - q^3\left(\zeta_5+\zeta_5^{-1}\right)D(q^5) + q^3\frac{\psi(q^5)}{q^5}\left(\zeta_5+\zeta_5^{-1}\right)\left(\zeta_5^2+\zeta_5^{-2}-2\right),
\end{aligned}
$$

and this is the assertion made by Entry 2.1.2. □

The argument above is not in the Ramanujan tradition; however, we are unable to replace it with something more appropriate.

2.5 Proof of Entry 2.1.4

The proof here is more direct than that for Entry 2.1.1 in that we do not require an analogue of (2.2.2). Recalling that $F_7(q)$ is defined by (2.1.28), we find that

$$
\begin{aligned}
F_7(q) &= \frac{(q;q)_\infty}{(\zeta_7 q;q)_\infty(\zeta_7^{-1}q;q)_\infty} \\
&= \frac{(q;q)_\infty^2(\zeta_7^2 q;q)_\infty(\zeta_7^{-2}q;q)_\infty(\zeta_7^3 q;q)_\infty(\zeta_7^{-3}q;q)_\infty}{(q;q)_\infty(\zeta_7 q;q)_\infty(\zeta_7^{-1};q)_\infty(\zeta_7^2 q;q)_\infty(\zeta_7^{-2}q,q)_\infty(\zeta_7^3 q;q)_\infty(\zeta_7^{-3}q;q)_\infty} \\
&= \frac{\displaystyle\sum_{n=-\infty}^{\infty}(-1)^n\zeta_7^{2n}q^{n(n-1)/2}\sum_{m=-\infty}^{\infty}(-1)^m\zeta_7^{3m}q^{m(m-1)/2}}{(1-\zeta_7^2)\,(1-\zeta_7^3)\,(q^7;q^7)_\infty},
\end{aligned}
\tag{2.5.1}
$$

by Jacobi's triple product identity (2.1.3). Now for any primitive seventh root of unity (as are each of ζ_7, ζ_7^2, and ζ_7^3),

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n\zeta_7^n q^{n(n-1)/2} \\
&= \sum_{\nu=-3}^{3}\sum_{m=-\infty}^{\infty}(-1)^{7m+\nu}\zeta_7^{7m+\nu}q^{(7m+\nu)(7m+\nu-1)/2}
\end{aligned}
$$

$$= \sum_{\nu=-3}^{3} (-1)^\nu q^{\nu(\nu-1)/2} \zeta_7^\nu \sum_{m=-\infty}^{\infty} (-1)^m q^{7m\left(7m+(2\nu-1)\right)/2}$$
$$= \sum_{\nu=-3}^{3} (-1)^\nu q^{\nu(\nu-1)/2} \zeta_7^\nu f(-q^{21+7\nu}, -q^{28-7\nu}). \tag{2.5.2}$$

Now, by the Jacobi triple product identity (2.1.3) and the definitions (2.1.30)–(2.1.32),

$$\frac{f(-q^{21+7\nu}, -q^{28-7\nu})}{(q^7;q^7)_\infty} = \frac{\left(q^{7\nu+21}, q^{28-7\nu}, q^{49}; q^{49}\right)_\infty}{(q^7;q^7)_\infty} = \begin{cases} X(q^7), & \text{if } \nu = 0, 1, \\ Y(q^7), & \text{if } \nu = -1, 2, \\ Z(q^7), & \text{if } \nu = -2, 3, \\ 0, & \text{if } \nu = -3. \end{cases} \tag{2.5.3}$$

Therefore, by (2.5.2) and (2.5.3),

$$\sum_{n=-\infty}^{\infty} (-1)^n \zeta_7^n q^{n(n-1)/2}$$
$$= (1-\zeta_7)\, X(q^7) + q\left(-\zeta_7^{-1} + \zeta_7^2\right) Y(q^7) + q^3\left(-\zeta_7^3 + \zeta_7^{-2}\right) Z(q^7). \tag{2.5.4}$$

We now use this evaluation for both series on the far right side of (2.5.1). Hence,

$$\begin{aligned} F_7(q) &= \left\{(1-\zeta_7^2)X(q^7) + q(-\zeta_7^{-2} + \zeta_7^4)Y(q^7) + q^3\left(-\zeta_7^6 + \zeta_7^{-4}\right) Z(q^7)\right\} \\ &\quad \times \left\{(1-\zeta_7^3)X(q^7) + q\left(-\zeta_7^{-3} + \zeta_7^6\right) Y(q^7) + q^3\left(-\zeta_7^2 + \zeta_7\right) Z(q^7)\right\} \\ &\quad \times \frac{(q^7;q^7)_\infty}{(1-\zeta_7^2)(1-\zeta_7^3)} \\ &= (q^7;q^7)\Big\{X^2(q^7) + \left(\zeta_7 + \zeta_7^{-1} - 1\right) qX(q^7)Y(q^7) \\ &\quad + \left(\zeta_7^2 + \zeta_7^{-2}\right) q^2Y^2(q^7) + \left(\zeta_7^3 + \zeta_7^{-3} + 1\right) q^3X(q^7)Z(q^7) \\ &\quad - \left(\zeta_7 + \zeta_7^{-1}\right) q^4Y(q^7)Z(q^7) - \left(\zeta_7^2 + \zeta_7^{-2} + 1\right) q^6Z^2(q^7)\Big\}, \end{aligned}$$

which proves Entry 2.1.4.

2.6 Proof of Entry 2.1.5

Let

$$S_7(b) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{bn+n(3n+1)/2}}{1-q^{7n}}. \tag{2.6.1}$$

Replacing n by $-n$, we see that

$$S_7(b) = -S_7(6-b), \tag{2.6.2}$$

from which an immediate consequence is

$$S_7(3) = 0. \tag{2.6.3}$$

Furthermore,

$$\begin{aligned} S_7(b) - S_7(b+7) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{bn+n(3n+1)/2} - 1 = f(-q^{2+b}, -q^{1-b}) - 1 \\ &= \begin{cases} (-1)^b q^{-b(b+1)/6}(q;q)_\infty - 1, & \text{if } b \equiv 0 \pmod 3, \\ -1, & \text{if } b \equiv 1 \pmod 3, \\ (-1)^{b-1} q^{-b(b+1)/6}(q;q)_\infty - 1, & \text{if } b \equiv 2 \pmod 3, \end{cases} \end{aligned} \tag{2.6.4}$$

as we have previously observed in (2.4.4).

Referring to (2.4.5), we are able to prove the following.

Lemma 2.6.1. *If ζ_7 be a primitive seventh root of unity, then*

$$\begin{aligned} &(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n (\zeta_7^{-1} q;q)_n} \\ &\quad = 1 - S_7(4) + 2S_7(0) + \left(\zeta_7 + \zeta_7^{-1}\right)\left(S_7(1) - S_7(4) - S_7(0)\right) \\ &\qquad + \left(\zeta_7^2 + \zeta_7^{-2}\right)\left(-S_7(1) - 2S_7(4)\right). \end{aligned} \tag{2.6.5}$$

Proof. Invoking (2.4.5), (2.6.2), and (2.6.3), we find that

$$\begin{aligned} &(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n (\zeta_7^{-1} q;q)_n} \\ &= 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &\quad \times \frac{(1-\zeta_7)(1-q^n)(1-\zeta_7^2 q^n)(1-\zeta_7^3 q^n)(1-\zeta_7^4 q^n)(1-\zeta_7^5 q^n)(1-\zeta_7^6 q^n)}{1-q^{7n}} \\ &= 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-q^{7n}} \Big((1-q^{6n}) + (q^n-1)\zeta_7 + q^n(q^n-1)\zeta_7^2 \\ &\quad + q^{2n}(q^n-1)\zeta_7^3 + q^{3n}(q^n-1)\zeta_7^4 + q^{4n}(q^n-1)\zeta_7^5 + q^{5n}(q^n-1)\zeta_7^6 \Big) \\ &= 1 + (1-\zeta_7)S_7(0) + (\zeta_7 - \zeta_7^2)S_7(1) + (\zeta_7^2 - \zeta_7^3)S_7(2) + (\zeta_7^3 - \zeta_7^4)S_7(3) \\ &\quad + (\zeta_7^4 - \zeta_7^5)S_7(4) + (\zeta_7^5 - \zeta_7^6)S_7(5) + (\zeta_7^6 - 1)S_7(6) \\ &= 1 + S_7(0)\left(2 - \zeta_7 - \zeta_7^{-1}\right) + \left(\zeta_7 + \zeta_7^{-1} - \zeta_7^2 - \zeta_7^{-2}\right) S_7(1) \end{aligned}$$

$$
\begin{aligned}
&+\left(\zeta_7^3+\zeta_7^{-3}-\zeta_7^2-\zeta_7^{-2}\right) S_7(4)\\
&=1-S_7(4)+2S_7(0)+\left(\zeta_7+\zeta_7^{-1}\right)\left(S_7(1)-S_7(4)-S_7(0)\right)\\
&\quad+\left(\zeta_7^2+\zeta_7^{-2}\right)\left(-S_7(1)-2S_7(4)\right).
\end{aligned}
$$

□

Recalling the notation (2.1.34)–(2.1.37), we define

$$
\begin{aligned}
g_7(a) &:= g(q^{7a}, q^{49}) \qquad (2.6.6)\\
&= \frac{q^{7a}P_7(2a)}{P_7(a)}\Sigma_7(a,0) - q^{21a}\Sigma_7(2a,3a) - \Sigma_7(0,-3a),
\end{aligned}
$$

by (2.3.11).

Lemma 2.6.2. *With $S_7(b)$ defined by* (2.6.1),

$$
S_7(1) = -g_7(3) + q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}(q;q)_\infty + \frac{q^9 P_7^2(0)P_7(1)}{P_7^2(3)} - \frac{q^3 P_7^2(0)}{P_7(1)}. \qquad (2.6.7)
$$

Proof. We begin by dissecting the series for $S_7(1)$ modulo 7. To that end,

$$
\begin{aligned}
S_7(1) &= \sum_{b=0}^{6}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty} \frac{(-1)^{m+b}q^{(7m+b)(21m+3b+1)/2+7m+b}}{1-q^{49m+7b}}\\
&= \sum_{b=0}^{6}(-1)^b q^{3b(b+1)/2}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^m q^{147m(m+1)/2+7m(3b-9)}}{1-q^{49m+7b}}\\
&= \Sigma_7(0,-9) - q^3\Sigma_7(1,-6) + q^9\Sigma_7(2,-3) - q^{18}\Sigma_7(3,0) + q^{30}\Sigma_7(4,3)\\
&\quad - q^{45}\Sigma_7(5,6) + q^{63}\Sigma_7(6,9). \qquad (2.6.8)
\end{aligned}
$$

Now, by (2.6.6),

$$
g_7(3) - \frac{q^{21}P_7(6)}{P_7(3)}\Sigma_7(3,0) = -q^{63}\Sigma_7(6,9) - \Sigma_7(0,-9). \qquad (2.6.9)
$$

In addition, by Lemma 2.3.2, with q replaced by q^{49}, $\zeta = q^{14}$, and $z = q^{21}$,

$$
q^{42}\Sigma_7(5,6) + \Sigma_7(1,-6) = q^{14}\frac{P_7(4)}{P_7(2)}\Sigma_7(3,0) + \frac{P_7^2(0)P_7(2)P_7(4)}{P_7(5)P_7(3)P_7(1)}. \qquad (2.6.10)
$$

By Lemma 2.3.2 with q replaced by q^{49}, $\zeta = q^7$, and $z = q^{21}$,

$$
q^{21}\Sigma_7(4,3) + \Sigma_7(2,-3) = q^7\frac{P_7(2)}{P_7(1)}\Sigma_7(3,0) + \frac{P_7^2(0)P_7(1)P_7(2)}{P_7(4)P_7(3)P_7(2)}. \qquad (2.6.11)
$$

We now substitute the right-hand sides of (2.6.9), (2.6.10), and (2.6.11) for the appearances of their respective left-hand sides in (2.6.8). Hence,

$$S_7(1) = -g_7(3) + \Sigma_7(3,0)\left(\frac{q^{21}P_7(6)}{P_7(3)} - q^{17}\frac{P_7(4)}{P_7(2)} + q^{16}\frac{P_7(2)}{P_7(1)} - q^{18}\right)$$
$$- q^3\frac{P_7^2(0)}{P_7(1)} + q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)},$$

by the fact that $P_7(a) = P_7(7-a)$. We now invoke Ramanujan's identity [55, p. 303, Entry 17(v)]

$$\frac{(q;q)_\infty}{P_7(0)} = \frac{P_7(2)}{P_7(1)} - q\frac{P_7(4)}{P_7(2)} - q^2 + q^5\frac{P_7(6)}{P_7(3)} \tag{2.6.12}$$

to conclude that

$$S_7(1) = -g_7(3) + \frac{q^{16}\Sigma_7(3,0)(q;q)_\infty}{P_7(0)} + \frac{q^9P_7^2(0)P_7(1)}{P_7^2(3)} - \frac{q^3P_7^2(0)}{P_7(1)},$$

as desired. □

Lemma 2.6.3. *We have*

$$S_7(4) = -g_7(2) - \frac{q^{13}\Sigma_7(2,0)}{P_7(0)}(q;q)_\infty - \frac{q^6P_7^2(0)}{P_7(3)} + \frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)}. \tag{2.6.13}$$

Proof. We dissect the series for $S_7(4)$ modulo 7 to deduce that

$$\begin{aligned} S_7(4) &= \sum_{b=-1}^{5}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^{m+b}q^{(7m+b)(21m+3b+1)/2+28m+4b}}{1-q^{49m+7b}} \\ &= \sum_{b=-1}^{5}(-1)^bq^{3b(b+3)/2}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^mq^{147m(m+1)/2+7m(3b-6)}}{1-q^{49m+7b}} \\ &= -q^{-3}\Sigma_7(-1,-9) + \Sigma_7(0,-6) - q^6\Sigma_7(1,-3) + q^{15}\Sigma_7(2,0) \\ &\quad - q^{27}\Sigma_7(3,3) + q^{42}\Sigma_7(4,6) - q^{60}\Sigma_7(5,9). \end{aligned} \tag{2.6.14}$$

Now, by (2.6.6),

$$-g_7(2) + \frac{q^{14}P_7(4)}{P_7(2)}\Sigma_7(2,0) = q^{42}\Sigma_7(4,6) + \Sigma_7(0,-6). \tag{2.6.15}$$

By Lemma 2.3.2, with q replaced by q^{49}, $\zeta = q^{21}$, and $z = q^{14}$,

$$q^{63}\Sigma_7(5,9) + \Sigma_7(-1,-9) = \frac{q^{21}P_7(6)}{P_7(3)}\Sigma_7(2,0) + \frac{P_7^2(0)P_7(3)P_7(6)}{P_7(5)P_7(2)P_7(-1)}, \tag{2.6.16}$$

and by Lemma 2.3.2 with q replaced by q^{49}, $\zeta = q^7$, and $z = q^{14}$,

$$q^{21}\Sigma_7(3,3)+\Sigma_7(1,-3)=\frac{q^7P_7(2)}{P_7(1)}\Sigma_7(2,0)+\frac{P_7^2(0)P_7(1)P_7(2)}{P_7(3)P_7(2)P_7(1)}. \tag{2.6.17}$$

We now substitute the right-hand sides of (2.6.15), (2.6.16), and (2.6.17) for the appearances of their respective left-hand sides in (2.6.14). Hence,

$$\begin{aligned}S_7(4)=-g_7(2)+\frac{q^{14}P_7(4)}{P_7(2)}\Sigma_7(2,0)-q^{-3}\left(\frac{q^{21}P_7(6)}{P_7(3)}\Sigma_7(2,0)\right.\\ \left.-\frac{q^7P_7^2(0)P_7(3)}{P_7^2(2)}\right)-q^6\left(\frac{q^7P_7(2)}{P_7(1)}\Sigma_7(2,0)+\frac{P_7^2(0)}{P_7(3)}\right)+q^{15}\Sigma_7(2,0),\end{aligned}$$

since $P_7(-1)=-q^{-7}P_7(1)$. Hence,

$$\begin{aligned}S_7(4)&=-g_7(2)+\Sigma_7(2,0)\left(\frac{q^{14}P_7(4)}{P_7(2)}+q^{15}-\frac{q^{18}P_7(6)}{P_7(3)}-\frac{q^{13}P_7(2)}{P_7(1)}\right)\\&\quad+\frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)}-\frac{q^6P_7^2(0)}{P_7(3)}\\&=-g_7(2)-q^{13}\Sigma_7(2,0)\left(\frac{P_7(2)}{P_7(1)}-\frac{qP_7(4)}{P_7(2)}-q^2+\frac{q^5P_7(6)}{P_7(3)}\right)\\&\quad-\frac{q^6P_7^2(0)}{P_7(3)}+\frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)},\end{aligned}$$

and by (2.6.12),

$$S_7(4)=-g_7(2)-\frac{q^{13}\Sigma_7(2,0)(q;q)_\infty}{P_7(0)}-\frac{q^6P_7^2(0)}{P_7(3)}+\frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)},$$

as desired. □

Lemma 2.6.4. *Recalling that $S_7(b)$ is defined by (2.6.1), we have*

$$S_7(7)=-g_7(1)+\frac{q^7\Sigma_7(1,0)}{P_7(0)}(q;q)_\infty+\frac{qP_7^2(0)P_7(2)}{P_7^2(1)}-\frac{q^5P_7^2(0)}{P_7(2)}. \tag{2.6.18}$$

Proof. As before, we begin by dissecting the series for $S_7(7)$ modulo 7 to arrive at

$$\begin{aligned}S_7(7)&=\sum_{b=-2}^{4}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^{m+b}q^{(7m+b)(21m+3b+1)/2+49m+7b}}{1-q^{49m+7b}}\\&=\sum_{b=-2}^{4}(-1)^bq^{3b(b+5)/2}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^mq^{147m(m+1)/2+7m(3b-3)}}{1-q^{49m+7b}}\\&=q^{-9}\Sigma_7(-2,-9)-q^{-6}\Sigma_7(-1,-6)+\Sigma_7(0,-3)\\&\quad-q^9\Sigma_7(1,0)+q^{21}\Sigma_7(2,3)-q^{36}\Sigma_7(3,6)+q^{54}\Sigma_7(4,9).\end{aligned} \tag{2.6.19}$$

Now, by (2.6.6),

$$-g_7(1)+\frac{q^7P_7(2)}{P_7(1)}\Sigma_7(1,0)=q^{21}\Sigma_7(2,3)+\Sigma_7(0,-3). \tag{2.6.20}$$

By Lemma 2.3.2 with q replaced by q^{49}, $\zeta=q^{14}$, and $z=q^7$,

$$q^{42}\Sigma_7(3,6)+\Sigma_7(-1,-6)=\frac{q^{14}P_7(4)}{P_7(2)}\Sigma_7(1,0)+\frac{P_7^2(0)P_7(2)P_7(4)}{P_7(3)P_7(1)P_7(-1)}, \tag{2.6.21}$$

and by Lemma 2.3.2 with q replaced by q^{49}, $\zeta=q^{21}$, and $z=q^7$,

$$q^{63}\Sigma_7(4,9)+\Sigma_7(-2,-9)=\frac{q^{21}P_7(6)\Sigma_7(1,0)}{P_7(3)}+\frac{P_7^2(0)P_7(3)P_7(6)}{P_7(4)P_7(1)P_7(-2)}. \tag{2.6.22}$$

We now substitute the right-hand sides of (2.6.20), (2.6.21), and (2.6.22) for the appearances of their respective left-hand sides in (2.6.19). Hence,

$$\begin{aligned}
S_7(7)&=-g_7(1)+\frac{q^7P_7(2)}{P_7(1)}\Sigma_7(1,0)-q^9\Sigma_7(1,0)\\
&\quad-q^{-6}\left(\frac{q^{14}P_7(4)}{P_7(2)}\Sigma_7(1,0)+\frac{P_7^2(0)P_7(2)P_7(4)}{P_7(3)P_7(1)P_7(-1)}\right)\\
&\quad+q^{-9}\left(\frac{q^{21}P_7(6)}{P_7(3)}\Sigma_7(1,0)+\frac{P_7^2(0)P_7(3)P_7(6)}{P_7(4)P_7(1)P_7(-2)}\right)\\
&=-g_7(1)+q^7\Sigma_7(1,0)\left(\frac{P_7(2)}{P_7(1)}-\frac{qP_7(4)}{P_7(2)}-q^2+\frac{q^5P_7(6)}{P_7(3)}\right)\\
&\quad+\frac{qP_7^2(0)P_7(2)}{P_7^2(1)}-\frac{q^5P_7^2(0)}{P_7(2)},
\end{aligned}$$

by the facts that $P_7(a)=P_7(7-a)$ and $P_7(-a)=-q^{-7a}P_7(a)$. We now invoke (2.6.12) to conclude that

$$S_7(7)=-g_7(1)+\frac{q^7\Sigma_7(1,0)(q;q)_\infty}{P_7(0)}+\frac{qP_7^2(0)P_7(2)}{P_7^2(1)}-\frac{q^5P_7^2(0)}{P_7(2)},$$

as desired. □

Lemma 2.6.5. *We have*

$$P_7(1)P_7^3(3)-P_7(3)P_7^3(2)+q^7P_7^3(1)P_7(2)=0.$$

Proof. This identity can be found in Ramanujan's second notebook [282, p. 300], where it is given as an identity involving quotients of theta functions. A proof can be found in Berndt's book [57], with the statement of Ramanujan's identity being given in Entry 32(ii) of Chapter 25 [57, p. 176]. (Unfortunately, there is a misprint in the definition of w in Entry 32; read 25/56 instead of 25/26.) Lemma 2.6.5 is also equivalent to a specialization of the three-term relation for the Weierstrass sigma function [339, p. 451, Exercise 5]. □

Lemma 2.6.6. *With $S_7(b)$ defined by* (2.6.1),

$$-2g_7(1)+g_7(2)-1+2q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}-2q^5\frac{P_7^2(0)}{P_7(2)}+q^6\frac{P_7^2(0)}{P_7(3)}$$
$$=\left\{-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}\right\}(q;q)_\infty. \qquad (2.6.23)$$

Proof. By Lemma 2.3.5 with q replaced by q^{49} and $z=q^7$, we see that

$$-2g_7(1)+g_7(2)-1=-\frac{P_7^2(0)P_7^2(3)}{P_7^2(1)P_7(4)}=-\frac{P_7^2(0)P_7(3)}{P_7^2(1)}.$$

Hence (2.6.23) is, with the use of (2.6.12), equivalent to the assertion

$$\frac{-P_7^2(0)P_7(3)}{P_7^2(1)}+2q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}-2q^5\frac{P_7^2(0)}{P_7(2)}+q^6\frac{P_7^2(0)}{P_7(3)}$$
$$=\left\{-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}\right\}$$
$$\times P_7(0)\left\{\frac{P_7(2)}{P_7(1)}-q\frac{P_7(4)}{P_7(2)}-q^2+q^5\frac{P_7(6)}{P_7(3)}\right\},$$

and if we collect terms on the left-hand side according to powers of q, we see that this last identity is equivalent (after cancellation of like terms) to

$$q\left\{\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-\frac{P_7^2(0)P_7(3)P_7(4)}{P_7(1)P_7^2(2)}-q^7\frac{P_7^2(0)P_7(6)}{P_7(3)P_7(2)}\right\}=0.$$

This last identity can be written in the form

$$\frac{qP_7^2(0)}{P_7^2(1)P_7^2(2)P_7(3)}\left\{P_7^3(2)P_7(3)-P_7^3(3)P_7(1)-q^7P_7^3(1)P_7(2)\right\}=0,$$

by Lemma 2.6.5. Thus (2.6.23) is proved. □

Lemma 2.6.7. *We have*

$$2g_7(2)+g_7(3)-q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}+q^3\frac{P_7^2(0)}{P_7(1)}-2q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}+2q^6\frac{P_7^2(0)}{P_7(3)}$$
$$=\left\{q^3\frac{P_7(0)}{P_7(2)}-q^4\frac{P_7(0)}{P_7(3)}+q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\right\}(q;q)_\infty. \qquad (2.6.24)$$

Proof. By Lemma 2.3.3, with $z=q^{21}$, Lemma 2.3.3, with $z=q^{14}$, and the fact that $P_7(8)=-q^{-7}P_7(1)$,

$$2g_7(2)+g_7(3)=-q^7\frac{P_7^2(0)P_7(1)}{P_7^2(2)}.$$

Therefore (2.6.24) is, by (2.6.12), equivalent to the assertion

$$-q^7\frac{P_7^2(0)P_7(1)}{P_7^2(2)}-q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}+q^3\frac{P_7^2(0)}{P_7(1)}-2q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}+2q^6\frac{P_7^2(0)}{P_7(3)}$$
$$=\left\{q^3\frac{P_7(0)}{P_7(2)}-q^4\frac{P_7(0)}{P_7(3)}+q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\right\}$$
$$\times P_7(0)\left\{\frac{P_7(2)}{P_7(1)}-q\frac{P_7(4)}{P_7(2)}-q^2+q^5\frac{P_7(6)}{P_7(3)}\right\}.$$

Multiply out the right-hand side above and cancel like terms. It then remains to show that

$$0=q^4\left\{-\frac{P_7^2(0)P_7(3)}{P_7^2(2)}+\frac{P_7^2(0)P_7(2)}{P_7(1)P_7(3)}-q^7\frac{P_7^2(0)P_7^2(1)}{P_7(2)P_7^2(3)}\right\}$$
$$=\frac{q^4P_7^2(0)}{P_7^2(2)P_7^2(3)P_7(1)}\left\{-P_7^3(3)P_7(1)+P_7^3(2)P_7(3)-q^7P_7^3(1)P_7(2)\right\}.$$

By Lemma 2.6.5, the equality above is indeed true. Thus (2.6.24) is proved. □

Lemma 2.6.8. *We have*

$$-g_7(1)-2g_7(3)+q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}+2q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}-2q^3\frac{P_7^2(0)}{P_7(1)}-q^5\frac{P_7^2(0)}{P_7(2)}$$
$$=\left\{q\frac{P_7(0)}{P_7(1)}+q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)}+q^4\frac{P_7(0)}{P_7(3)}\right\}(q;q)_\infty. \quad (2.6.25)$$

Proof. By Lemma 2.3.3, with $z=q^7$, Lemma 2.3.5, with $z=q^{21}$, and the facts that $P_7(9)=-q^{-14}P_7(2)$ and $P_7(12)=-q^{-35}P_7(2)$,

$$-g_7(1)-2g_7(3)=q^7\frac{P_7^2(0)P_7(2)}{P_7^2(3)}.$$

Thus, by (2.6.12), (2.6.25) is equivalent to the assertion

$$q^7\frac{P_7^2(0)P_7(2)}{P_7^2(3)}+q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}+2q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}-2q^3\frac{P_7^2(0)}{P_7(1)}-q^5\frac{P_7^2(0)}{P_7(2)}$$
$$=\left\{q\frac{P_7(0)}{P_7(1)}+q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)}+q^4\frac{P_7(0)}{P_7(3)}\right\}$$
$$\times P_7(0)\left\{\frac{P_7(2)}{P_7(1)}-q\frac{P_7(4)}{P_7(2)}-q^2+q^5\frac{P_7(6)}{P_7(3)}\right\}.$$

As before, we collect terms on the left-hand side according to powers of q. This reduces the last identity to the equivalent assertion

$$
\begin{aligned}
&q^2\left\{\frac{q^7 P_7^2(0)P_7(1)}{P_7^2(3)}+\frac{P_7^2(0)P_7(4)}{P_7(1)P_7(2)}-\frac{P_7^2(0)P_7^2(2)}{P_7^2(1)P_7(3)}\right\}\\
&\quad=\frac{q^2P_7^2(0)}{P_7^2(3)P_7^2(1)P_7(2)}\left\{q^7P_7^3(1)P_7(2)+P_7^3(3)P_7(1)-P_7^3(2)P_7(3)\right\}\\
&\quad=0,
\end{aligned}
$$

by Lemma 2.6.5. Thus (2.6.25) is proved. □

Lemma 2.6.9. *We have*

$$
\begin{aligned}
-1+(q;q)_\infty-S_7(4)+2S_7(7)=\Bigg\{&2q^7\frac{\Sigma_7(1,0)}{P_7(0)}-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}\\
&+1+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}+q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}\Bigg\}(q;q)_\infty. \qquad (2.6.26)
\end{aligned}
$$

Proof. By Lemmas 2.6.3 and 2.6.4,

$$
\begin{aligned}
&-1+(q;q)_\infty-S_7(4)+2S_7(7)\\
&\qquad=-1+(q;q)_\infty+g_7(2)+q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}(q;q)_\infty+q^6\frac{P_7^2(0)}{P_7(3)}\\
&\qquad\quad-q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}-2g_7(1)+2q^7\frac{\Sigma_7(1,0)}{P_7(0)}(q;q)_\infty\\
&\qquad\quad+2q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-2q^5\frac{P_7^2(0)}{P_7(2)}.
\end{aligned}
$$

We now note that the entire left-hand side of (2.6.23) appears on the right-hand side of the preceding expression. Hence, by (2.6.23),

$$
\begin{aligned}
-1+(q;q)_\infty-S_7(4)+2S_7(7)=\Bigg\{&2q^7\frac{\Sigma_7(1,0)}{P_7(0)}+q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}\\
&+1-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}\Bigg\}(q;q)_\infty,
\end{aligned}
$$

which is equivalent to (2.6.26). □

Lemma 2.6.10. *We have*

$$
\begin{aligned}
-S_7(1)-2S_7(4)=\Bigg\{&-q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}+q^3\frac{P_7(0)}{P_7(2)}-q^4\frac{P_7(0)}{P_7(3)}\\
&+2q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}+q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\Bigg\}(q;q)_\infty. \qquad (2.6.27)
\end{aligned}
$$

Proof. By Lemmas 2.6.2 and 2.6.3,

$$
\begin{aligned}
-S_7(1)-2S_7(4) &= g_7(3) - q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}(q;q)_\infty - q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)} + q^3\frac{P_7^2(0)}{P_7(1)} \\
&\quad + 2g_7(2) + 2q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}(q;q)_\infty + 2q^6\frac{P_7^2(0)}{P_7(3)} - 2q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}.
\end{aligned}
$$

We observe that the entire left-hand side of (2.6.24) appears in the preceding expression's right-hand side. Therefore, by (2.6.24),

$$
\begin{aligned}
-S_7(1)-2S_7(4) = \Bigg\{ &- q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} + 2q^{13}\frac{\Sigma_7(2,0)}{P_7(0)} + q^3\frac{P_7(0)}{P_7(2)} - q^4\frac{P_7(0)}{P_7(3)} \\
&+ q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\Bigg\}(q;q)_\infty,
\end{aligned}
$$

which is equivalent to (2.6.27). □

Lemma 2.6.11. *We have*

$$
\begin{aligned}
2S_7(1)+S_7(7) = \Bigg\{ &q^7\frac{\Sigma_7(1,0)}{P_7(0)} + q\frac{P_7(0)}{P_7(1)} + q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} \\
&+ 2q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} + q^4\frac{P_7(0)}{P_7(3)}\Bigg\}(q;q)_\infty. \qquad (2.6.28)
\end{aligned}
$$

Proof. By Lemmas 2.6.2 and 2.6.4,

$$
\begin{aligned}
2S_7(1)+S_7(7) &= -2g_7(3) + 2q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}(q;q)_\infty + 2q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)} \\
&\quad - 2q^3\frac{P_7^2(0)}{P_7(1)} - g_7(1) + q^7\frac{\Sigma_7(1,0)}{P_7(0)}(q;q)_\infty + q\frac{P_7^2(0)P_7(2)}{P_7^2(1)} - q^5\frac{P_7^2(0)}{P_7(2)}.
\end{aligned}
$$

As before, we see that the entire left-hand side of (2.6.25) appears on the right-hand side of the preceding expression. So, by (2.6.25),

$$
\begin{aligned}
2S_7(1)+S_7(7) = \Bigg\{ &2q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} + q^7\frac{\Sigma_7(1,0)}{P_7(0)} + q\frac{P_7(0)}{P_7(1)} \\
&+ q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} + q^4\frac{P_7(0)}{P_7(3)}\Bigg\}(q;q)_\infty,
\end{aligned}
$$

which is equivalent to (2.6.28). □

Lemma 2.6.12. *We have*

$$
\begin{aligned}
&1-(q;q)_\infty + S_7(1) - S_7(4) - S_7(7) \\
&= \Bigg\{\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)} - 1 - q^7\frac{\Sigma_7(1,0)}{P_7(0)} + q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} + q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} \\
&\quad + q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)} + q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}\Bigg\}(q;q)_\infty. \qquad (2.6.29)
\end{aligned}
$$

Proof. If we add together the left-hand sides of (2.6.27) and (2.6.28) and subtract the left-hand side of (2.6.26), we obtain the left-hand side of (2.6.29). The same combination of right-hand sides produces the right-hand side of (2.6.29). □

Lemma 2.6.13. *We have*

$$S_7(0) = S_7(7) + (q;q)_\infty - 1.$$

Proof. This is (2.6.4) with $b = 0$. □

Finally we are ready to prove Entry 2.1.5.

Proof of Entry 2.1.5. By Lemma 2.6.1,

$$\begin{aligned}
(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n}
&= 1 - S_7(4) + 2S_7(0) + (\zeta_7+\zeta_7^{-1})\big(S_7(1) - S_7(4) - S_7(0)\big)\\
&\quad + (\zeta_7^2+\zeta_7^{-2})\big(-S_7(1) - 2S_7(4)\big)\\
&= \big(-1 + (q;q)_\infty - S_7(4) + 2S_7(7)\big) + (q;q)_\infty\\
&\quad + (\zeta_7+\zeta_7^{-1})\big(S_7(1) - S_7(4) - S_7(7) - (q;q)_\infty + 1\big)\\
&\quad + (\zeta_7^2+\zeta_7^{-2})\big(-S_7(1) - 2S_7(4)\big),
\end{aligned}$$

by Lemma 2.6.13.

We now apply Lemmas 2.6.9, 2.6.12, and 2.6.10 to the combinations of S_7's contained in parentheses. This yields a large expression multiplied by $(q;q)_\infty$. We cancel $(q;q)_\infty$ from each side, use (2.4.5), and recall the notation (2.1.43)–(2.1.47) to obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n}\\
&= 2 - \zeta_7 - \zeta_7^{-1} + \frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}\left(\zeta_7+\zeta_7^{-1}-1\right)\\
&\quad + q^7\frac{\Sigma_7(1,0)}{P_7(0)}\left(2-\zeta_7-\zeta_7^{-1}\right) + q\frac{P_7(0)}{P_7(1)}\\
&\quad + q^2\left\{\left(\zeta_7+\zeta_7^{-1}\right)\left(\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} + q^{14}\frac{\Sigma_7(3,0)}{P_7(0)}\right) - \left(\zeta_7^2+\zeta_7^{-2}\right)q^{14}\frac{\Sigma_7(3,0)}{P_7(0)}\right\}\\
&\quad + q^3\frac{P_7(0)}{P_7(2)}\left(1+\zeta_7^2+\zeta_7^{-2}\right) - q^4\frac{P_7(0)}{P_7(3)}\left(\zeta_7^2+\zeta_7^{-2}\right)\\
&\quad + q^6\left\{q^7\frac{\Sigma_7(2,0)}{P_7(0)}\left(1+\zeta_7+\zeta_7^{-1}+2\zeta_7^2+2\zeta_7^{-2}\right)\right.\\
&\qquad\left. + \frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\left(\zeta_7+\zeta_7^{-1}+\zeta_7^2+\zeta_7^{-2}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left(2-\zeta_7-\zeta_7^{-1}\right)\left(1-A_7(q^7)+q^7Q_1(q^7)\right)+qT_1(q^7)+A_7(q^7)\\
&\quad + q^2\left\{\left(\zeta_7+\zeta_7^{-1}\right)B_7(q^7)+q^{14}Q_3(q^7)\left(\zeta_7+\zeta_7^{-1}-\zeta_7^2-\zeta_7^{-2}\right)\right\}\\
&\quad + q^3T_2(q^7)\left(1+\zeta_7^2+\zeta_7^{-2}\right)-q^4\left(\zeta_7^2+\zeta_7^{-2}\right)T_3(q^7)\\
&\quad + q^6\left\{q^7Q_2(q^7)\left(\zeta_7^2+\zeta_7^{-2}-\zeta_7^3-\zeta_7^{-3}\right)-C_7(q^7)\left(1+\zeta_7^3+\zeta_7^{-3}\right)\right\},
\end{aligned}
$$

as desired. $\square$

3

Ranks and Cranks, Part II

3.1 Introduction

In his lost notebook [283], Ramanujan recorded several entries on the generating function for cranks (2.1.27). In this chapter, we employ the notation for cranks that Ramanujan gives at the top of page 179 in his lost notebook [283] and which we partially gave in (2.1.14). More precisely, Ramanujan defines the function $F(q)$ and coefficients $\lambda_n, n \geq 0$, by

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n. \tag{3.1.1}$$

Thus, by (2.1.27), for $n > 1$,

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n) a^m.$$

The goal of this chapter is to establish, in terms of congruences, all five of Ramanujan's dissections for $F_a(q)$ offered by him in his lost notebook. Ramanujan gave one of these, Entry 2.1.1, in the form of an identity. Two were presented by Ramanujan in terms of congruences. The last pair were not explicitly stated by Ramanujan; only the quotients of theta functions appearing in the dissections are given.

On page 179, Ramanujan states the two aforementioned congruences for $F(q)$. These congruences, like others in this chapter, are to be regarded as congruences in the ring of power series in the two variables a and q. The two congruences are given by

$$F(\sqrt{q}) \equiv \frac{f(-q^3,-q^5)}{(-q^2;q^2)_\infty} + \left(a - 1 + \frac{1}{a}\right) \sqrt{q} \frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \pmod{a^2 + a^{-2}} \tag{3.1.2}$$

and

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$$
\begin{aligned}
F(q^{1/3}) \equiv& \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\
&+ \left(a-1+\frac{1}{a}\right) q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\
&+ \left(a^2+\frac{1}{a^2}\right) q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \pmod{a^3+a^{-3}}. \quad (3.1.3)
\end{aligned}
$$

Note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \pmod{\lambda_2}$ and $a^8 \equiv 1 \pmod{\lambda_2}$. Thus, in (3.1.2), a behaves like a primitive eighth root of unity modulo λ_2. On the other hand, $\lambda_3 = a^3+1+a^{-3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1 \pmod{\lambda_3}$. So in (3.1.3), a behaves like a primitive ninth root of unity modulo λ_3.

Thus, if we let $a = \exp(2\pi i/8)$ and replace q by q^2, (3.1.2) implies the 2-dissection of $F(q)$, while if we let $a = \exp(2\pi i/9)$ and replace q by q^3, (3.1.3) implies the 3-dissection of $F(q)$.

As we saw in Chapter 2, Ramanujan gives the 5-dissection of $F(q)$ on page 20 of his lost notebook [283]. It is interesting that Ramanujan does not give the alternative form, analogous to those in (3.1.2) and (3.1.3), from which the 5-dissection would follow by setting a to be a primitive fifth root of unity. Proofs of the 5-dissection have been given by F. Garvan [146] and A.B. Ekin [133].

The first explicit statement and proof of the 7-dissection of $F(q)$ was given by Garvan [146, Theorem 5.1]. Although Ramanujan did not state the 7-dissection of $F(q)$, he clearly knew it, because the six quotients of theta functions that appear in the 7-dissection are found on the bottom of page 71 (written upside down) in his lost notebook. The first appearance of the 11-dissection of $F(q)$ in the literature also can be found in Garvan's paper [146, Theorem 6.7]. Further proofs have been given by M.D. Hirschhorn [173] and Ekin [132], [133], who also gave a different proof of the 7-dissection. However, again, it is very likely that Ramanujan knew the 11-dissection, since he offers the quotients of theta functions that appear in the 11-dissection on page 70 of his lost notebook [283].

On page 59 in his lost notebook [283], Ramanujan records a quotient of two power series, with the highest power of the numerator being q^{21} and the highest power of the denominator being q^{22}. Underneath, he records another power series with the highest power being q^5. Although not claimed by Ramanujan, the two expressions are equal. We state Ramanujan's "claim" in the following theorem.

Entry 3.1.1 (p. 59). *If*

$$A_n := a^n + a^{-n}, \tag{3.1.4}$$

then

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} = \frac{1 - \displaystyle\sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn}(A_{n+1} - A_n)}{(q;q)_\infty}. \quad (3.1.5)$$

In this chapter, we provide uniform proofs of the 2-, 3-, 5-, 7-, and 11-dissections of $F(q)$, when expressed in terms of congruences, by two different methods. The first method, "rationalization," is similar, but a bit shorter, than that used to prove Entries 2.1.1 and 2.1.4 in Chapter 2. Our systematic procedure relies on Ramanujan's addition formula for theta functions in Lemma 3.2.1. In the second method, we employ an alternative version of Entry 3.1.1 to give uniform proofs of the aforementioned dissections of $F(q)$. An interesting byproduct of our work is that several interesting q-series identities naturally arise in our proofs. Some of these identities appeared for the first time in [62] (see (3.4.9)–(3.4.11)), while others (see Theorems 3.4.1, 3.5.1, and 3.7.1) can also be proved using identities discovered by Ekin [133]. We emphasize that the approach here to these q-series identities is much simpler than that of Ekin. For example, Ekin's proof of Entry 3.7.1 requires the verifications of 55 identities [133, p. 2154], while in our proof, only Winquist's identity and Entry 3.2.1 are needed.

In Section 3.8, we in fact show that the formulations in terms of congruences are equivalent to those in terms of roots of unity. This was claimed without proof by W.-C. Liaw [215, pp. 85–86], but the first proof was provided by Garvan in [62]; a modification of Garvan's proof is given in Section 3.8. An advantage of the formulations in terms of congruences is that they yield congruences like those of Atkin and Swinnerton-Dyer [28] as corollaries.

The content of this chapter is based on a paper [62] that the second author coauthored with H.H. Chan, S.H. Chan, and Liaw.

3.2 Preliminary Results

It is easily seen that Ramanujan's Entry 3.1.1, which we prove in Chapter 4, is equivalent to Entry 3.2.1 below, which was independently discovered by R.J. Evans [136, Equation (3.1)], V.G. Kač and D.H. Peterson [187, Equation (5.26)], and Kač and M. Wakimoto [188, middle of p. 438]. As remarked in [187], the identity in fact appears in the classic text of J. Tannery and J. Molk [331, Section 486]. The notation ρ_k in the theorem below will be used throughout the chapter.

Entry 3.2.1. *Let* $\rho_k = (-1)^k q^{k(k+1)/2}$. *Then*

$$\frac{(q;q)_\infty^2}{(qa;q)_\infty(q/a;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{\rho_k(1-a)}{1-aq^k}. \quad (3.2.1)$$

Several times in the sequel we shall use an addition theorem for theta functions found in Chapter 16 of Ramanujan's second notebook [282], [55, p. 48, Entry 31].

Lemma 3.2.1. *Let* $|\alpha\beta| < 1$. *If* $U_n = \alpha^{n(n+1)/2}\beta^{n(n-1)/2}$ *and* $V_n = \alpha^{n(n-1)/2}\beta^{n(n+1)/2}$ *for each integer* n *and if* N *is any positive integer, then*

$$f(U_1, V_1) = \sum_{k=0}^{N-1} U_k f\left(\frac{U_{N+k}}{U_k}, \frac{V_{N-k}}{U_k}\right). \tag{3.2.2}$$

Also useful for us is the quintuple product identity [55, p. 80, Equation (38.2)].

Lemma 3.2.2. (*Quintuple product identity.*) *Let* $f(a,b)$ *be defined as in* (2.1.2), *and let*

$$f(-q) := f(-q, -q^2) = (q;q)_\infty, \tag{3.2.3}$$

by (2.1.3). *Then*

$$f(P^3Q, Q^5/P^3) - P^2 f(Q/P^3, P^3Q^5) = f(-Q^2)\frac{f(-P^2, -Q^2/P^2)}{f(PQ, Q/P)}. \tag{3.2.4}$$

Lastly, we need Winquist's identity [342]. From [146, Equation (6.15)], Winquist's identity can be put in the following form.

Lemma 3.2.3. (*Winquist's identity.*) *In the notation* (2.1.1),

$$\begin{aligned}
&(\alpha, q/\alpha, \beta, q/\beta, \alpha\beta, q/(\alpha\beta), \alpha/\beta, \beta q/\alpha, q, q; q)_\infty \\
&\quad = f(-\alpha^3, -q^3/\alpha^3)\left\{f(-\beta^3 q, -q^2/\beta^3) - \beta f(-\beta^3 q^2, -q/\beta^3)\right\} \\
&\qquad - \alpha\beta^{-1} f(-\beta^3, -q^3/\beta^3)\left\{f(-\alpha^3 q, -q^2/\alpha^3) - \alpha f(-\alpha^3 q^2, -q/\alpha^3)\right\}.
\end{aligned} \tag{3.2.5}$$

3.3 The 2-Dissection for $F(q)$

Entry 3.3.1 (p. 179). *Recall that* $F(q) = F_a(q)$ *is defined by* (3.1.1) *and that* $f(a,b)$ *is defined by* (2.1.2). *Then*

$$F_a(q) \equiv \frac{f(-q^6, -q^{10})}{(-q^4;q^4)_\infty} + \left(a - 1 + \frac{1}{a}\right) q \frac{f(-q^2, -q^{14})}{(-q^4;q^4)_\infty} \pmod{A_2}, \tag{3.3.1}$$

where A_2 *is defined in* (3.1.4).

Note that (3.3.1) is equivalent to (3.1.2), with $\sqrt{q}$ in (3.1.2) replaced by q.

The first proof of Theorem 3.3.1 that we give uses the method of "rationalization" and is an extension of Garvan's proof [146]. This method does not work in general, but only for those n-dissections for which n is "small." The method used in our second proof is longer, but it is more general. Furthermore, we obtain very interesting identities, (3.3.12) and (3.3.13), along the way.

First Proof of Entry 3.3.1. Throughout the proof, we assume that $|q| < |a| < 1/|q|$. We also shall frequently use the facts that $a^4 \equiv -1 \pmod{A_2}$ and that $a^8 \equiv 1 \pmod{A_2}$.

Write

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} = (q;q)_\infty \prod_{n=1}^\infty \left(\sum_{k=0}^\infty (aq^n)^k\right)\left(\sum_{k=0}^\infty (q^n/a)^k\right). \tag{3.3.2}$$

We now subdivide the series under the product sign into residue classes modulo 8 and then sum the series. Using repeatedly congruences modulo 8 for the powers of a, we readily find from (3.3.2) that

$$\begin{aligned}
&\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \\
&\equiv (q;q)_\infty \prod_{n=1}^\infty \frac{\left(1+aq^n+a^2q^{2n}+a^3q^{3n}\right)\left(1+a^{-1}q^n+a^{-2}q^{2n}+a^{-3}q^{3n}\right)}{(1+q^{4n})^2} \\
&\equiv \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} \prod_{n=1}^\infty (1+aq^n)(1+a^{-1}q^n) \pmod{A_2},
\end{aligned} \tag{3.3.3}$$

upon multiplying out the polynomials in the product on the previous line and using congruences for powers of a modulo A_2.

Next, using Lemma 3.2.1 with $\alpha = a$, $\beta = q/a$, and $N = 4$, (2.1.4), and congruences for powers of a modulo A_2, we find that

$$\begin{aligned}
(q;q)_\infty(-aq;q)_\infty(-q/a;q)_\infty &= (q;q)_\infty \frac{(-a;q)_\infty}{1+a}(-q/a;q)_\infty \\
&= \frac{1}{1+a}\{f(a^4q^6,q^{10}/a^4) + af(q^6/a^4,a^4q^{10}) \\
&\quad + a^2qf(q^2/a^4,a^4q^{14}) + (q/a)f(a^4q^2,q^{14}/a^4)\} \\
&\equiv \frac{1}{1+a}\left\{(1+a)f(-q^6,-q^{10}) + (a^2+1/a)qf(-q^2,-q^{14})\right\} \\
&\equiv f(-q^6,-q^{10}) + (A_1-1)\,qf(-q^2,-q^{14}) \pmod{A_2}.
\end{aligned} \tag{3.3.4}$$

Using (3.3.4) in (3.3.3), we complete the proof of Entry 3.3.1. □

Second Proof of Entry 3.3.1. From (3.2.1) and under the temporary conditions $|q| < |a| < |1/q|$, we deduce that

$$\begin{aligned}
\frac{(q;q)_\infty^2}{(qa;q)_\infty(q/a;q)_\infty} &= 1 + \sum_{k=1}^\infty \rho_k \frac{1-a}{1-aq^k} + \sum_{k=1}^\infty \rho_k \frac{1-a^{-1}}{1-q^k/a} \\
&= 1 + (1-a)\sum_{k=1,m=0}^\infty \rho_k q^{km}a^m + (1-a^{-1})\sum_{k=1,m=0}^\infty \rho_k q^{km}a^{-m}.
\end{aligned}$$

Hence, we deduce that

$$(q;q)_\infty F_a(q) = \frac{(q;q)_\infty^2}{(qa;q)_\infty(q/a;q)_\infty} = 1 + \sum_{k=1,m=0}^{\infty} \rho_k q^{km}\left(A_m - A_{m+1}\right), \tag{3.3.5}$$

where A_m is defined in (3.1.4). Observe that

$$A_m - A_{m+1} \equiv A_j - A_{j+1} \pmod{A_2},$$

whenever $m \equiv j \pmod 8$. Therefore, if

$$S_{i,j} := \sum_{\substack{k=1,m=0 \\ m\equiv i,j \pmod 8}}^{\infty} \rho_k q^{km},$$

we conclude that

$$\begin{aligned} 1+ &\sum_{k=1,m=0}^{\infty} \rho_k q^{km}\left(A_m - A_{m+1}\right) \\ &\equiv 1 + (2-A_1)\left\{S_{0,3} - S_{4,7}\right\} + A_1\left\{S_{1,2} - S_{5,6}\right\} \\ &\equiv 1 + (A_1 - 1)\left\{S_{1,2} - S_{5,6} - S_{0,3} + S_{4,7}\right\} \\ &\quad + \left\{S_{1,2} - S_{5,6} + S_{0,3} - S_{4,7}\right\} \pmod{A_2}, \end{aligned}$$

where we added and subtracted $S_{1,2} - S_{5,6}$. Summing the series on m, and then converting the sums into bilateral series, we conclude that

$$\begin{aligned} 1 + &\sum_{k=1,m=0}^{\infty} \rho_k q^{km}\left(A_m - A_{m+1}\right) \\ &\equiv (A_1 - 1)\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^{4k}} + \sum_{k=-\infty}^{\infty} \rho_k \frac{q^k+1}{1+q^{4k}} \pmod{A_2}. \end{aligned} \tag{3.3.6}$$

We are now ready to complete the proof of (3.3.1). Let $\omega = e^{\pi i/4}$. Calculating the partial fraction decomposition, we find that

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^{4k}} \\ &\quad = -\frac{1}{4}\sum_{k=-\infty}^{\infty} \rho_k \left(\frac{1+\omega^3}{1-\omega q^k} + \frac{1+\omega}{1-\omega^3 q^k} + \frac{1+\omega^7}{1-\omega^5 q^k} + \frac{1+\omega^5}{1-\omega^7 q^k}\right). \end{aligned} \tag{3.3.7}$$

By (3.2.1), we may rewrite (3.3.7) as

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^{4k}} = -\frac{1}{4}(q;q)_\infty^2 \Bigg\{ &\frac{1+\omega^3}{(\omega;q)_\infty(q/\omega;q)_\infty} + \frac{1+\omega}{(\omega^3;q)_\infty(q/\omega^3;q)_\infty} \\ &+ \frac{1+\omega^7}{(\omega^5;q)_\infty(q/\omega^5;q)_\infty} + \frac{1+\omega^5}{(\omega^7;q)_\infty(q/\omega^7;q)_\infty}\Bigg\}. \end{aligned} \tag{3.3.8}$$

Since $1-\omega^j q^k = 1 - q^k/\omega^{8-j}$, we may simplify (3.3.8) and obtain

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^{4k}} = -\frac{\sqrt{2}}{4}(q;q)_\infty^2 \left(\frac{1}{(\omega^3 q;q)_\infty (q/\omega^3;q)_\infty} - \frac{1}{(\omega q;q)_\infty (q/\omega;q)_\infty} \right)$$

$$= -\frac{\sqrt{2}}{4} \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} \left((q;q)_\infty (\omega q;q)_\infty (q/\omega;q)_\infty - (q;q)_\infty (\omega^3 q;q)_\infty (q/\omega^3;q)_\infty \right). \tag{3.3.9}$$

From the second equality of (3.3.4), with a replaced by $-\omega$ and $-\omega^3$, respectively, we find that

$$(q;q)_\infty (q\omega;q)_\infty (q/\omega;q)_\infty = f(-q^6,-q^{10}) + \left(-\omega-\omega^7-1\right) q f(-q^2,-q^{14}), \tag{3.3.10}$$

$$(q;q)_\infty (q\omega^3;q)_\infty (q/\omega^3;q)_\infty = f(-q^6,-q^{10}) + \left(-\omega^3-\omega^5-1\right) q f(-q^2,-q^{14}). \tag{3.3.11}$$

Employing (3.3.10) and (3.3.11) in (3.3.9) and simplifying yields

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^{4k}} = q\frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^2,-q^{14}). \tag{3.3.12}$$

Using exactly the same method, we can show that

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k+1}{1+q^{4k}} = \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^6,-q^{10}). \tag{3.3.13}$$

Substituting (3.3.12) and (3.3.13) into (3.3.6), we obtain (3.3.1) by eliminating the factor $(q;q)_\infty$ in (3.3.5). □

3.4 The 3-Dissection for $F(q)$

As in the case of (3.3.1), we prove instead the congruence given below. Surprisingly, the 3-dissection is considerably more difficult to prove than the 2- and 5-dissections, for example. We give two proofs. The first uses the method of "rationalization" and is shorter than our second proof, which depends on Ramanujan's key theorem, Entry 3.2.1. However, we were able to find the first proof only because of insights gained from the second proof. When $a = e^{2\pi i/9}$, Entry 3.4.1 yields the 3-dissection of $F_a(q)$, which was first proved by Garvan [147] using the Macdonald identity for the root system A_2. Garvan's proof can be modified to give another proof of Entry 3.4.1.

Entry 3.4.1 (p. 179). *If A_n is given by* (3.1.4), *then*

$$F_a(q) \equiv \frac{f(-q^6,-q^{21})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty} + (A_1-1)\,q\frac{f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}$$
$$+ A_2q^2\frac{f(-q^3,-q^{24})f(-q^6,-q^{21})}{(q^{27};q^{27})_\infty} \pmod{A_3+1}. \tag{3.4.1}$$

First Proof of Entry 3.4.1. We first record the identities that we need in our proof.

Substituting $(P,Q)=(-q^{3/2},q^{27/2})$, $(-q^{15/2},q^{27/2})$, and $(-q^{21/2},q^{27/2})$ into the quintuple product identity (3.2.4), we find that

$$B-q^3C=\frac{f(-q^3,-q^{24})(q^{27};q^{27})_\infty}{f(-q^{12},-q^{15})}, \tag{3.4.2}$$

$$A+q^6C=A-q^{15}f(-q^{-9},-q^{90})=\frac{f(-q^{12},-q^{15})(q^{27};q^{27})_\infty}{f(-q^6,-q^{21})}, \tag{3.4.3}$$

$$A+q^3B=A-q^{21}f(-q^{-18},-q^{99})=\frac{f(-q^6,-q^{21})(q^{27};q^{27})_\infty}{f(-q^3,-q^{24})}, \tag{3.4.4}$$

respectively, where $A=f(-q^{45},-q^{36})$, $B=f(-q^{63},-q^{18})$, and $C=f(-q^{72},-q^9)$.

Substituting $\alpha=-a^2$, $\beta=-q/a^2$, and $N=9$ into (3.2.2) and simplifying, we deduce that

$$(q;q)_\infty(a^2q;q)_\infty(q/a^2;q)_\infty \tag{3.4.5}$$
$$\equiv -(1+A_2)\,q(q^{27};q^{27})_\infty + A - (A_1-1)\,q^3B + A_1q^6C \pmod{A_3+1}.$$

After these preliminary steps, we now complete our proof of the 3-dissection.

From the generating function (3.1.1),

$$\begin{aligned}
F_a(q) &\equiv \frac{(q;q)_\infty}{(aq;q)_\infty(a^8q;q)_\infty}\\
&\equiv \frac{(q;q)_\infty(q^3;q^3)_\infty(a^2q;q)_\infty(a^4q;q)_\infty(a^5q;q)_\infty(a^7q;q)_\infty}{(q^9;q^9)_\infty}\\
&\equiv \frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\\
&\quad\times\left\{-\left(1+a^2+a^7\right)q(q^{27};q^{27})_\infty + A - \left(a+a^8-1\right)q^3B + \left(a+a^8\right)q^6C\right\}\\
&\quad\times\left\{-\left(1+a^4+a^5\right)q(q^{27};q^{27})_\infty + A - \left(a^2+a^7-1\right)q^3B\right.\\
&\quad\left.+\left(a^2+a^7\right)q^6C\right\} \pmod{A_3+1},
\end{aligned}$$

where we have applied (3.4.5) in the last equality.

Arranging the terms in the "right" order, with knowledge from our second proof being helpful, we find that

$$F_a(q) \equiv \frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\Big\{ \big(-q^2(q^{27};q^{27})_\infty^2 - q(q^{27};q^{27})_\infty(A+q^3B) \\ +(A+q^3B)(A+q^6C)\big) + \left[a-1+a^8\right]\big(-q^2(q^{27};q^{27})_\infty^2 \\ +q(q^{27};q^{27})_\infty(A+q^6C) - q^3(B-q^3C)(A+q^6C)\big) \\ + \left[a^2+a^7\right]\big(q^2(q^{27};q^{27})_\infty^2 - q^4(q^{27};q^{27})_\infty(B-q^3C) \\ -q^3(A+q^3B)(B-q^3C)\big)\Big\} \pmod{A_3+1}.$$

Substituting (3.4.2)–(3.4.4) into the terms on the right-hand side and simplifying, we find that

$$F_a(q) \equiv \frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty} \\ \times\Big\{\frac{f(-q^{12},-q^{15})-qf(-q^6,-q^{21})-q^2f(-q^3,-q^{24})}{f(-q^3,-q^{24})} \\ + \left[a-1+a^8\right]q\frac{f(-q^{12},-q^{15})-qf(-q^6,-q^{21})-q^2f(-q^3,-q^{24})}{f(-q^6,-q^{21})} \\ + \left[a^2+a^7\right]q^2\frac{f(-q^{12},-q^{15})-qf(-q^6,-q^{21})-q^2f(-q^3,-q^{24})}{f(-q^{12},-q^{15})}\Big\} \\ \equiv \frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty}\Big\{\frac{(q;q)_\infty}{f(-q^3,-q^{24})} + \left[a-1+a^8\right]q\frac{(q;q)_\infty}{f(-q^6,-q^{21})} \\ + \left[a^2+a^7\right]q^2\frac{(q;q)_\infty}{f(-q^{12},-q^{15})}\Big\} \pmod{A_3+1}, \tag{3.4.6}$$

where we have applied [55, p. 349, Entry 2(v)] in the last equality, namely,

$$(q;q)_\infty = f(-q^{12},-q^{15}) - qf(-q^6,-q^{21}) - q^2f(-q^3,-q^{24}). \tag{3.4.7}$$

Finally, note that [55, p. 349, Entry 2(vi)]

$$\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^3,-q^{24})} = \frac{f(-q^6,-q^{21})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}, \\ \frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^6,-q^{21})} = \frac{f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}, \\ \frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^{12},-q^{15})} = \frac{f(-q^3,-q^{24})f(-q^6,-q^{21})}{(q^{27};q^{27})_\infty}.$$

Employing these identities in (3.4.6), we complete the first proof of the 3-dissection given in Entry 3.4.1. □

Second Proof of Entry 3.4.1. First, we observe that

$$A_m - A_{m+1} \equiv A_j - A_{j+1} \pmod{A_3+1},$$

whenever $m \equiv j \pmod 9$, where A_m is defined in (3.1.4). Proceeding as before, if we set

$$T_{i,j,l} := \sum_{\substack{k=1,m=0\\ m\equiv i,j,l \pmod 9}}^{\infty} \rho_k q^{km},$$

we find that

$$\begin{aligned} 1+\sum_{k=1,m=0}^{\infty} \rho_k q^{km}(A_m - A_{m+1}) &\equiv 1+\{T_{0,1,2}-T_{6,7,8}\} \\ &+(A_1-1)\{T_{1,3,8}-T_{0,5,7}\}+A_2\{T_{2,3,7}-T_{1,5,6}\} \pmod{A_3+1}. \end{aligned}$$

Simplifying, we find that

$$\begin{aligned} 1+\sum_{k=1,m=0}^{\infty} \rho_k q^{km}(A_m - A_{m+1}) &\equiv \sum_{k=-\infty}^{\infty} \rho_k \frac{1+q^k+q^{2k}}{1+q^{3k}+q^{6k}} \\ +(A_1-1)\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^{3k}+q^{6k}} &+ A_2 \sum_{k=-\infty}^{\infty} \rho_k \frac{q^{2k}-q^k}{1+q^{3k}+q^{6k}} \pmod{A_3+1}. \end{aligned} \tag{3.4.8}$$

The proof of (3.4.1) now follows from Entry 3.1.1 and the following identities, which are analogues of (3.3.12) and (3.3.13). □

Theorem 3.4.1. *We have*

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^{3k}+q^{6k}} = q(q;q)_\infty \frac{f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}, \tag{3.4.9}$$

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{q^{2k}-q^k}{1+q^{3k}+q^{6k}} = q^2(q;q)_\infty \frac{f(-q^3,-q^{24})f(-q^{6},-q^{21})}{(q^{27};q^{27})_\infty}, \tag{3.4.10}$$

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{1+q^k+q^{2k}}{1+q^{3k}+q^{6k}} = (q;q)_\infty \frac{f(-q^6,-q^{21})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}. \tag{3.4.11}$$

We give only the proof of (3.4.9). The other two identities can be established using the same method.

Proof of Theorem 3.4.1. Let $\zeta = e^{2\pi i/9}$. Proceeding as in the second proof of the 2-dissection, we calculate the partial fraction decomposition

$$\frac{9(1-q^k)}{1+q^{3k}+q^{6k}} = (1-\zeta^6)\left(\frac{1-\zeta^8}{1-\zeta q^k}+\frac{1-\zeta^5}{1-\zeta^4 q^k}+\frac{1-\zeta^2}{1-\zeta^7 q^k}\right) \\ +(1-\zeta^3)\left(\frac{1-\zeta}{1-\zeta^8 q^k}+\frac{1-\zeta^4}{1-\zeta^5 q^k}+\frac{1-\zeta^7}{1-\zeta^2 q^k}\right).$$

Hence, we find that

$$9\sum_{k=-\infty}^{\infty}\rho_k\frac{1-q^k}{1+q^{3k}+q^{6k}} = (1-\zeta^6)\left[\frac{1-\zeta^8}{1-\zeta}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta}{1-\zeta q^k} \right. \\ \left. +\frac{1-\zeta^5}{1-\zeta^4}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^4}{1-\zeta^4 q^k}+\frac{1-\zeta^2}{1-\zeta^7}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^7}{1-\zeta^7 q^k}\right] \\ +(1-\zeta^3)\left[\frac{1-\zeta}{1-\zeta^8}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^8}{1-\zeta^8 q^k}\right. \\ \left. +\frac{1-\zeta^4}{1-\zeta^5}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^5}{1-\zeta^5 q^k}+\frac{1-\zeta^7}{1-\zeta^2}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^2}{1-\zeta^2 q^k}\right]. \tag{3.4.12}$$

Using (3.2.1) and the identity $1-\zeta^6=-\zeta^6(1-\zeta^3)$, we rewrite (3.4.12) as

$$9\sum_{k=-\infty}^{\infty}\rho_k\frac{1-q^k}{1+q^{3k}+q^{6k}} = -(1-\zeta^3)(q;q)_\infty^2 \\ \times\left(\frac{\zeta-\zeta^5}{(\zeta q;q)_\infty(\zeta^8 q;q)_\infty}+\frac{\zeta^4-\zeta^2}{(\zeta^4 q;q)_\infty(\zeta^5 q;q)_\infty}+\frac{\zeta^7-\zeta^8}{(\zeta^2 q;q)_\infty(\zeta^7 q;q)_\infty}\right) \\ = -(1-\zeta^3)\frac{(q;q)_\infty^2(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\Big[(\zeta-\zeta^5)(\zeta^2 q;q)_\infty(\zeta^4 q;q)_\infty(\zeta^5 q;q)_\infty(\zeta^7 q;q)_\infty \\ +(\zeta^4-\zeta^2)(\zeta q;q)_\infty(\zeta^2 q;q)_\infty(\zeta^7 q;q)_\infty(\zeta^8 q;q)_\infty \\ +(\zeta^7-\zeta^8)(\zeta q;q)_\infty(\zeta^4 q;q)_\infty(\zeta^5 q;q)_\infty(\zeta^8 q;q)_\infty\Big]. \tag{3.4.13}$$

Note that when $a=\zeta^j$ and $\gcd(j,9)=1$, we can deduce from (3.4.5) that

$$(q;q)_\infty(\zeta^{2j}q;q)_\infty(q/\zeta^{2j};q)_\infty = -\left(1+\frac{1}{\zeta^{2j}}+\zeta^{2j}\right)q(q^{27};q^{27})_\infty \\ +A-\left(\zeta^j+\frac{1}{\zeta^j}-1\right)q^3B+\left(\zeta^j+\frac{1}{\zeta^j}\right)q^6C. \tag{3.4.14}$$

Using (3.4.14) six times, we rewrite (3.4.13) as

$$9\sum_{k=-\infty}^{\infty}\rho_k\frac{1-q^k}{1+q^{3k}+q^{6k}} = -9q\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\Big[-q(q^{27};q^{27})_\infty^2+q^6(q^{27};q^{27})_\infty C \\ +(q^{27};q^{27})_\infty A-q^8BC-q^2AB+q^{11}C^2+q^5AC\Big]$$

$$= -9q\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\Big[-q(q^{27};q^{27})_\infty^2 + (q^{27};q^{27})_\infty\left(A+q^6C\right)$$
$$- q^2\left(A+q^6C\right)\left(B-q^3C\right)\Big]. \tag{3.4.15}$$

Substituting (3.4.2) and (3.4.3) into (3.4.15), we deduce that

$$\sum_{k=-\infty}^{\infty}\rho_k\frac{1-q^k}{1+q^{3k}+q^{6k}}$$
$$= -q\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\Bigg(-q(q^{27};q^{27})_\infty^2 + \frac{f(-q^{12},-q^{15})(q^{27};q^{27})_\infty^2}{f(-q^6,-q^{21})}$$
$$- q^2\frac{f(-q^3,-q^{24})(q^{27};q^{27})_\infty}{f(-q^{12},-q^{15})}\frac{f(-q^{12},-q^{15})(q^{27};q^{27})_\infty}{f(-q^6,-q^{21})}\Bigg)$$
$$= -q\frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q^9;q^9)_\infty f(-q^6,-q^{21})}$$
$$\times\Big(-qf(-q^6,-q^{21}) + f(-q^{12},-q^{15}) - q^2f(-q^3,-q^{24})\Big)$$
$$= -q\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^6,-q^{21})}$$
$$= -q\frac{(q;q)_\infty f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty},$$

where we have applied (3.4.7) in the penultimate equality. This completes the proof of (3.4.9). □

3.5 The 5-Dissection for $F(q)$

For this and the remaining sections, it will be convenient to define

$$S_n := S_n(a) := \sum_{k=-n}^{n} a^k. \tag{3.5.1}$$

Note that when p is an odd prime,

$$S_{(p-1)/2}(a) = a^{(1-p)/2}\Phi_p(a),$$

where $\Phi_n(a)$ is the minimal polynomial for a primitive nth root of unity.

In this section, $\zeta = e^{2\pi i/5}$. We provide two proofs of the congruence corresponding to the 5-dissection. The first proof is similar to Garvan's proof [146] of the 5-dissection of $F_\zeta(q)$ that we gave in Chapter 2; however, our argument is somewhat shorter. Note that if we set $a = 1$ in Theorem 3.5.1, we recover Atkin and Swinnerton-Dyer's result [28, Theorem 1].

Entry 3.5.1 (pp. 18, 20). *With $f(-q)$ defined by (3.2.3), S_2 defined by (3.5.1), and A_n defined by (3.1.4),*

$$\begin{aligned} F_a(q) \equiv \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})} f^2(-q^{25}) + (A_1-1)q \frac{f^2(-q^{25})}{f(-q^5,-q^{20})} \\ + A_2 q^2 \frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} - A_1 q^3 \frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})} f^2(-q^{25}) \pmod{S_2}. \end{aligned} \tag{3.5.2}$$

In his lost notebook [283, pp. 58, 59, 182], Ramanujan factored the coefficients of $F_a(q)$ as functions of a. In particular, he sought factors S_2 in the coefficients. Details may be found in Chapter 4.

First Proof of Entry 3.5.1. It is easy to see that

$$F_a(q) \equiv \frac{(q;q)_\infty^2 (a^2q;q)_\infty (a^3q;q)_\infty}{(q^5;q^5)_\infty} \pmod{S_2}. \tag{3.5.3}$$

We shall use later a famous formula for the Rogers–Ramanujan continued fraction $R(q)$ defined by

$$R(q) := \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \cdots, \qquad |q|<1,$$

namely [55, p. 265, Entry 11(iii)],

$$\frac{1}{R(q)} - R(q) - 1 = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}. \tag{3.5.4}$$

Using the well-known fact [55, p. 266, Entry 11(iii)],

$$R(q) = q^{1/5} \frac{f(-q,-q^4)}{f(-q^2,-q^3)},$$

we can rewrite (3.5.4) in the form

$$\frac{f(-q)}{qf(-q^{25})} = \frac{f(-q^{10},-q^{15})}{qf(-q^5,-q^{20})} - \frac{qf(-q^5,-q^{20})}{f(-q^{10},-q^{15})} - 1. \tag{3.5.5}$$

By (2.1.3) and Lemma 3.2.1 with $(\alpha,\beta,n) = (-a^2,-q/a^2,5)$, we find that

$$\begin{aligned} (q;q)_\infty (a^2q;q)_\infty (a^3q;q)_\infty &\equiv \frac{f(-a^2,-q/a^2)}{(1-a^2)} \\ &\equiv f(-q^{10},-q^{15}) + qA_1 f(-q^{20},-q^5) \pmod{S_2}. \end{aligned} \tag{3.5.6}$$

Substituting (3.5.6) and (3.5.5) into (3.5.3) yields (3.5.2). □

Second Proof of Entry 3.5.1. We apply Entry 3.2.1. Define

$$T_i := \sum_{\substack{k=1,m=0\\ m\equiv i \pmod 5}}^{\infty} \rho_k q^{km}.$$

Then

$$\begin{aligned}\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} &= 1 + \sum_{k=1,m=0}^{\infty} \rho_k q^{km}\left(A_m - A_{m+1}\right)\\ &\equiv 1 + (2-A_1)\{T_0 - T_4\} + (A_1 - A_2)\{T_1 - T_3\}\\ &\equiv 1 + \{2T_0 - 2T_4 + T_1 - T_3\}\\ &\quad + A_1\{T_4 - T_0 + 2T_1 - 2T_3\} \pmod{S_2},\end{aligned} \tag{3.5.7}$$

where A_m is defined in (3.1.4). Note that

$$\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \rho_k \frac{q^{2k}}{1-q^{5k}} = 0.$$

This enables us to simplify (3.5.7) to conclude that

$$\begin{aligned}\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} &\equiv \sum_{k=-\infty}^{\infty} \rho_k \frac{2+3q^k}{1+q^k+q^{2k}+q^{3k}+q^{4k}}\\ &\quad + A_1 \sum_{k=-\infty}^{\infty} \rho_k \frac{q^k - 1}{1+q^k+q^{2k}+q^{3k}+q^{4k}} \pmod{S_2}.\end{aligned}$$

The proof of the 5-dissection now follows from the following identities. □

Theorem 3.5.1. *We have*

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \rho_k \frac{2+3q^k}{1+q^k+q^{2k}+q^{3k}+q^{4k}} &= \frac{(q;q)_\infty(q^{25};q^{25})_\infty^2 f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})}\\ &\quad - q\frac{(q;q)_\infty(q^{25};q^{25})_\infty^2}{f(-q^5,-q^{20})} - q^2\frac{(q;q)_\infty(q^{25};q^{25})_\infty^2}{f(-q^{10},-q^{15})},\end{aligned} \tag{3.5.8}$$

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \rho_k \frac{q^k-1}{1+q^k+q^{2k}+q^{3k}+q^{4k}} &= q\frac{(q;q)_\infty(q^{25};q^{25})_\infty^2}{f(-q^5,-q^{20})}\\ &\quad - q^2\frac{(q;q)_\infty(q^{25};q^{25})_\infty^2}{f(-q^{10},-q^{15})} - q^3\frac{(q;q)_\infty(q^{25};q^{25})_\infty^2 f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})}.\end{aligned} \tag{3.5.9}$$

Proof. We prove only (3.5.8), since the proof of (3.5.9) is similar.

We begin with the partial fraction decomposition

$$\frac{5(2+3q^k)}{1+q^k+q^{2k}+q^{3k}+q^{4k}} = \frac{(1-\zeta^4)(2+3\zeta^4)}{1-\zeta q^k} + \frac{(1-\zeta^3)(2+3\zeta^3)}{1-\zeta^2 q^k} + \frac{(1-\zeta^2)(2+3\zeta^2)}{1-\zeta^3 q^k} + \frac{(1-\zeta)(2+3\zeta)}{1-\zeta^4 q^k}.$$

Therefore,

$$\begin{aligned} G(q) &:= 5\sum_{k=-\infty}^{\infty} \rho_k \frac{2+3q^k}{1+q^k+q^{2k}+q^{3k}+q^{4k}} \\ &= \frac{(1-\zeta^4)(2+3\zeta^4)}{1-\zeta}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta}{1-\zeta q^k} + \frac{(1-\zeta^3)(2+3\zeta^3)}{1-\zeta^2}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^2}{1-\zeta^2 q^k} \\ &\quad + \frac{(1-\zeta^2)(2+3\zeta^2)}{1-\zeta^3}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^3}{1-\zeta^3 q^k} + \frac{(1-\zeta)(2+3\zeta)}{1-\zeta^4}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^4}{1-\zeta^4 q^k}. \end{aligned}$$

Applying Entry 3.2.1 on the right-hand side and simplifying, we find that

$$\begin{aligned} G(q) &= -\left(2\zeta^4+3\zeta^3+2\zeta+3\zeta^2\right)\frac{(q;q)_\infty^2}{(\zeta q;q)_\infty(\zeta^4 q;q)_\infty} \\ &\quad - \left(2\zeta^3+3\zeta+2\zeta^2+3\zeta^4\right)\frac{(q;q)_\infty^2}{(\zeta^2 q;q)_\infty(\zeta^3 q;q)_\infty} \\ &= \left(2-\zeta^2-\zeta^3\right)\frac{(q;q)_\infty^3(\zeta^2 q;q)_\infty(\zeta^3 q;q)_\infty}{(q^5;q^5)_\infty} \\ &\quad + \left(2-\zeta-\zeta^4\right)\frac{(q;q)_\infty^3(\zeta q;q)_\infty(\zeta^4 q;q)_\infty}{(q^5;q^5)_\infty}. \end{aligned} \tag{3.5.10}$$

Applying (3.5.6) two times on the right-hand side of (3.5.10) with $a=\zeta$ and $a=\zeta^3$, respectively, we find that

$$\begin{aligned} G(q) &= \frac{(q;q)_\infty^2}{(q^5;q^5)_\infty}\Bigg\{5f(-q^{10},-q^{15}) \\ &\quad + \left[\left(2-\zeta^2-\zeta^3\right)\frac{\zeta^4-\zeta^3}{1-\zeta^2} + \left(2-\zeta-\zeta^4\right)\frac{\zeta^2-\zeta^4}{1-\zeta}\right] f(-q^5,-q^{20})\Bigg\} \\ &= 5\frac{(q;q)_\infty^2 f(-q^{10},-q^{15})}{(q^5;q^5)_\infty}. \end{aligned} \tag{3.5.11}$$

From (3.5.5), we find that

$$(q;q)_\infty = -q(q^{25};q^{25})_\infty + (q^{25};q^{25})_\infty\left\{\frac{f(-q^{10},-q^{15})}{f(-q^5,-q^{20})} - q^2\frac{f(-q^5,-q^{20})}{f(-q^{10},-q^{15})}\right\}. \tag{3.5.12}$$

Substituting (3.5.12) into (3.5.11) and dividing by 5, we find that $G(q)/5$ equals the right-hand side of (3.5.8). $\square$

Theorem 3.5.1 can also be proved using identities established by Ekin [133, bottom of p. 2149].

3.6 The 7-Dissection for $F(q)$

We offer two proofs of the 7-dissection of $F_a(q)$. The first is an extension of that of Garvan [146], while the second uses the theorem of Ramanujan, Kač and Wakimoto [188], and Evans[136], Entry 3.2.1. Note that if we substitute $a = 1$ in Entry 3.6.1, we immediately obtain [28, Theorem 2]. In this section, $\zeta = e^{2\pi i/7}$.

Entry 3.6.1 (p. 19). *With* $f(a,b)$ *defined by* (2.1.2), $f(-q)$ *defined by* (3.2.3), A_n *defined by* (3.1.4), *and* S_m *defined by* (3.5.1),

$$\frac{(q;q)_\infty}{(qa;q)_\infty(q/a;q)_\infty} \equiv \frac{1}{f(-q^7)}\biggl(A^2 + (A_1-1)qAB + A_2q^2B^2 + (A_3+1)q^3AC \\ - A_1q^4BC - (A_2+1)q^6C^2\biggr) \pmod{S_3}, \tag{3.6.1}$$

where $A = f(-q^{21}, -q^{28})$, $B = f(-q^{35}, -q^{14})$, *and* $C = f(-q^{42}, -q^7)$.

First Proof of Entry 3.6.1. Rationalizing and using Jacobi's triple product identity (2.1.3), we find that

$$\begin{aligned}\frac{(q;q)_\infty}{(qa;q)_\infty(q/a;q)_\infty} &\equiv \frac{(q;q)_\infty^2(qa^2;q)_\infty(qa^{-2};q)_\infty(qa^3;q)_\infty(qa^{-3};q)_\infty}{(q^7;q^7)_\infty} \\ &\equiv \frac{1}{f(-q^7)}\frac{f(-a^2,-q/a^2)}{(1-a^2)}\frac{f(-a^3,-q/a^3)}{(1-a^3)} \pmod{S_3}. \end{aligned} \tag{3.6.2}$$

Using Lemma 3.2.1, with $(\alpha,\beta,N) = (-a^2,-q/a^2,7)$ and $(-a^3,-q/a^3,7)$, respectively, we find that

$$\frac{f(-a^2,-q/a^2)}{(1-a^2)} \equiv A - q\frac{(a^5-a^4)}{(1-a^2)}B + q^3\frac{(a^3-a^6)}{(1-a^2)}C \pmod{S_3} \tag{3.6.3}$$

and

$$\frac{f(-a^3,-q/a^3)}{(1-a^3)} \equiv A - q\frac{(a^4-a^6)}{(1-a^3)}B + q^3\frac{(a-a^2)}{(1-a^3)}C \pmod{S_3}. \tag{3.6.4}$$

Substituting (3.6.3) and (3.6.4) into (3.6.2) and simplifying, we complete the proof of Entry 3.6.1. $\square$

Second Proof of Entry 3.6.1. Set

$$T_i := \sum_{\substack{k=1,m=0 \\ m\equiv i \pmod 7}}^{\infty} \rho_k q^{km}.$$

As in our proofs of the 2-, 3-, and 5-dissections, we begin by using Entry 3.2.1 and recalling the notation A_m from (3.1.4) to deduce that

$$\begin{aligned}
\frac{(q;q)_\infty^2}{(aq;q)_\infty (q/a;q)_\infty} &= 1 + \sum_{k=1,m=0}^{\infty} \rho_k q^{km} \left(A_m - A_{m+1}\right) \\
&\equiv 1 + (2 - A_1)\left\{T_0 - T_6\right\} + (A_1 - A_2)\left\{T_1 - T_5\right\} \\
&\quad + (A_2 - A_3)\left\{T_2 - T_4\right\} \\
&\equiv (2 - A_1) \sum_{k=-\infty}^{\infty} \rho_k \frac{1 + q^k + q^{2k}}{1 + q^k + \cdots + q^{6k}} \\
&\quad + (A_1 - A_2) \sum_{k=-\infty}^{\infty} \rho_k \frac{q^k + q^{2k}}{1 + q^k + \cdots + q^{6k}} \\
&\quad + (A_2 - A_3) \sum_{k=-\infty}^{\infty} \rho_k \frac{q^{2k}}{1 + q^k + \cdots + q^{6k}} \\
&\equiv \sum_{k=-\infty}^{\infty} \rho_k \frac{2 + 2q^k + 3q^{2k}}{1 + q^k + \cdots + q^{6k}} \\
&\quad + A_1 \sum_{k=-\infty}^{\infty} \rho_k \frac{q^{2k} - 1}{1 + q^k + \cdots + q^{6k}} \\
&\quad + A_2 \sum_{k=-\infty}^{\infty} \rho_k \frac{q^{2k} - q^k}{1 + q^k + \cdots + q^{6k}} \pmod{S_3}. \qquad (3.6.5)
\end{aligned}$$

The proof of Entry 3.6.1 now follows from the following identities. Indeed, if we substitute the identities of Theorem 3.6.1 into (3.6.5) and collect terms, we complete the second proof of Entry 3.6.1. □

Theorem 3.6.1. *We have*

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \rho_k \frac{7}{1 + q^k + \cdots + q^{6k}} \qquad (3.6.6) \\
&\qquad = \frac{(q;q)_\infty}{(q^7;q^7)_\infty} \left\{A^2 - 6qAB + 2q^2B^2 + 3q^3AC + 5q^4BC - 3q^6C^2\right\}, \\
&\sum_{k=-\infty}^{\infty} \rho_k \frac{7q^k}{1 + q^k + \cdots + q^{6k}} \qquad (3.6.7)
\end{aligned}$$

$$= \frac{(q;q)_\infty}{(q^7;q^7)_\infty} \left\{ A^2 + qAB - 5q^2B^2 + 3q^3AC - 2q^4BC + 4q^6C^2 \right\},$$

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{7q^{2k}}{1+q^k+\cdots+q^{6k}} \tag{3.6.8}$$

$$= \frac{(q;q)_\infty}{(q^7;q^7)_\infty} \left\{ A^2 + qAB + 2q^2B^2 - 4q^3AC - 2q^4BC - 3q^6C^2 \right\},$$

where $A, B,$ and C are given in Entry 3.6.1.

Proof. We prove only (3.6.6), since the proofs of the remaining two identities are similar. We first calculate the partial fraction decomposition

$$\frac{7}{1+q^k+\cdots+q^{6k}} = \frac{1-\zeta^6}{1-\zeta q^k} + \frac{1-\zeta^5}{1-\zeta^2 q^k} + \frac{1-\zeta^4}{1-\zeta^3 q^k} + \frac{1-\zeta^3}{1-\zeta^4 q^k} + \frac{1-\zeta^2}{1-\zeta^5 q^k} + \frac{1-\zeta}{1-\zeta^6 q^k}.$$

Therefore we deduce that

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \rho_k \frac{7}{1+q^k+\cdots+q^{6k}} &= \frac{1-\zeta^6}{1-\zeta} \sum_{k=-\infty}^{\infty} \rho_k \frac{1-\zeta}{1-\zeta q^k} \\ &+ \frac{1-\zeta^5}{1-\zeta^2} \sum_{k=-\infty}^{\infty} \rho_k \frac{1-\zeta^2}{1-\zeta^2 q^k} + \frac{1-\zeta^4}{1-\zeta^3} \sum_{k=-\infty}^{\infty} \rho_k \frac{1-\zeta^3}{1-\zeta^3 q^k} \\ &+ \frac{1-\zeta^3}{1-\zeta^4} \sum_{k=-\infty}^{\infty} \rho_k \frac{1-\zeta^4}{1-\zeta^4 q^k} + \frac{1-\zeta^2}{1-\zeta^5} \sum_{k=-\infty}^{\infty} \rho_k \frac{1-\zeta^5}{1-\zeta^5 q^k} \\ &+ \frac{1-\zeta}{1-\zeta^6} \sum_{k=-\infty}^{\infty} \rho_k \frac{1-\zeta^6}{1-\zeta^6 q^k}. \end{aligned} \tag{3.6.9}$$

From the identity (3.6.9) and Entry 3.2.1, we find that

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \rho_k \frac{7}{1+q^k+\cdots+q^{6k}} &= \left\{ \frac{1-\zeta^6}{1-\zeta} + \frac{1-\zeta}{1-\zeta^6} \right\} \frac{(q;q)_\infty^2}{(\zeta q;q)_\infty (\zeta^6 q;q)_\infty} \\ &+ \left\{ \frac{1-\zeta^5}{1-\zeta^2} + \frac{1-\zeta^2}{1-\zeta^5} \right\} \frac{(q;q)_\infty^2}{(\zeta^2 q;q)_\infty (\zeta^5 q;q)_\infty} \\ &+ \left\{ \frac{1-\zeta^4}{1-\zeta^3} + \frac{1-\zeta^3}{1-\zeta^4} \right\} \frac{(q;q)_\infty^2}{(\zeta^3 q;q)_\infty (\zeta^4 q;q)_\infty}.\end{aligned}$$

Rationalizing the denominators of the infinite products on the right-hand side and applying the elementary identity $1-\zeta^n = -\zeta^n(1-\zeta^{7-n})$, we deduce that

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{7}{1+q^k+\cdots+q^{6k}} \tag{3.6.10}$$
$$= \frac{(q;q)_\infty^3}{(q^7;q^7)_\infty}\Big\{(-\zeta^6-\zeta)(\zeta^2 q;q)_\infty(\zeta^3 q;q)_\infty(\zeta^4 q;q)_\infty(\zeta^5 q;q)_\infty$$
$$+(-\zeta^5-\zeta^2)(\zeta q;q)_\infty(\zeta^3 q;q)_\infty(\zeta^4 q;q)_\infty(\zeta^6 q;q)_\infty$$
$$+(-\zeta^4-\zeta^3)(\zeta q;q)_\infty(\zeta^2 q;q)_\infty(\zeta^5 q;q)_\infty(\zeta^6 q;q)_\infty\Big\}.$$

Applying Lemma 3.2.1 with $\alpha=-a$, $\beta=-q/a$, and $N=7$, we deduce that

$$(aq;q)_\infty(q/a;q)_\infty(q;q)_\infty \equiv A+\frac{a^2-a^6}{1-a}qB+\frac{a^5-a^3}{1-a}q^3C \pmod{S_3}. \tag{3.6.11}$$

Applying (3.6.11) six times with $a=\zeta^2,\zeta^3,\zeta,\zeta^3,\zeta,\zeta^2$ in (3.6.10) and simplifying, we complete the proof of the first identity in Theorem 3.6.1. ☐

Theorem 3.6.1 can also be found in [133, Equations (4.13)–(4.15)]. Our method of proof is different from that of Ekin.

3.7 The 11-Dissection for $F(q)$

In this section, $\zeta=e^{2\pi i/11}$. If we set $a=1$ in Entry 3.7.1 below, we recover [28, Theorem 3]. An elementary proof of [28, Theorem 3] has been given by Hirschhorn [178].

Entry 3.7.1 (p. 70). *With A_m defined by (3.1.4) and S_5 defined by (3.5.1), we have*

$$F_a(q) \equiv \frac{1}{(q^{11};q^{11})_\infty(q^{121};q^{121})_\infty^2}\Big(ABCD+\{A_1-1\}\,qA^2BE$$
$$+A_2\,q^2AC^2D+\{A_3+1\}\,q^3ABD^2$$
$$+\{A_2+A_4+1\}\,q^4ABCE-\{A_2+A_4\}\,q^5B^2CE$$
$$+\{A_1+A_4\}\,q^7ABDE-\{A_2+A_5+1\}\,q^{19}CDE^2$$
$$-\{A_4+1\}\,q^9ACDE-\{A_3\}\,q^{10}BCDE\Big) \pmod{S_5},$$

where $A=f(-q^{55},-q^{66})$, $B=f(-q^{77},-q^{44})$, $C=f(-q^{88},-q^{33})$, $D=f(-q^{99},-q^{22})$, and $E=f(-q^{110},-q^{11})$.

Before we begin our proofs of Entry 3.7.1, we first state some results that will be useful in our proofs.

Specializing (3.2.5) with $\alpha=a^m$ and $\beta=a^n$, we find that

$$\begin{aligned}(a^m q, &a^{11-m}q, a^n q, a^{11-n}q, a^{m+n}q, a^{11-m-n}q, a^{m-n}q, a^{11-m+n}q, q, q; q)_\infty \\ &\equiv \frac{1}{(1-a^m)(1-a^n)(1-a^{m+n})(1-a^{m-n})} \\ &\quad \times \left\{\mathbf{G}(a^{3m})\mathbf{H}(a^n) - a^{m-n}\mathbf{G}(a^{3n})\mathbf{H}(a^m)\right\} \pmod{S_5}, \qquad (3.7.1)\end{aligned}$$

where

$$\mathbf{G}(x) := f(-x, -x^{10}q^3) \quad \text{and} \quad \mathbf{H}(x) := f(-x^3q, -x^8q^2) - xf(-x^3q^2, -x^8q).$$

Using Lemma 3.2.1 with $N = 11$ and $(\alpha, \beta) = (-x, -x^{10}q^3), (-x^3q, -x^8q^2)$, and $(-x^3q^2, -x^8q)$, taking congruences modulo S_5, and using the fact that $f(-1, b) = 0$ for every complex number b with $|b| < 1$ [55, p. 34, Entry 18(iii)], we find that for every positive integer n,

$$\begin{aligned}\mathbf{G}(a^n) \equiv (1-a^n)P(15) &+ (a^{2n} - a^{10n})q^3P(12) + (a^{9n} - a^{3n})q^9P(9) \qquad (3.7.2)\\ &+ (a^{4n} - a^{8n})q^{18}P(6) + (a^{7n} - a^{5n})q^{30}P(3) \pmod{S_5}\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}(a^n) &\equiv (1-a^n)[P(16) - q^{22}P(5)] + q(a^{9n} - a^{3n})[P(14) - q^{11}P(8)] \\ &+ q^2(a^{4n} - a^{8n})[P(13) - q^{33}P(2)] + q^{15}(a^{10n} - a^{2n})[P(7) + q^{11}P(4)] \\ &+ q^7(a^{5n} - a^{7n})[P(10) + q^{33}P(1)] \pmod{S_5}, \qquad (3.7.3)\end{aligned}$$

where

$$P(k) := f(-q^{11k}, -q^{363-11k}). \qquad (3.7.4)$$

Furthermore, we obtain the following ten identities (3.7.5)–(3.7.14) from Winquist's identity (3.2.5) by replacing (α, β, q) by $(q^{55}, q^{22}, q^{121})$, $(q^{55}, q^{11}, q^{121})$, $(q^{55}, q^{33}, q^{121})$, $(q^{44}, q^{22}, q^{121})$, $(q^{44}, q^{11}, q^{121})$, $(q^{44}, q^{33}, q^{121})$, $(q^{55}, q^{44}, q^{121})$, $(q^{22}, q^{11}, q^{121})$, $(q^{33}, q^{22}, q^{121})$, and $(q^{33}, q^{11}, q^{121})$:

$$P(15)[P(16) - q^{22}P(5)] - q^{33}P(6)[P(7) + q^{11}P(4)] = \frac{ABCD}{(q^{121}; q^{121})_\infty^2}, \qquad (3.7.5)$$

$$P(15)[P(14) - q^{11}P(8)] - q^{44}P(3)[P(7) + q^{11}P(4)] = \frac{A^2BE}{(q^{121}; q^{121})_\infty^2}, \qquad (3.7.6)$$

$$P(15)[P(13) - q^{33}P(2)] - q^{22}P(9)[P(7) + q^{11}P(4)] = \frac{AC^2D}{(q^{121}; q^{121})_\infty^2}, \qquad (3.7.7)$$

$$P(12)[P(16) - q^{22}P(5)] - q^{22}P(6)[P(10) + q^{33}P(1)] = \frac{ABD^2}{(q^{121}; q^{121})_\infty^2}, \qquad (3.7.8)$$

$$P(12)[P(14)-q^{11}P(8)]-q^{33}P(3)[P(10)+q^{33}P(1)]=\frac{ABCE}{(q^{121};q^{121})_\infty^2}, \tag{3.7.9}$$

$$P(12)[P(13)-q^{33}P(2)]-q^{11}P(9)[P(10)+q^{33}P(1)]=\frac{B^2CE}{(q^{121};q^{121})_\infty^2}, \tag{3.7.10}$$

$$P(15)[P(10)+q^{33}P(1)]-q^{11}P(12)[P(7)+q^{11}P(4)]=\frac{ABDE}{(q^{121};q^{121})_\infty^2}, \tag{3.7.11}$$

$$P(6)[P(14)-q^{11}P(8)]-q^{11}P(3)[P(16)-q^{22}P(5)]=\frac{CDE^2}{(q^{121};q^{121})_\infty^2}, \tag{3.7.12}$$

$$P(9)[P(16)-q^{22}P(5)]-q^{11}P(6)[P(13)-q^{33}P(2)]=\frac{ACDE}{(q^{121};q^{121})_\infty^2}, \tag{3.7.13}$$

$$P(9)[P(14)-q^{11}P(8)]-q^{22}P(3)[P(13)-q^{33}P(2)]=\frac{BCDE}{(q^{121};q^{121})_\infty^2}. \tag{3.7.14}$$

We now begin our first proof of the 11-dissection of the generating function $F_a(q)$ for cranks.

First Proof of Entry 3.7.1. Beginning, as usual, with the generating function for $F_a(q)$ and rationalizing, we find that

$$\begin{aligned}
F_a(q) &\equiv \frac{(q;q)_\infty}{(aq;q)_\infty(a^{10}q;q)_\infty} \equiv \frac{(a^2q,a^3q,a^4q,a^5q,a^6q,a^7q,a^8q,a^9q,q,q;q)_\infty}{(q^{11};q^{11})_\infty} \\
&\equiv \frac{1}{(q^{11};q^{11})_\infty}\Bigg(\frac{1}{(1-a^2)(1-a^5)(1-a^7)(1-a^8)} \\
&\quad \times \left\{G(a^6)H(a^5)-a^8G(a^4)H(a^2)\right\}\Bigg) \\
&\equiv \frac{1}{(q^{11};q^{11})_\infty}\Big(P(15)[P(16)-q^{22}P(5)]-q^{33}P(6)[P(7)+q^{11}P(4)] \\
&\quad +\{A_1-1\}\,q\left(P(15)[P(14)-q^{11}P(8)]-q^{44}P(3)[P(7)+q^{11}P(4)]\right) \\
&\quad +A_2q^2\left(P(15)[P(13)-q^{33}P(2)]-q^{22}P(9)[P(7)+q^{11}P(4)]\right) \\
&\quad +\{A_3+1\}\,q^3\left(P(12)[P(16)-q^{22}P(5)]-q^{22}P(6)[P(10)+q^{33}P(1)]\right) \\
&\quad +\{A_2+A_4+1\}\,q^4\left(P(12)[P(14)-q^{11}P(8)]-q^{33}P(3)[P(10)+q^{33}P(1)]\right) \\
&\quad -\{A_2+A_4\}\,q^5\left(P(12)[P(13)-q^{33}P(2)]-q^{11}P(9)[P(10)+q^{33}P(1)]\right)
\end{aligned}$$

$$
\begin{aligned}
&+\{A_1+A_4\}\,q^7\left(P(15)[P(10)+q^{33}P(1)]-q^{11}P(12)[P(7)+q^{11}P(4)]\right)\\
&-\{A_2+A_5+1\}\,q^{19}\left(P(6)[P(14)-q^{11}P(8)]-q^{11}P(3)[P(16)-q^{22}P(5)]\right)\\
&-\{A_4+1\}\,q^{9}\left(P(9)[P(16)-q^{22}P(5)]-q^{11}P(6)[P(13)-q^{33}P(2)]\right)\\
&-A_3q^{10}\left(P(9)[P(14)-q^{11}P(8)]-q^{22}P(3)[P(13)-q^{33}P(2)]\right)\Big) \pmod{S_5},
\end{aligned}
$$

where in the last congruence, we applied (3.7.1) with $m=5$ and $n=2$, (3.7.2) with $n=4,6$, and (3.7.3) with $n=5,2$.

Applying (3.7.5)–(3.7.14) to each of the dissection factors, respectively, above, we complete the first proof of Entry 3.7.1. □

Second Proof of Entry 3.7.1. As in our second proofs of the 2-, 3-, 5-, and 7-dissections, we apply Entry 3.2.1 and divide the series into residue classes modulo 11. If we set

$$T_i := \sum_{\substack{k=1,m=0\\ m\equiv i \pmod{11}}}^{\infty} \rho_k q^{km},$$

we deduce that

$$
\begin{aligned}
\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} &= 1+\sum_{k=1,m=0}^{\infty}\rho_k q^{km}\left(A_m-A_{m+1}\right) \qquad (3.7.15)\\
&\equiv 1+(2-A_1)\{T_0-T_{10}\}+(A_1-A_2)\{T_1-T_9\}\\
&\quad+(A_2-A_3)\{T_2-T_8\}+(A_3-A_4)\{T_3-T_7\}\\
&\quad+(A_4-A_5)\{T_4-T_6\}\\
&\equiv \sum_{k=-\infty}^{\infty}\rho_k\frac{2+2q^k+2q^{2k}+2q^{3k}+3q^{4k}}{1+q^k+\cdots+q^{10k}}\\
&\quad+A_1\sum_{k=-\infty}^{\infty}\rho_k\frac{q^{4k}-1}{1+q^k+\cdots+q^{10k}}\\
&\quad+A_2\sum_{k=-\infty}^{\infty}\rho_k\frac{q^{4k}-q^k}{1+q^k+\cdots+q^{10k}}\\
&\quad+A_3\sum_{k=-\infty}^{\infty}\rho_k\frac{q^{4k}-q^{2k}}{1+q^k+\cdots+q^{10k}}\\
&\quad+A_4\sum_{k=-\infty}^{\infty}\rho_k\frac{q^{4k}-q^{3k}}{1+q^k+\cdots+q^{10k}} \pmod{S_5}.
\end{aligned}
$$

The second proof of Entry 3.7.1 now follows from the following identities. Indeed, if we substitute the identities of Theorem 3.7.1 into (3.7.15) and collect terms, we complete the second proof of Entry 3.7.1. □

Theorem 3.7.1. *We have*

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{11}{1+q^k+\cdots+q^{10k}} = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2} \times \Big\{ ABCD - 10qA^2BE + 2q^2AC^2D + 3q^3ABD^2 + 5q^4ABCE - 4q^5B^2CE - 7q^7ABDE - 5q^{19}CDE^2 - 3q^9ACDE - 2q^{10}BCDE \Big\}, \tag{3.7.16}$$

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{11q^k}{1+q^k+\cdots+q^{10k}} = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2} \times \Big\{ ABCD + qA^2BE - 9q^2AC^2D + 3q^3ABD^2 - 6q^4ABCE + 7q^5B^2CE + 4q^7ABDE + 6q^{19}CDE^2 - 3q^9ACDE - 2q^{10}BCDE \Big\}, \tag{3.7.17}$$

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{11q^{2k}}{1+q^k+\cdots+q^{10k}} = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2} \times \Big\{ ABCD + qA^2BE + 2q^2AC^2D - 8q^3ABD^2 + 5q^4ABCE - 4q^5B^2CE + 4q^7ABDE - 5q^{19}CDE^2 - 3q^9ACDE + 9q^{10}BCDE \Big\}, \tag{3.7.18}$$

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{11q^{3k}}{1+q^k+\cdots+q^{10k}} = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2} \times \Big\{ ABCD + qA^2BE + 2q^2AC^2D + 3q^3ABD^2 - 6q^4ABCE + 7q^5B^2CE - 7q^7ABDE - 5q^{19}CDE^2 + 8q^9ACDE - 2q^{10}BCDE \Big\}, \tag{3.7.19}$$

$$\sum_{k=-\infty}^{\infty} \rho_k \frac{11q^{4k}}{1+q^k+\cdots+q^{10k}} = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2} \times \Big\{ ABCD + qA^2BE + 2q^2AC^2D + 3q^3ABD^2 + 5q^4ABCE - 4q^5B^2CE + 4q^7ABDE + 6q^{19}CDE^2 - 3q^9ACDE - 2q^{10}BCDE \Big\}, \tag{3.7.20}$$

where $A = f(-q^{55}, -q^{66})$, $B = f(-q^{77}, -q^{44})$, $C = f(-q^{88}, -q^{33})$, $D = f(-q^{99}, -q^{22})$, *and* $E = f(-q^{110}, -q^{11})$.

We present the proof of only (3.7.16), since the proofs of the remaining four identities are similar.

Proof. We calculate the partial fraction decomposition

$$\begin{aligned}\frac{11}{1+q^k+\cdots+q^{10k}} &= \frac{1-\zeta^{10}}{1-\zeta q^k}+\frac{1-\zeta^{9}}{1-\zeta^2 q^k}+\frac{1-\zeta^{8}}{1-\zeta^3 q^k}+\frac{1-\zeta^{7}}{1-\zeta^4 q^k}+\frac{1-\zeta^{6}}{1-\zeta^5 q^k}\\ &\quad+\frac{1-\zeta^{5}}{1-\zeta^6 q^k}+\frac{1-\zeta^{4}}{1-\zeta^7 q^k}+\frac{1-\zeta^{3}}{1-\zeta^8 q^k}+\frac{1-\zeta^{2}}{1-\zeta^9 q^k}+\frac{1-\zeta}{1-\zeta^{10} q^k}. \qquad (3.7.21)\end{aligned}$$

From the identity (3.7.21) and Entry 3.2.1, we find that

$$\begin{aligned}I(q) &=: \sum_{k=-\infty}^{\infty}\rho_k\frac{11}{1+q^k+\cdots+q^{10k}} \qquad (3.7.22)\\ &= \frac{1-\zeta^{10}}{1-\zeta}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta}{1-\zeta q^k}+\frac{1-\zeta^{9}}{1-\zeta^2}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^2}{1-\zeta^2 q^k}\\ &\quad+\frac{1-\zeta^{8}}{1-\zeta^3}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^3}{1-\zeta^3 q^k}+\frac{1-\zeta^{7}}{1-\zeta^4}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^4}{1-\zeta^4 q^k}\\ &\quad+\frac{1-\zeta^{6}}{1-\zeta^5}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^5}{1-\zeta^5 q^k}+\frac{1-\zeta^{5}}{1-\zeta^6}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^6}{1-\zeta^6 q^k}\\ &\quad+\frac{1-\zeta^{4}}{1-\zeta^7}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^7}{1-\zeta^7 q^k}+\frac{1-\zeta^{3}}{1-\zeta^8}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^8}{1-\zeta^8 q^k}\\ &\quad+\frac{1-\zeta^{2}}{1-\zeta^9}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^9}{1-\zeta^9 q^k}+\frac{1-\zeta}{1-\zeta^{10}}\sum_{k=-\infty}^{\infty}\rho_k\frac{1-\zeta^{10}}{1-\zeta^{10} q^k}\\ &= \left\{\frac{1-\zeta^{10}}{1-\zeta}+\frac{1-\zeta}{1-\zeta^{10}}\right\}\frac{(q;q)_\infty^2}{(\zeta q;q)_\infty(\zeta^{10}q;q)_\infty}\\ &\quad+\left\{\frac{1-\zeta^{9}}{1-\zeta^2}+\frac{1-\zeta^2}{1-\zeta^{9}}\right\}\frac{(q;q)_\infty^2}{(\zeta^2 q;q)_\infty(\zeta^{9}q;q)_\infty}\\ &\quad+\left\{\frac{1-\zeta^{8}}{1-\zeta^3}+\frac{1-\zeta^3}{1-\zeta^{8}}\right\}\frac{(q;q)_\infty^2}{(\zeta^3 q;q)_\infty(\zeta^{8}q;q)_\infty}\\ &\quad+\left\{\frac{1-\zeta^{7}}{1-\zeta^4}+\frac{1-\zeta^4}{1-\zeta^{7}}\right\}\frac{(q;q)_\infty^2}{(\zeta^4 q;q)_\infty(\zeta^{7}q;q)_\infty}\\ &\quad+\left\{\frac{1-\zeta^{6}}{1-\zeta^5}+\frac{1-\zeta^5}{1-\zeta^{6}}\right\}\frac{(q;q)_\infty^2}{(\zeta^5 q;q)_\infty(\zeta^{6}q;q)_\infty}.\end{aligned}$$

Applying the elementary identity $1-\zeta^n=-\zeta^n(1-\zeta^{11-n})$ and rationalizing the denominator, we find that

$$\begin{aligned}I(q) = \frac{(q;q)_\infty^3}{(q^{11};q^{11})_\infty}\Big(&\left\{-\zeta-\zeta^{10}\right\}(\zeta^2q,\zeta^3q,\zeta^4q,\zeta^5q,\zeta^6q,\zeta^7q,\zeta^8q,\zeta^9q;q)_\infty\\ &+\left\{-\zeta^2-\zeta^{9}\right\}(\zeta q,\zeta^3q,\zeta^4q,\zeta^5q,\zeta^6q,\zeta^7q,\zeta^8q,\zeta^{10}q;q)_\infty\\ &+\left\{-\zeta^3-\zeta^{8}\right\}(\zeta q,\zeta^2q,\zeta^4q,\zeta^5q,\zeta^6q,\zeta^7q,\zeta^9q,\zeta^{10}q;q)_\infty\end{aligned}$$

$$+\left\{-\zeta^4-\zeta^7\right\}(\zeta q,\zeta^2 q,\zeta^3 q,\zeta^5 q,\zeta^6 q,\zeta^8 q,\zeta^9 q,\zeta^{10} q;q)_\infty$$
$$+\left\{-\zeta^5-\zeta^6\right\}(\zeta q,\zeta^2 q,\zeta^3 q,\zeta^4 q,\zeta^7 q,\zeta^8 q,\zeta^9 q,\zeta^{10} q;q)_\infty\Big). \tag{3.7.23}$$

Next, applying (3.7.1) with $(a,m,n)=(\zeta,5,2)$, $(\zeta,4,1)$, $(\zeta,5,4)$, $(\zeta,3,2)$, and $(\zeta,3,1)$, respectively, on each summand of (3.7.23) and simplifying, we find that

$$\begin{aligned}
I(q)=\frac{(q;q)_\infty}{(q^{11};q^{11})_\infty}\Big(&P(15)[P(16)-q^{22}P(5)]-q^{33}P(6)[P(7)+q^{11}P(4)]\\
&-10q\left\{P(15)[P(14)-q^{11}P(8)]-q^{44}P(3)[P(7)+q^{11}P(4)]\right\}\\
&+2q^2\left\{P(15)[P(13)-q^{33}P(2)]-q^{22}P(9)[P(7)+q^{11}P(4)]\right\}\\
&+3q^3\left\{P(12)[P(16)-q^{22}P(5)]-q^{22}P(6)[P(10)+q^{33}P(1)]\right\}\\
&+5q^4\left\{P(12)[P(14)-q^{11}P(8)]-q^{33}P(3)[P(10)+q^{33}P(1)]\right\}\\
&-4q^5\left\{P(12)[P(13)-q^{33}P(2)]-q^{11}P(9)[P(10)+q^{33}P(1)]\right\}\\
&-7q^7\left\{P(15)[P(10)+q^{33}P(1)]-q^{11}P(12)[P(7)+q^{11}P(4)]\right\}\\
&-5q^{19}\left\{P(6)[P(14)-q^{11}P(8)]-q^{11}P(3)[P(16)-q^{22}P(5)]\right\}\\
&-3q^9\left\{P(9)[P(16)-q^{22}P(5)]-q^{11}P(6)[P(13)-q^{33}P(2)]\right\}\\
&-2q^{10}\left\{P(9)[P(14)-q^{11}P(8)]-q^{22}P(3)[P(13)-q^{33}P(2)]\right\}\Big).
\end{aligned}$$

Finally, applying (3.7.5)–(3.7.14) to each of the dissection factors, respectively, we obtain the right-hand side of (3.7.16), which completes the proof of Theorem 3.7.1. □

If we let a be a primitive 11th root of unity in Entry 3.7.1, then we recover the identity discovered by Hirschhorn [173]. Hirschhorn's identity is a simplification of Garvan's identity given in [146, Theorem 6.7]. A proof of Hirschhorn's identity was given by Ekin [132, pp. 286–287]. The idea illustrated in our first proof here is similar to that of Ekin.

Entry 3.7.1 can also be proved using identities found in Ekin's paper [133, p. 2153, equations (5.13)–(5.17)]. Our approach to Entry 3.7.1 is different from that of Ekin.

3.8 Conclusion

In the beginning of this chapter, we mentioned that by substituting a by the corresponding primitive root of unity, we obtain Garvan's identities proved in [146] and [147]. Garvan informed us that the identities in [146] and [147] imply the congruences established in this paper. We briefly explain his observation here.

Suppose that for some function $G_a(q)$, we want to show that

$$F_a(q)\equiv G_a(q)\,(\operatorname{mod} S_{(p-1)/2}).$$

Let $H_a(q) = F_a(q) - G_a(q)$. Then

$$H_a(q) = \sum_{n=0}^{\infty} h(a,n)q^n,$$

where $h(a,n) \in \mathbf{Z}[a, 1/a]$. Let $h(a,n) = a^{-t(n)}\tilde{h}(a,n)$, where now, $\tilde{h}(a,n)$ is a polynomial in $\mathbf{Z}[a]$ and $t(n)$ is the largest integer k for which $1/a^k$ appears in $h(a,n)$. Garvan's identities show that $\tilde{h}(\zeta,n) = 0$ for all roots of the cyclotomic polynomial $\Phi_p(a)$. Since $\tilde{h}(a,n) \in \mathbf{Z}[a]$, this implies that $\Phi_p(a)$ divides $\tilde{h}(a,n)$. Therefore,

$$h(a,n) = a^{-t(n)}\Phi_p(a)Q(a,n) = a^{-t(n)+(p-1)/2}S_{(p-1)/2}(a)Q(a,n),$$

where $Q(a,n) \in \mathbf{Z}[a]$. This implies that

$$H_a(q) \equiv 0 \pmod{S_{(p-1)/2}}.$$

Garvan's observation allows us to deduce from [147, Equation (2.16)] and [146, Theorem 8.16], respectively, the 5-dissection of $F_a(q) \pmod{a^{-4}\Phi_{10}(a)}$ and the congruence

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{q^{m^2}}{(aq;q)_m(q/a;q)_m} &\equiv \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})} f^2(-q^{25}) + \left(a + \frac{1}{a} - 2\right)\phi(q^5) \\ &+ q\frac{f^2(-q^{25})}{f(-q^5,-q^{20})} + \left(a + \frac{1}{a}\right) q^2 \frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} \\ &- \left(a + \frac{1}{a}\right) q^3 \frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})} f^2(-q^{25}) - \left(2a + \frac{2}{a} + 1\right) \frac{\psi(q^5)}{q^2} \pmod{S_2},\end{aligned}$$

which is given in a slightly different form in Entry 2.1.2. A direct proof of the congruence above in the spirit of our second method illustrated in this chapter has not been found. As we remarked in Chapter 2, if we substitute $a = 1$ in the congruence above, we recover the Atkin and Swinnerton-Dyer congruences [28, Theorem 1]. This also provides an explanation to the "curious fact" raised by Garvan [146, second paragraph, p. 52].

4

Ranks and Cranks, Part III

4.1 Introduction

In the introduction to this book, we conjectured that Ramanujan had focused his attention on cranks in the days before he died. Much of the material on cranks in the lost notebook is rough, preliminary, and devoted to extensive calculations. In this chapter, we discuss the remaining entries on cranks, many of which are not in polished form, that we did not examine in Chapters 2 and 3. We begin by proving Ramanujan's forerunner of Entry 3.2.1, which was a key result in our proofs of crank dissections in Chapter 3.

4.2 Key Formulas on Page 59

On page 59, Ramanujan offers the quotient (with one misprint corrected)

$$\begin{aligned}\Big(1 &+ q(a_1-2) + q^2(a_2-a_1) + q^3(a_3-a_2) + q^4(a_4-a_3) + \cdots \\ &- \big(q^3(a_1-2) + q^5(a_2-a_1) + q^7(a_3-a_2) + q^9(a_4-a_3) + \cdots\big) \\ &+ \big(q^6(a_1-2) + q^9(a_2-a_1) + q^{12}(a_3-a_2) + q^{15}(a_4-a_3) + \cdots\big) \\ &- \big(q^{10}(a_1-2) + q^{14}(a_2-a_1) + q^{18}(a_3-a_2) + q^{22}(a_4-a_3) + \cdots\big) \\ &+ \big(q^{15}(a_1-2) + q^{20}(a_2-a_1) + q^{25}(a_3-a_2) + \cdots\big) \\ &- \big(q^{21}(a_1-2) + \cdots\big)\Big)\Big/ \\ &\quad (1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + \cdots). \qquad (4.2.1)\end{aligned}$$

In more succinct notation, (4.2.1) can be rewritten as

$$\frac{1 - \displaystyle\sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn}(a_{n+1}-a_n)}{(q;q)_\infty}, \qquad (4.2.2)$$

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where

$$a_n := a^n + a^{-n}, \qquad n \geq 0. \tag{4.2.3}$$

Scribbled underneath (4.2.1) are the first few terms of (4.4.1) through q^5. Thus, although not claimed by Ramanujan, (4.2.1) is, in fact, equal to $F_a(q)$. We state this in the next theorem.

Entry 4.2.1 (p. 59). *If a_n is given by (4.2.3), and if $|q| < \min(|a|, 1/|a|)$, then*

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = 1 - \sum_{m=1,n=0}^{\infty}(-1)^m q^{m(m+1)/2+mn}(a_{n+1} - a_n). \tag{4.2.4}$$

It is easily seen that Ramanujan's Entry 4.2.1 is equivalent to Entry 3.2.1, which we repeat here and whose interesting history we detailed in Chapter 3.

Entry 4.2.2. *Let*

$$\rho_k = (-1)^k q^{k(k+1)/2}. \tag{4.2.5}$$

Then

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{\rho_k(1-a)}{1-aq^k}. \tag{4.2.6}$$

On page 59, below a list of factors and above the aforementioned quotient of two series, Ramanujan records two further series, namely,

$$S_1(a,q) := \frac{1}{1+a} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} + \frac{(-1)^n q^{n(n+1)/2}}{a+q^n}\right) \tag{4.2.7}$$

and

$$S_2(a,q) := 1 + \sum_{m=1,n=0}^{\infty}(-1)^{m+n} q^{m(m+1)/2+mn}(a_{n+1} + a_n), \tag{4.2.8}$$

where here $a_0 := 1$. No theorem is claimed by Ramanujan, but the following theorem, to be proved in the next section, holds.

Entry 4.2.3 (p. 59). *With $S_1(a,q)$ and $S_2(a,q)$ defined by (4.2.7) and (4.2.8), respectively,*

$$(1+a)S_1(a,q) = S_2(a,q) = F_{-a}(q).$$

4.3 Proofs of Entries 4.2.1 and 4.2.3

Proof of Entry 4.2.1. Our proof is different from that of Evans [136], Kač and Peterson [187], and Kač and Wakimoto [188]. We employ the partial fraction decomposition

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = 1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}(1+q^n) \times\left\{1-\frac{1-q^n}{1+q^n}\sum_{m=0}^{\infty}a^m q^{mn}-\frac{1-q^n}{1+q^n}\sum_{m=1}^{\infty}a^{-m}q^{mn}\right\}, \tag{4.3.1}$$

found in Garvan's paper [146, Equation (7.16)]. From (4.3.1), we find that

$$\begin{aligned}
&\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty}\\
&=1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\Bigg((1+q^n)\\
&\qquad -(1-q^n)\sum_{m=0}^{\infty}a^m q^{mn}-(1-q^n)\sum_{m=1}^{\infty}a^{-m}q^{mn}\Bigg)\\
&=1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left(2-(1-q^n)\sum_{m=0}^{\infty}q^{mn}(a^m+a^{-m})\right)\\
&=1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left(2-\sum_{m=0}^{\infty}q^{mn}a_m+\sum_{m=0}^{\infty}q^{(m+1)n}a_m\right)\\
&=1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left(-\sum_{m=1}^{\infty}q^{mn}a_m+\sum_{m=1}^{\infty}q^{mn}a_{m-1}\right)\\
&=1+\sum_{m,n=1}^{\infty}(-1)^n q^{n(n-1)/2+mn}\left(a_{m-1}-a_m\right)\\
&=1-\sum_{m=0,n=1}^{\infty}(-1)^n q^{n(n+1)/2+mn}\left(a_{m+1}-a_m\right),
\end{aligned}$$

which is (4.2.4), but with the roles of m and n reversed. □

Proof of Entry 4.2.3. Multiply (4.2.7) throughout by $(1+a)$ to deduce that

$$\begin{aligned}
(1+a)S_1(a,q)&=1+(1+a)\sum_{n=1}^{\infty}\left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n}+\frac{(-1)^n q^{n(n+1)/2}}{a+q^n}\right)\\
&=1+(1+a)\sum_{n=1}^{\infty}\left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n}+\frac{(-1)^{-n} q^{n(n-1)/2}}{1+aq^{-n}}\right)\\
&=1+(1+a)\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n}\\
&=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(1+a)}{1+aq^n}
\end{aligned}$$

$$= \frac{(q;q)_\infty^2}{(-aq;q)_\infty(-q/a;q)_\infty}, \tag{4.3.2}$$

by an application of (4.2.6).

Secondly,

$$\begin{aligned} S_2(a,q) &= 1 + \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn} \\ &\qquad \times \Big(-(-a)^{n+1} - (-a)^{-n-1} + (-a)^n + (-a)^{-n} \Big) \\ &= \frac{(q;q)_\infty^2}{(-aq;q)_\infty(-q/a;q)_\infty}, \end{aligned} \tag{4.3.3}$$

by Entry 4.2.1. Thus, (4.3.2) and (4.3.3) yield Entry 4.2.3. □

4.4 Further Entries on Pages 58 and 59

On page 58 in his lost notebook [283], Ramanujan recorded the following power series:

$$\begin{aligned} &1 + q(a_1-1) + q^2a_2 + q^3(a_3+1) + q^4(a_4+a_2+1) \\ &+ q^5(a_5+a_3+a_1+1) + q^6(a_6+a_4+a_3+a_2+a_1+1) \\ &+ q^7(a_3+1)(a_4+a_2+1) + q^8a_2(a_6+a_4+a_3+a_2+a_1+1) \\ &+ q^9a_2(a_3+1)(a_4+a_2+1) + q^{10}a_2(a_3+1)(a_5+a_3+a_1+1) \\ &+ q^{11}a_1a_2(a_8+a_5+a_4+a_3+a_2+a_1+2) \\ &+ q^{12}(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\ &\quad \times (a_4-2a_3+2a_2-a_1+1) \\ &+ q^{13}(a_1-1)(a_2-a_1+1) \\ &\quad \times (a_{10}+2a_9+2a_8+2a_7+2a_6+4a_5+6a_4+8a_3+9a_2+9a_1+9) \\ &+ q^{14}(a_2+1)(a_3+1)(a_4+a_2+1)(a_5-a_3+a_1+1) \\ &+ q^{15}a_1a_2(a_5+a_4+a_3+a_2+a_1+1)(a_7-a_6+a_4+a_1) \\ &+ q^{16}(a_3+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\ &\quad \times (a_5-2a_4+2a_3-2a_2+3a_1-3) \\ &+ q^{17}(a_2+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1)(a_7-a_6+a_3+a_1-1) \\ &+ q^{18}(a_4+a_2+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\ &\quad \times (a_6-2a_5+a_4+a_3-a_2+1) \\ &+ q^{19}a_2(a_1-1)(a_4+a_2+1)(a_3+a_2+a_1+1) \\ &\quad \times (a_9-a_7+a_4+2a_3+a_2-1) \\ &+ q^{20}(a_2-a_1+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1) \end{aligned}$$

$$
\begin{aligned}
&\quad \times (a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3) \\
&+ q^{21} a_1 a_2 (a_3 + 1)(a_2 - a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\quad \times (a_8 - a_6 + a_4 + a_1 + 2) \\
&+ \cdots . \qquad (4.4.1)
\end{aligned}
$$

Recall that a_n is defined in (4.2.3). Thus, Ramanujan wrote out the first 21 coefficients in the power series representation of the crank generating function $F_a(q)$ given in (3.1.1). (We have corrected a misprint in the coefficient of q^{21}.)

On the following page, beginning with the coefficient of q^{13}, Ramanujan listed some (but not necessarily all) of the factors of the coefficients up to q^{26}. The factors he recorded are

$$
\begin{aligned}
&13. \quad (a_1 - 1)(a_2 - a_1 + 1) \qquad (4.4.2)\\
&14. \quad (a_2 + 1)(a_3 + 1)(a_4 + a_2 + 1)\\
&15. \quad a_1 a_2 (a_5 + a_4 + a_3 + a_2 + a_1 + 1)\\
&16. \quad (a_3 + 1)(a_3 + a_2 + a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)\\
&17. \quad (a_2 + 1)(a_3 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)\\
&18. \quad (a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)\\
&19. \quad a_2(a_1 - 1)(a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1)\\
&20. \quad (a_3 + 1)(a_2 - a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)\\
&21. \quad a_1 a_2 (a_3 + 1)(a_2 - a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)\\
&22. \quad a_2(a_3 + 1)(a_1 - 1)\\
&23. \quad (a_1 - 1)(a_4 + a_2 + 1)\\
&24. \quad (a_3 + 1)(a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1)\\
&25. \quad a_2(a_1 - 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)\\
&26. \quad a_2(a_3 + 1)(a_3 + a_2 + a_1 + 1).
\end{aligned}
$$

Ramanujan did not indicate why he recorded only these factors. However, it can be noted that in each case he recorded linear factors only when the leading index is ≤ 5. To the left of each $n, 15 \leq n \leq 26$, are the unexplained numbers 16×16, undecipherable, $27 \times 27, -25, 49, -7 \cdot 19, 9, -7, -9, -11 \cdot 15, -11$, and -4, respectively.

4.5 Congruences for the Coefficients λ_n on Pages 179 and 180

On pages 179 and 180 in his lost notebook [283], Ramanujan offers ten tables of indices of coefficients λ_n satisfying certain congruences. On page 61 in [283], he offers rougher drafts of nine of the ten tables; Table 6 is missing on page 61. In contrast to the tables on pages 179 and 180, no explanations are given

for the tables on page 61. Clearly, Ramanujan calculated factors well beyond those recorded on pages 58 and 59 given in Section 4.4. To verify Ramanujan's claims, we calculated λ_n up to $n = 500$ with the use of Maple V. Ramanujan evidently thought that each table was complete in that there are no further values of n for which the prescribed divisibility property holds.

Table 1. $\lambda_n \equiv 0 \pmod{a^2 + a^{-2}}$

Thus, Ramanujan indicates which coefficients λ_n have a_2 as a factor. The 47 values of n with a_2 as a factor of λ_n are

$$2, 8, 9, 10, 11, 15, 19, 21, 22, 25, 26, 27, 28, 30, 31, 34, 40, 42, 45,$$
$$46, 47, 50, 55, 57, 58, 59, 62, 66, 70, 74, 75, 78, 79, 86, 94, 98,$$
$$106, 110, 122, 126, 130, 142, 154, 158, 170, 174, 206.$$

Replacing q by q^2 in (3.1.2), we see that Table 1 contains the degree of q for those terms with zero coefficients for both

$$\frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty} \qquad \text{and} \qquad q\frac{f(-q^2, -q^{14})}{(-q^4; q^4)_\infty}. \tag{4.5.1}$$

Table 2. $\lambda_n \equiv 1 \pmod{a^2 + a^{-2}}$

To interpret this table properly, we return to the congruence given in (3.1.2). Replacing q by q^2, we see that Ramanujan has recorded all the degrees of q of the terms (except for the constant term) with coefficients equal to 1 in the power series expansion of

$$\frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty}. \tag{4.5.2}$$

The 27 values of n given by Ramanujan are

$$14, 16, 18, 24, 32, 48, 56, 72, 82, 88, 90, 104, 114, 138, 146,$$
$$162, 178, 186, 194, 202, 210, 218, 226, 234, 242, 250, 266.$$

Table 3. $\lambda_n \equiv -1 \pmod{a^2 + a^{-2}}$

This table is to be understood in the same way as the previous table, except that now Ramanujan is recording the indices of those terms with coefficients equal to -1 in the power series expansion of (4.5.2). Here Ramanujan missed one value, namely, $n = 214$. The 27 (not 26) values of n are then given by

$$4, 6, 12, 20, 36, 38, 44, 52, 54, 60, 68, 76, 92, 102, 118,$$
$$134, 150, 166, 182, 190, 214, 222, 238, 254, 270, 286, 302.$$

Table 4. $\lambda_n \equiv a - 1 + \dfrac{1}{a} \pmod{a^2 + a^{-2}}$

We again return to the congruence given in (3.1.2). Note that $a - 1 + 1/a$ occurs as a factor of the second expression on the right side. Thus, replacing q by q^2, Ramanujan records the indices of all terms of

$$q\frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty} \tag{4.5.3}$$

with coefficients that are equal to 1. The 22 values of n that give the coefficient 1 are equal to

$$1, 7, 17, 23, 33, 39, 41, 49, 63, 71, 73, 81,$$
$$87, 89, 95, 105, 111, 119, 121, 127, 143, 159.$$

Table 5. $\lambda_n \equiv -(a-1+a^{-1}) \pmod{a^2+a^{-2}}$

The interpretation of this table is analogous to the preceding one. Now Ramanujan determines those coefficients in the expansion of (4.5.3) that are equal to -1. His table of 23 values of n includes

$$3, 5, 13, 29, 35, 37, 43, 51, 53, 61, 67, 69, 77,$$
$$83, 85, 91, 93, 99, 107, 115, 123, 139, 155.$$

Table 6. $\lambda_n \equiv 0 \pmod{a+a^{-1}}$

Ramanujan thus gives here those coefficients that have a_1 as a factor. There are only three values, namely, when n equals

$$11, 15, 21.$$

These three values can be discerned from the table on page 59 of the lost notebook.

From the calculation

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q;q)_\infty}{(-q^2;q^2)_\infty} = \frac{f(-q)f(-q^2)}{f(-q^4)} \pmod{a+a^{-1}},$$

where $f(-q)$ is defined by (3.2.3), we see that in Table 6 Ramanujan recorded the degree of q for the terms with zero coefficients in the power series expansion of

$$\frac{f(-q)f(-q^2)}{f(-q^4)}. \tag{4.5.4}$$

For the next three tables, it is clear from the calculation

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q^2;q^2)_\infty}{(-q^3;q^3)_\infty} = \frac{f(-q^2)f(-q^3)}{f(-q^6)} \pmod{a-1+a^{-1}}$$

that Ramanujan recorded the degree of q for the terms with coefficients 0, 1, and -1, respectively, in the power series expansion of

$$\frac{f(-q^2)f(-q^3)}{f(-q^6)}. \tag{4.5.5}$$

Table 7. $\lambda_n \equiv 0 \pmod{a-1+a^{-1}}$

The 19 values satisfying the congruence above are, according to Ramanujan,

$$1, 6, 8, 13, 14, 17, 19, 22, 23, 25,$$
$$33, 34, 37, 44, 46, 55, 58, 61, 82.$$

Table 8. $\lambda_n \equiv 1 \pmod{a - 1 + a^{-1}}$

The 26 values of n found by Ramanujan are

$$5, 7, 10, 11, 12, 18, 24, 29, 30, 31, 35, 41, 42, 43,$$
$$47, 49, 53, 54, 59, 67, 71, 73, 85, 91, 97, 109.$$

As in Table 2, Ramanujan ignored the value $n = 0$.

Table 9. $\lambda_n \equiv -1 \pmod{a - 1 + a^{-1}}$

The 26 values of n found by Ramanujan are

$$2, 3, 4, 9, 15, 16, 20, 21, 26, 27, 28, 32, 38, 39,$$
$$40, 52, 56, 62, 64, 68, 70, 76, 94, 106, 118, 130.$$

Table 10. $\lambda_n \equiv 0 \pmod{a + 1 + a^{-1}}$

Ramanujan has but two values of n such that λ_n satisfies the congruence above, and they are when n equals

$$14, 17.$$

From the calculation

$$\frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} \equiv \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty} = \frac{f^2(-q)}{f(-q^3)} \pmod{a + 1 + a^{-1}},$$

it is clear that Ramanujan recorded the degree of q for the terms with zero coefficients in the power series expansion of

$$\frac{f^2(-q)}{f(-q^3)}. \tag{4.5.6}$$

The infinite products in (4.5.2)–(4.5.6) do not appear to have monotonic coefficients for sufficiently large n. However, if these infinite products are dissected properly, then we conjecture that the coefficients in the dissections are indeed monotonic. Hence, for (4.5.2), (4.5.3), (4.5.4), (4.5.5), and (4.5.6), we must study, respectively, the dissections of

$$\frac{f(-q^6, -q^{10})}{(-q^4;q^4)_\infty}, \qquad \frac{f(-q^2, -q^{14})}{(-q^4;q^4)_\infty},$$
$$\frac{f(-q)f(-q^2)}{f(-q^4)}, \qquad \frac{f(-q^2)f(-q^3)}{f(-q^6)}, \qquad \frac{f^2(-q)}{f(-q^3)}.$$

For each of the five products given above, we have determined certain dissections.

We require the addition formula for theta functions given in Lemma 3.2.1 to prove the desired dissections.

Setting $(\alpha, \beta, N) = (-q^6, -q^{10}, 4)$ and $(-q^4, -q^{12}, 2)$ in (3.2.2), we obtain, respectively,

$$f(-q^6, -q^{10}) = A - q^6 B - q^{10} C + q^{28} D, \tag{4.5.7}$$

$$f(-q^4, -q^{12}) = f(q^{24}, q^{40}) - q^4 f(q^8, q^{56}), \tag{4.5.8}$$

where $A := f(q^{120}, q^{136})$, $B := f(q^{72}, q^{184})$, $C := f(q^{56}, q^{200})$, and $D := f(q^8, q^{248})$.

Setting $(\alpha, \beta, N) = (-q, -q^2, 3)$ in (3.2.2), we obtain

$$f(-q) = f(-q^{12}, -q^{15}) - q f(-q^6, -q^{21}) - q^2 f(-q^3, -q^{24}). \tag{4.5.9}$$

For (4.5.2), the 8-dissection (with, of course, the odd powers missing) is given by

$$\begin{aligned}
\frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty} &= \frac{f(-q^6, -q^{10}) f(-q^4, -q^{12})}{f(-q^{16})} \\
&= \frac{1}{f(-q^{16})} \left\{A - q^6 B - q^{10} C + q^{28} D\right\} \\
&\quad \times \left\{f(q^{24}, q^{40}) - q^4 f(q^8, q^{56})\right\} \\
&= \frac{1}{f(-q^{16})} \Big\{ A f(q^{24}, q^{40}) - q^{32} D f(q^8, q^{56}) \\
&\quad + q^2 \left[q^8 B f(q^8, q^{56}) - q^8 C f(q^{24}, q^{40})\right] \\
&\quad + q^4 \left[-A f(q^{24}, q^{40}) + q^{24} D f(q^8, q^{56})\right] \\
&\quad + q^6 \left[-B f(q^{24}, q^{40}) + q^8 C f(q^8, q^{56})\right] \Big\},
\end{aligned}$$

where we have applied (4.5.7) and (4.5.8) in the penultimate equality.

For (4.5.6), we have the 3-dissection

$$\begin{aligned}
\frac{f^2(-q)}{f(-q^3)} &= \frac{1}{(q^3; q^3)_\infty} \left\{ f(-q^{12}, -q^{15}) - q f(-q^6, -q^{21}) - q^2 f(-q^3, -q^{24}) \right\}^2 \\
&= \frac{1}{(q^3; q^3)_\infty} \Big\{ f^2(-q^{12}, -q^{15}) + 2q^3 f(-q^6, -q^{21}) f(-q^3, -q^{24}) \\
&\quad - q \left[2 f(-q^{12}, -q^{15}) f(-q^6, -q^{21}) - q^3 f^2(-q^3, -q^{24})\right] \\
&\quad + q^2 \left[f^2(-q^6, -q^{21}) - 2 f(-q^{12}, -q^{15}) f(-q^3, -q^{24})\right] \Big\},
\end{aligned}$$

where we have applied (4.5.9) in the first equality. For (4.5.3), (4.5.4), and (4.5.5), we have derived an 8-dissection, a 4-dissection, and a 6-dissection, respectively. Furthermore, we make the following conjecture.

Conjecture 4.5.1. Each component of each of the dissections for the five products given above has monotonic coefficients for powers of q above 1400.

We have checked the coefficients for each of the five products up to $n = 2000$. For each product, we give below the values of n after which their dissections appear to be monotonic and strictly monotonic, respectively.

(4.5.2)	1262	1374
(4.5.3)	719	759
(4.5.4)	149	169
(4.5.5)	550	580
(4.5.6)	95	95

Our conjectures on the dissections of (4.5.4), (4.5.5), and (4.5.6) have motivated the following stronger conjecture.

Conjecture 4.5.2. For any positive integers α and β, each component of the $(\alpha+\beta+1)$-dissection of the product

$$\frac{f(-q^{\alpha})f(-q^{\beta})}{f(-q^{\alpha+\beta+1})}$$

has monotonic coefficients for sufficiently large powers of q.

We remark that our conjectures for (4.5.4), (4.5.5), and (4.5.6) are then the special cases of Conjecture 4.5.2 when we set $(\alpha,\beta) = (1,2)$, $(2,3)$, and $(1,1)$, respectively.

Setting $(\alpha,\beta,N) = (-q^6,-q^{10},2)$ and $(-q^2,-q^{14},2)$ in (3.2.2), we obtain, respectively,

$$f(-q^6,-q^{10}) = f(q^{28},q^{36}) - q^6 f(q^4,q^{60}) \tag{4.5.10}$$

and

$$f(-q^2,-q^{14}) = f(q^{20},q^{44}) - q^2 f(q^{12},q^{52}). \tag{4.5.11}$$

After reading the conjectures for (4.5.2) and (4.5.3), Garvan made the following stronger conjecture.

Conjecture 4.5.3. Define b_n by

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^n &= \frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty} + q\frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty}\\ &= \frac{f(q^{28},q^{36})}{(-q^4;q^4)_\infty} + q\frac{f(q^{20},q^{44})}{(-q^4;q^4)_\infty} - q^6\frac{f(q^4,q^{60})}{(-q^4;q^4)_\infty}\\ &\quad - q^3\frac{f(q^{12},q^{52})}{(-q^4;q^4)_\infty},\end{aligned}$$

where we have applied (4.5.10) and (4.5.11) in the last equality. Then

$$
\begin{aligned}
(-1)^n b_{4n} &\geq 0, && \text{for all } n \geq 0,\\
(-1)^n b_{4n+1} &\geq 0, && \text{for all } n \geq 0,\\
(-1)^n b_{4n+2} &\geq 0, && \text{for all } n \geq 0, n \neq 3,\\
(-1)^{n+1} b_{4n+3} &\geq 0, && \text{for all } n \geq 0.
\end{aligned}
$$

Furthermore, each of these subsequences is eventually monotonic.

It is clear that the monotonicity of the subsequences in Conjecture 4.5.3 implies the monotonicity of the dissections of (4.5.2) and (4.5.3) as stated in Conjecture 4.5.1.

In [19], Andrews and R. Lewis made three conjectures on inequalities between the rank counts $N(m,t,n)$ and between the crank counts $M(m,t,n)$. Two of them, [19, Conjecture 2 and Conjecture 3] directly imply that Tables 10 and 6, respectively, are complete. Using the circle method, D.M. Kane [189] proved the former conjecture. More precisely, it follows immediately from [189, Corollary 2] that Table 10 is complete.

In his excellent, carefully prepared paper [102], O.-Y. Chan used the circle method and a careful error analysis to prove all of the conjectures of Berndt, H.H. Chan, S.H. Chan, and W.-C. Liaw [63], the conjecture of Garvan, and five conjectures of Andrews and Lewis [19] on ranks and cranks. We present his main theorem.

Theorem 4.5.1. *Let*

$$a_1(n), a_2(n), a_3(n), a_4(n)$$

be the coefficients of q^n in the expansions of

$$\frac{f(-q)f(-q^2)}{f(-q^4)}, \frac{f(-q^2)f(-q^3)}{f(-q^6)}, \frac{f(-q^6,-q^{10})}{(-q^4,q^4)_\infty}, q\frac{f(-q^2,-q^{14})}{(-q^4,q^4)_\infty}, \tag{4.5.12}$$

respectively. Then,

$$
\begin{aligned}
a_1(n) &= c_1(n)\frac{e^{\frac{\pi}{4}\sqrt{2(n-1/24)/3}}}{2\sqrt{2(n-1/24)}} + E_1(n),\\
a_2(n) &= c_2(n)\frac{e^{\frac{\pi}{6}\sqrt{2(n-1/24)/3}}}{2\sqrt{3(n-1/24)}} + E_2(n),\\
a_3(2n) &= (-1)^n c_3(2n)\frac{e^{\frac{\pi}{8}\sqrt{2(2n-1/24)/3}}}{4\sqrt{2n-1/24}} + E_3(2n),\\
a_4(2n+1) &= (-1)^{n+1} c_3(2n+1)\frac{e^{\frac{\pi}{8}\sqrt{2(2n+1-1/24)/3}}}{4\sqrt{2n+1-1/24}} + E_3(2n+1),\\
a_3(2n+1) &= a_4(2n) = 0,
\end{aligned}
$$

where $c_1(n)$, $c_2(n)$, and $c_3(n)$ are approximately

$$c_1(n) = \begin{cases} (-1)^{n/2}(1.847759\ldots), & \textit{if } n \equiv 0 \pmod 2, \\ (-1)^{(n+1)/2}(0.765366\ldots), & \textit{if } n \equiv 1 \pmod 2, \end{cases}$$

$$c_2(n) = \begin{cases} (-1)^{n/3}(1.9696155\ldots), & \textit{if } n \equiv 0 \pmod 3, \\ (-1)^{(n-1)/3}(0.6840402\ldots), & \textit{if } n \equiv 1 \pmod 3, \\ (-1)^{(n-2)/3}(-1.2855752\ldots), & \textit{if } n \equiv 2 \pmod 3, \end{cases}$$

$$c_3(n) = \begin{cases} (-1)^{n/4}(2.77407969\ldots), & \textit{if } n \equiv 0 \pmod 4, \\ (-1)^{(n-1)/4}(2.3517512\ldots), & \textit{if } n \equiv 1 \pmod 4, \\ (-1)^{(n-2)/4}(-0.5517987\ldots), & \textit{if } n \equiv 2 \pmod 4, \\ (-1)^{(n-3)/4}(1.5713899\ldots), & \textit{if } n \equiv 3 \pmod 4, \end{cases}$$

and

$$|E_1(n)| \leq 150(n-1/24)^{1/4} + 0.71(n-1/24)^{1/4} e^{\frac{\pi}{8}\sqrt{2(n-1/24)/3}},$$
$$|E_2(n)| \leq 1879(n-1/24)^{1/4} + 0.468(n-1/24)^{1/4} e^{\frac{\pi}{12}\sqrt{2(n-1/24)/3}},$$
$$|E_3(n)| \leq 68793(n-1/24)^{1/4} + 0.1754(n-1/24)^{1/4} e^{\frac{\pi}{16}\sqrt{2(n-1/24)/3}}.$$

In each case, O.-Y. Chan determined where the coefficients of the theta quotients become strictly monotonic. Then he checked Ramanujan's tables up to the values of n where the coefficients become strictly monotonic. In particular, monotonicity ensures that the coefficients are bounded away from $0, 1$, and -1. He verified all of Ramanujan's tables in PARI-GP 2.0.20(beta) and found them to be complete. In summary, Chan showed that Conjectures 4.5.1, 4.5.2, and 4.5.3 are all correct.

4.6 Page 181: Partitions and Factorizations of Crank Coefficients

On page 181 in his lost notebook [283], Ramanujan returns to the coefficients λ_n in the generating function (3.1.1) of the crank. He factors λ_n, $1 \leq n \leq 21$, as before, but singles out nine particular factors by giving them special notation. The criterion that Ramanujan apparently uses is that of multiple occurrence, i.e., each of these nine factors appears more than once in the 21 factorizations, while other factors not favorably designated appear only once. Ramanujan uses these factorizations to compute $p(n)$, which, of course, arises from the special case $a = 1$ in (3.1.1), i.e.,

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n, \qquad |q| < 1.$$

Ramanujan evidently was searching for some general principles or theorems on the factorization of λ_n so that he could not only compute $p(n)$ but say

something about the divisibility of $p(n)$. No theorems are stated by Ramanujan. Is it possible to determine that certain factors appear in some precisely described infinite family of values of λ_n? It would be interesting to speculate on the motivations that led Ramanujan to make these factorizations.

The factors designated by Ramanujan are

$$\begin{aligned}
\rho_1 &= a_1 - 1, \\
\rho &= a_2 - a_1 + 1, \\
\rho_2 &= a_2, \\
\rho_3 &= a_3 + 1, \\
\rho_4 &= a_1 a_2, \\
\rho_5 &= a_4 + a_2 + 1, \\
\rho_7 &= a_3 + a_2 + a_1 + 1, \\
\rho_9 &= (a_2 + 1)(a_3 + 1), \\
\rho_{11} &= a_5 + a_4 + a_3 + a_2 + a_1 + 1.
\end{aligned}$$

At first glance, there does not appear to be any reasoning behind the choice of subscripts; note that there is no subscript for the second value. However, observe that in each case, the subscript

> n equals (as a sum of powers of a) the number of terms with positive coefficients minus the number of terms with negative coefficients in the representation of ρ_n, when all expressions are expanded out, or if $\rho_n = \rho_n(a)$, we see that $\rho_n(1) = n$.

The reason ρ does not have a subscript is that the value of n in this case would be $3 - 2 = 1$, which has been reserved for the first factor. These factors then lead to rapid calculations of values for $p(n)$. For example, since $\lambda_{10} = \rho\rho_2\rho_3\rho_7$, then

$$p(10) = 1 \cdot 2 \cdot 3 \cdot 7 = 42.$$

In the table below, we provide the content of this page:

$$\begin{aligned}
&p(1) = 1, \quad \lambda_1 = \rho_1, \\
&p(2) = 2, \quad \lambda_2 = \rho_2, \\
&p(3) = 3, \quad \lambda_3 = \rho_3, \\
&p(4) = 5, \quad \lambda_4 = \rho_5, \\
&p(5) = 7, \quad \lambda_5 = \rho_7\rho, \\
&p(6) = 11, \quad \lambda_6 = \rho_1\rho_{11}, \\
&p(7) = 15, \quad \lambda_7 = \rho_3\rho_5, \\
&p(8) = 22, \quad \lambda_8 = \rho_1\rho_2\rho_{11}, \\
&p(9) = 30, \quad \lambda_9 = \rho_2\rho_3\rho_5,
\end{aligned}$$

$$
\begin{aligned}
&p(10) = 42, \quad \lambda_{10} = \rho\rho_2\rho_3\rho_7,\\
&p(11) = 56, \quad \lambda_{11} = \rho_4\rho_7(a_5 - a_4 + a_2),\\
&p(12) = 77, \quad \lambda_{12} = \rho_7\rho_{11}(a_4 - 2a_3 + 2a_2 - a_1 + 1),\\
&p(13) = 101, \quad \lambda_{13} = \rho\rho_1\left(a_{10} + 2a_9 + 2a_8 + 2a_7 + 3a_6\right.\\
&\qquad\qquad\qquad\left. +4a_5 + 6a_4 + 8a_3 + 9a_2 + 9a_1 + 9\right),\\
&p(14) = 135, \quad \lambda_{14} = \rho_5\rho_9(a_5 - a_3 + a_1 + 1),\\
&p(15) = 176, \quad \lambda_{15} = \rho_4\rho_{11}(a_7 - a_6 + a_4 + a_1),\\
&p(16) = 231, \quad \lambda_{16} = \rho_3\rho_7\rho_{11}(a_5 - 2a_4 + 2a_3 - 2a_2 + 3a_1 - 3),\\
&p(17) = 297, \quad \lambda_{17} = \rho_9\rho_{11}(a_7 - a_6 + a_3 + a_1 - 1),\\
&p(18) = 385, \quad \lambda_{18} = \rho_5\rho_7\rho_{11}(a_6 - 2a_5 + a_4 + a_3 - a_2 + 1),\\
&p(19) = 490, \quad \lambda_{19} = \rho_1\rho_2\rho_5\rho_7(a_9 - a_7 + a_4 + 2a_3 + a_2 - 1),\\
&p(20) = 627, \quad \lambda_{20} = \rho\rho_3\rho_{11}(a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3),\\
&p(21) = 792, \quad \lambda_{21} = \rho\rho_3\rho_4\rho_{11}(a_8 - a_6 + a_4 + a_1 + 2).
\end{aligned}
$$

4.7 Series on Pages 63 and 64 Related to Cranks

On pages 63 and 64 of his lost notebook [283], Ramanujan computed the coefficients up to q^{100} and q^{33}, respectively, of two particular quotients of q-series. We are uncertain about Ramanujan's intent in recording these two series expansions. We observe that the coefficients are nonnegative and "almost" increasing. In this section, we reproduce the content of a paper by Liaw [216], in which he identifies the series on both pages, gives a partition-theoretic interpretation that explains the nonnegativity of the coefficients, and proves a slightly more general version of the observation on the coefficients that are almost increasing.

First, we recall the well-known q-binomial theorem [12, p. 17], [55, p. 14]

$$
\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(q)_n} z^n = \frac{(\alpha z)_\infty}{(z)_\infty}, \tag{4.7.1}
$$

where $|q| < 1$, $|z| < 1$. An application of (4.7.1) with $\alpha = z^{-1}q$, $z \neq 0$, gives

$$
L(z,q) := \frac{(q)_\infty}{(z)_\infty (z^{-1}q)_\infty} = \sum_{j=0}^{\infty} \frac{z^j}{(q)_j (z^{-1}q^{j+1})_\infty}. \tag{4.7.2}
$$

Replacing q by q^p and z by q^r, respectively, in (4.7.2), where p and r are positive integers with $p \geq 2$ and $r < p$, we deduce that

$$
L_{p,r}(q) := \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty (q^{p-r};q^p)_\infty} = \sum_{j=0}^{\infty} \frac{q^{rj}}{(q^p;q^p)_j (q^{pj+p-r};q^p)_\infty}. \tag{4.7.3}
$$

Ramanujan's series on pages 63 and 64 are $L_{11,1}(q)$ and $L_{11,2}(q)$, respectively. Observe that the function $L(z,q)$ is closely related to the generating function for the crank (3.1.1).

In view of (4.7.3), the coefficients of $L_{p,r}(q)$ are always nonnegative. But with $p \geq 3$ and $r < p/2$, (4.7.3) can be interpreted combinatorially. It shows that $L_{p,r}(q)$ is the generating function of partitions into r's and parts congruent to 0 or $-r$ modulo p, and the largest part that is a multiple of p is no more than p times the number of r's, which in turn is not greater than the smallest part that is congruent to $-r$ modulo p. In [145, Theorem 7], the special cases $p = 5$ and $r = 1, 2$ are treated. This partition-theoretic approach yields part of Liaw's main result, which we now state and prove.

Theorem 4.7.1. *Let p and r be positive integers with $p \geq 2$ and $r < p$. Let*

$$L_{p,r}(q) = \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty(q^{p-r};q^p)_\infty} := \sum_{n=0}^{\infty} b_{p,r}(n)q^n. \tag{4.7.4}$$

Then $b_{p,r}(n) \geq 0$ for all n. Moreover, we let

$$L_{p,r}(q) + q^p := \sum_{n=0}^{\infty} c_{p,r}(n)q^n := \Sigma_0 + \Sigma_1 + \cdots + \Sigma_{r-1}, \tag{4.7.5}$$

where we subdivide the series in (4.7.5) according to the residue of the exponent modulo r, so that for $0 \leq i \leq r-1$,

$$\Sigma_i = \sum_{n=0}^{\infty} c_{p,r}(nr+i)q^{nr+i}.$$

Then for each i, the coefficient sequence $\{c_{p,r}(nr+i)\}_{n=0}^{\infty}$ is nondecreasing.

Proof. First, we note that $L_{p,r}(q) = L_{p,p-r}(q)$. Therefore, without loss of generality, we assume that $r \leq p/2$. By Equations (5.3) and (5.4) in Garvan's paper [145], we have

$$\begin{aligned} L_{p,r}(q) &= \frac{1}{1-q^r}\frac{(q^p;q^p)_\infty}{(q^{p+r};q^p)_\infty(q^{p-r};q^p)_\infty} \\ &= \frac{1}{1-q^r}\sum_{k=0}^{\infty}\Bigg(\sum_{\substack{k=pn+rm \\ 0 \leqq n < \infty \\ |m| \leqq n}} N_V(m,n)\Bigg)q^k \\ &:= \frac{1}{1-q^r}\sum_{k=0}^{\infty} a_{p,r}(k)q^k, \end{aligned} \tag{4.7.6}$$

where $N_V(m,n)$ is the number of vector partitions of n with crank m. It is known [145] that $N_V(m,n)$ is nonnegative except for $N_V(0,1) = -1$. The

equation $p = pn + rm$ has a unique solution $(m, n) = (0, 1)$ if $r < p/2$, while if $r = p/2$, it has two solutions $(m, n) = (0, 1)$ and $(-2, 2)$. It follows that $a_{p,r}(k)$ is nonnegative for all $k \geq 0$, except

$$a_{p,r}(p) = \begin{cases} N_V(0,1) + N_V(-2,2) = 0, & \text{for } r = p/2, \\ N_V(0,1) = -1, & \text{for } r < p/2. \end{cases}$$

It will be convenient to let $a_{p,r}(k) = 0$ for $k < 0$. Thus, by (4.7.4) and (4.7.6), $b_{p,r}(n) = \sum_{j=0}^{\infty} a_{p,r}(n - jr)$ and

$$b_{p,r}(n) - b_{p,r}(n - r) = a_{p,r}(n) \geq 0,$$

except when $r < p/2$ and $n = p$. Now, since $b_{p,r}(n)$ and $c_{p,r}(n)$ differ only at $n = p$, the difference

$$c_{p,r}(n) - c_{p,r}(n - r) \tag{4.7.7}$$

is nonnegative, except if $r < p/2$ *and* $n = p$ or $n = p + r$. The proof will be complete if we can show that the difference is also nonnegative when $r < p/2$ and $n = p$ or $n = p + r$. To this end, we observe that under the assumption $r < p/2$,

$$a_{p,r}(p + r) = \begin{cases} N_V(1,1) + N_V(-2,2) = 2, & \text{for } r = p/3; \\ N_V(1,1) = 1, & \text{for } r \neq p/3. \end{cases}$$

Therefore,

$$c_{p,r}(p) - c_{p,r}(p - r) = 1 + b_{p,r}(p) - b_{p,r}(p - r) = 1 + a_{p,r}(p) = 0$$

and

$$c_{p,r}(p + r) - c_{p,r}(p) = b_{p,r}(p + r) - 1 - b_{p,r}(p) = a_{p,r}(p + r) - 1 \geq 0.$$

Thus, since (4.7.7) has also been proved in these two exceptional cases, the proof is complete. □

4.8 Ranks and Cranks: Ramanujan's Influence Continues

From the abundance of material in the lost notebook on factors of the coefficients λ_n of the generating function (3.1.1) for cranks, Ramanujan clearly was eager to find some general theorems with the likely intention of applying them in the special case $a = 1$ to determine arithmetic properties of the partition function $p(n)$. Indeed, general theorems on the divisibility of λ_n by sums of powers of a appear extremely difficult to obtain. As we saw in the first two chapters, Ramanujan was able to derive five beautiful congruences for $F_a(q)$, but further arithmetic theorems that he was evidently seeking eluded him before his early death.

Although he was unable to complete his investigations of ranks and cranks of partitions, Ramanujan's work has been continued by many researchers. As Dyson [128, p. 10] astutely observed, "That was the wonderful thing about Ramanujan. He discovered so much, and yet he left so much more in his garden for other people to discover."

In recent decades, there have been many papers published on ranks and cranks. These fall roughly into five categories, which we label as follows:

1. Congruences and related arithmetic properties
2. Asymptotics and related analysis
3. Combinatorics
4. Inequalities
5. Generalizations.

It should be stressed that these categories are more than a little arbitrary. Historically, Dyson's original paper [127] was devoted to the discovery of combinatorial explanations of Ramanujan's congruences for the partition function. Thus 1 and 3 automatically overlap. When the rank and crank functions differ, one is naturally led to inequalities and category 4. So our five groupings should be viewed as a convenient, albeit somewhat arbitrary, means of outlining recent work.

4.8.1 Congruences and Related Work

One of the central themes in recent work has been the linking by K. Ono and his collaborators and colleagues of ranks and cranks with the mock theta functions and weak Maass forms. A good survey is given in [259]. There are related papers by S. Ahlgren and S. Treneer [8], K. Bringmann in several collaborations [84], [86], [92], M. Dewar [124], and M. Monks [225]. It is noteworthy that K. Mahlburg's paper [221] (introduced in [20]) on the crank won the first "Paper of the Year" award from the *Proceedings of the National Academy of Sciences*.

The work of Garvan goes back to his Ph.D. thesis [144]. Subsequently (in collaboration with D.S. Kim and D. Stanton) [153] he discovered new cranks, and in [145], [147], [149], [150] extended all these discoveries.

R. Lewis [209]–[214] provided early looks at rank and crank congruences for moduli other than 5, 7, and 11. N. Santa-Gadea's [307] work was also influenced by Lewis.

In addition, D. Choi, S.-Y. Kang, and J. Lovejoy [110], Ekin [131], [132], [133], S.J. Kaavya [186], and A.E. Patkowski [261] have contributed further to the arithmetic properties of ranks and cranks.

4.8.2 Asymptotics and Related Analysis

There are really two branches of analytic work on ranks and cranks. A.O.L. Atkin and Garvan [26] produced a partial differential equation relating the

generating functions of ranks and cranks. This surprising work was followed up by Bringmann and S. Zwegers in [94] and [356], and also by S.H. Chan, A. Dixit, and Garvan [107].

Relevant asymptotics have been established by Bringmann [85] and by Bringmann, Mahlburg, and R.C. Rhoades [91]. Earlier, in this chapter, we featured the asymptotic analysis by O.-Y. Chan.

4.8.3 Combinatorics

As mentioned previously, the line between categories 1 and 3 is not bright. However, there are a number of papers in which Dyson's original combinatorial view is paramount. Among these is Dyson's early study [129] of related symmetry questions. Much of Garvan's early work cited in Section 4.8.1 might also be cited here. In addition, A. Berkovich and Garvan [48]–[51] extended the work of Dyson and made further combinatorial discoveries. In [14], the rank moments introduced in [26] were interpreted combinatorially with a variety of related new congruences arising. This was followed up by Bringmann [83] and W.J. Keith [191].

4.8.4 Inequalities

It was observed early on by Lewis that some rank/crank enumerating functions were always larger than others. This was made explicit in [19]. The conjectures raised there have been proved by D.M. Kane [189], and more general results have been found by Bringmann and B. Kane [87], Bringmann and Mahlburg [90], Ekin [132], and Berkovich and Garvan [51].

Special note should be made of Garvan's proof that the $2k$th crank moment exceeds the $2k$th rank moment [152].

4.8.5 Generalizations

Here we are considering rank/crank questions related to restricted partition functions.

In [88], [89], [218], [220], Bringmann, Lovejoy, and R. Osburn extend rank and crank configurations to overpartitions. In [148] and [151], Garvan studies colored partitions. L.W. Kolitsch [200] considers generalized Frobenius partitions, and E. Mortenson [232] and S.H. Chan [106] examine broken diamond partitions.

The ranks and cranks for a variety of other restricted partition functions are considered by Berkovich and Garvan [49], Garvan [148], B. Kim [192], and Lovejoy and Osburn [219].

5

Ramanujan's Unpublished Manuscript on the Partition and Tau Functions

When Ramanujan died in 1920, he left behind an incomplete, unpublished manuscript in two parts on the partition function $p(n)$ and, in contemporary terminology, Ramanujan's tau function $\tau(n)$. The first part, beginning with the Roman numeral I, is written on 43 pages [283, pp. 135–177], with the last nine pages comprising material for insertion at various junctures in the first 34 pages of the manuscript. G.H. Hardy extracted a portion of Part I, providing proofs of Ramanujan's congruences for $p(n)$ modulo 5, 7, and 11, and published it in 1921 [280], [281, pp. 232–238] under Ramanujan's name. In a footnote, Hardy remarks, "The manuscript contains a large number of further results. It is very incomplete, and will require very careful editing before it can be published in full. I have taken from it the three simplest and most striking results ...." In 1952, J.M. Rushforth [306] published several further results, mostly on $\tau(n)$, from Part I. In 1977, R.A. Rankin [289] discussed several congruences for $\tau(n)$ found in Part I. The manuscript was not made available to the public until 1988, when it was photocopied in its original handwritten form and published with Ramanujan's lost notebook [283]. The existence of Part II [283, pp. 238–243] was first pointed out by B.J. Birch [75] in 1975, but like Part I, it was also hidden from the public until 1988, when a handwritten copy made by G.N. Watson was photocopied for [283]. Several theorems and proofs in this manuscript had not previously appeared before 1988. Until the publication of [67], none of the contents of Part II had been examined in the literature.

The $p(n)/\tau(n)$ manuscript arises from the last three years of Ramanujan's life. It may have been written in nursing homes and sanitariums in 1917–1919, when, as we know from letters that Ramanujan wrote to Hardy during this time [68, pp. 192–193], Ramanujan was thinking deeply about partitions, or more likely, it may have been written in India during the last year of his life. According to Rushforth [306], the manuscript was sent to Hardy by "Ramanujan a few months before the latter's death in 1920." If this is true, then it probably was enclosed with Ramanujan's last letter to Hardy, dated January 12, 1920 [68, pp. 220–223]. There is no mention of the manuscript

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*, DOI 10.1007/978-1-4614-3810-6_5,

in the extant portion of that letter, but we emphasize that part of the letter has been lost. Rushforth's account seems to be correct, although Rankin [291] thought that Rushforth's claim "was open to doubt." Ramanujan departed England on March 13, 1919, the same day his paper [279] was received by the *Proceedings of the London Mathematical Society.* In this paper, Ramanujan states his congruences for $p(n)$ modulo $5, 7, 11, 5^2, 7^2$, and 11^2. He furthermore offers his congruences for $\tau(n)$ modulo 5, 7, and 23. As we shall see, Ramanujan discovered further congruences for $p(n)$ and $\tau(n)$, and so it is likely that he continued to think about this topic while sailing back to India on the *Nagoya* from March 13 to March 27, 1919, the day the ship arrived in Bombay, and also in at least the early days after his arrival home. Recall also that Hardy extracted a portion of the $p(n)/\tau(n)$ manuscript for the posthumous paper [280], and so Hardy must have had the manuscript in his possession by sometime in 1920. In conclusion, either the manuscript was given to Hardy when Ramanujan departed England in 1919, or it was sent to Hardy with his last letter on January 12, 1920.

The manuscript was given by Hardy in 1928 to Watson, who had it in his possession until he died in 1965. At the suggestion of Rankin, Part I was sent shortly thereafter to the library of Trinity College, Cambridge, where it still resides. Watson's copy of Part II can be found in the library of Oxford's Mathematical Institute. When [67] was written, it was not realized that the original manuscript for Part II also resides at Trinity College. For further historical information, see Rankin's two papers [289], [291].

Since many of the proofs in this manuscript had not been published before their appearance in handwritten form with the lost notebook [283], since many details were omitted by Ramanujan, since mathematicians have established results either proved or asserted in the manuscript since it was written, and since the manuscript contains many unproved claims, the purpose of this chapter is to present the manuscript in its entirety, offer some additional details, and provide extensive commentary on it. Although many of the results in this manuscript have been proved or explained within a greater context in the work of P. Deligne, J.-P. Serre, H.P.F. Swinnerton-Dyer, and others, we were delighted to find a number of surprising new gems. For example, Ramanujan's claims (5.14.1)–(5.14.6) and many of the assertions in Sections 15 and 16 were unexpected and entirely new. Moreover, in proving the claims in Section 14, K. Ono was led, by the "shape of Ramanujan's claims," to several new general results regarding the distribution of the partition function modulo every prime $m \geq 5$ [258]. Part II, beginning with Section 20, is also fascinating, for it contains Ramanujan's proof, albeit lacking in many details, of his conjectured congruences for $p(n)$ modulo arbitrary integral powers of 5. It had not been previously known that Ramanujan had found a proof of his general conjecture for powers of 5, and therefore over the years, he had not been given credit for it.

Several editorial decisions needed to be made in our presentation of the manuscript.

(1) The nine pages of insertions at the end of Part I were interposed at their intended positions.

(2) None of Ramanujan's footnotes, such as "For a direct proof of this see," were completed in the manuscript. We have executed their completions, but we do not claim with certainty that they are what Ramanujan had in mind.

(3) Due to Ramanujan's failure to tag certain equalities, the manuscript contains incomplete references, such as "... deduce from ( ) and ( ) ...." We have added the tags and inserted the equation numbers. Difficulties arose when tags needed to be inserted at places between already existing tags with consecutive numbers. Thus, our account of the manuscript contains additional tags; generally, equation numbers here are not identical to the corresponding ones in Ramanujan's manuscript. However, we have preserved Ramanujan's numbering of sections, and so readers should have no difficulty identifying material read here with that in Ramanujan's original manuscript.

(4) As with most of his mathematics, Ramanujan provided very few details in this manuscript. In Part I, Ramanujan indicates, at more than one place, that this is the first of two papers that he intends to write on $p(n)$ and $\tau(n)$. It is clear that as Ramanujan wrote the manuscript, he continued to discover more and more theorems on the subject, and so he more and more frequently recorded his results without details with the promise that he would provide them in his next paper. Thus, details become more sparse as the manuscript progresses, so that in the last third of the manuscript there are hardly any details at all. Instead of returning in Part II to the details omitted in Part I, Ramanujan sketched his proofs of the congruences for $p(n)$ modulo any power of 5 or 7. In Hardy's extraction [280], he considerably amplified Ramanujan's arguments. Similarly, Rushforth [306] provided many details omitted by Ramanujan. In his paper providing proofs of the general congruences modulo 5^n and $7^{[n/2]+1}$, Watson [336] had to supply many details omitted by Ramanujan. We have followed their leads and have supplied more details for some of Ramanujan's arguments. However, for those parts of the manuscript examined by Hardy, Rushforth, and Watson, we have not added details here, since readers can find them in the aforementioned papers. We were faced with further difficult decisions about details. If Ramanujan presents a proof, but with modest deficiencies in details, we have placed additional details within square brackets, so that readers can remain clear about what was written by Ramanujan. On the other hand, many unproved claims can be found in the manuscript. Since Ramanujan's death, some have been discovered and proved by others, often without realizing that Ramanujan had originally found them. Some claims are false, and others have not been proved until recently. Because of the desire to make minimal additions within Ramanujan's manuscript, we have deferred discussions of most of Ramanujan's unproved claims to the end of this chapter, where many references to the literature are cited.

(5) We have taken the liberty of making minor editorial changes without comment. Such alterations include correcting misprints, adding punctuation,

and introducing notation. In particular, Ramanujan generally wrote infinite series in expanded form without resorting to summation signs, which we utilize here. We have also added titles to sections.

PROPERTIES OF $p(n)$ AND $\tau(n)$
DEFINED BY THE FUNCTIONS

$$\sum_{n=0}^{\infty} p(n)q^n = (q;q)_\infty^{-1},$$
$$\sum_{n=1}^{\infty} \tau(n)q^n = q(q;q)_\infty^{24}$$

S. RAMANUJAN

I

5.0 Congruences for $\tau(n)$

I have shown elsewhere by very simple arguments that

$$p(5n-1) \equiv 0 \pmod 5,$$
$$p(7n-2) \equiv 0 \pmod 7.$$

In the case of $\tau(n)$ such simple arguments give the following results.

Modulus 2

It is easy to see that the coefficients of q^n in the expansion of

$$q(q;q)_\infty^{24} \qquad \text{and} \qquad q(q^8;q^8)_\infty^3$$

are both odd or both even. But [by Jacobi's identity [168, p. 285, Theorem 357], [55, p. 39, Entry 24(ii)]],

$$q(q^8;q^8)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}.$$

It follows that $\tau(n)$ is odd or even according as n is an odd square or not. Thus we see that the number of values of n not exceeding n for which $\tau(n)$ is odd is only

$$\left[\frac{1+\sqrt{n}}{2}\right].$$

Modulus 5

Further, let J be any function of q with integral coefficients but not the same function throughout. It is easy to see that

$$q(q;q)_\infty^{24} = q(q;q)_\infty^4 (q^5;q^5)_\infty^4 + 5J.$$

But the coefficient of q^{5n} in

$$q(q;q)_\infty^4$$

is a multiple of 5.[1] It follows that

$$\tau(5n) \equiv 0 \pmod 5.$$

Modulus 7

This is the simplest of all cases. Here we have

$$q(q;q)_\infty^{24} = q(q;q)_\infty^3 (q^7;q^7)_\infty^3 + 7J.$$

But since

$$q(q;q)_\infty^3 = q\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2},$$

it is easy to see that the coefficients of q^{7n}, q^{7n-1}, q^{7n-2}, and q^{7n-4} are all multiples of 7. It follows that

$$\tau(7n),\ \tau(7n-1),\ \tau(7n-2),\ \tau(7n-4) \equiv 0 \pmod 7.$$

Modulus 23

We have

$$q(q;q)_\infty^{24} = q(q;q)_\infty (q^{23};q^{23})_\infty + 23J.$$

But [by Euler's pentagonal number theorem [168, p. 284, Theorem 353], [55, p. 36, Entry 22(iii)]],

$$q(q;q)_\infty = \sum (-1)^\nu q^{1+\nu(3\nu+1)/2},$$

where the summation extends over all values of ν from $-\infty$ to ∞. Now

$$1 + \frac{1}{2}\nu(3\nu+1) = (6\nu+1)^2 - \frac{23\nu(3\nu+1)}{2}.$$

The residues of a square number for modulus 23 cannot be

$$5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22.$$

It follows from this that

$$\begin{cases} \tau(23n-1), \tau(23n-2), \tau(23n-3), \tau(23n-4), \\ \tau(23n+5), \tau(23n-6), \tau(23n+7), \tau(23n-8), \\ \tau(23n-9), \tau(23n+10), \tau(23n+11) \equiv 0 \pmod{23}. \end{cases}$$

[1] Recall that $p(5n+4) \equiv 0 \pmod 5$.

5.1 The Congruence $p(5n+4) \equiv 0 \pmod 5$

Modulus 5

Let

$$P := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \tag{5.1.1}$$

$$Q := 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n}, \tag{5.1.2}$$

and

$$R := 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}, \tag{5.1.3}$$

so that[2]

$$Q^3 - R^2 = 1728q(q;q)_\infty^{24}. \tag{5.1.4}$$

Let $\sigma_s(n)$ denote the [sum of the] sth powers of the divisors of n. Then it is easy to see that

$$Q = 1 + 5J; \qquad R = P + 5J. \tag{5.1.5}$$

Hence,

$$Q^3 - R^2 = Q - P^2 + 5J. \tag{5.1.6}$$

But[3]

$$Q - P^2 = 288\sum_{n=1}^{\infty} n\sigma_1(n)q^n; \tag{5.1.7}$$

and it is obvious that

$$(q;q)_\infty^{24} = \frac{(q^{25};q^{25})_\infty}{(q;q)_\infty} + 5J. \tag{5.1.8}$$

It follows from (5.1.4) and (5.1.6)–(5.1.8) that

$$q\frac{(q^{25};q^{25})_\infty}{(q;q)_\infty} = \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 5J. \tag{5.1.9}$$

In other words,

$$(q^{25};q^{25})_\infty \sum_{n=0}^{\infty} p(n)q^{n+1} = \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 5J. \tag{5.1.10}$$

---

2 For an elementary proof, see [275, Equation (44)].

3 See [275, Equation (36)].

But the coefficient of q^{5n} in the right-hand side is a multiple of 5. It follows that

$$p(5n-1) \equiv 0 \pmod 5. \tag{5.1.11}$$

It also follows from (5.1.10) that

$$\begin{aligned} &p(n-1) - p(n-26) - p(n-51) + p(n-126) \\ &+ p(n-176) - p(n-301) - \cdots - n\sigma_1(n) \equiv 0 \pmod 5, \end{aligned} \tag{5.1.12}$$

where $1, 26, 51, 126, \ldots$ are numbers of the form $\frac{1}{2}(5\nu+1)(15\nu+2)$ and $\frac{1}{2}(5\nu-1)(15\nu-2)$. The number of values of n not exceeding 200 for which $p(n) \equiv 0, 1, 2, 3, 4 \pmod 5$ is 69, 33, 34, 34, 30, respectively; and the least value of n for which $p(n) \equiv 4 \pmod 5$ is 30. These being so, it appears that $p(n) \equiv 0 \pmod 5$ for about $\frac{1}{3}$ of the values of n, while $p(n) \equiv 1, 2, 3$ or $4 \pmod 5$ for about $\frac{1}{6}$ of the values of n each. It seems extremely difficult to prove any result in this direction concerning $p(n)$, but the problem is much easier concerning $\tau(n)$.

5.2 Divisibility of $\tau(n)$ by 5

It follows from (5.1.8) and (5.1.9) that

$$\tau(n) - n\sigma_1(n) \equiv 0 \pmod 5, \quad \lambda(n) - n\sigma_1(n) \equiv 0 \pmod 5, \tag{5.2.1}$$

where

$$\sum_{n=1}^{\infty} \lambda(n) q^n = q\frac{(q^{25};q^{25})_\infty}{(q;q)_\infty},$$

so that $\lambda(n+1)$ is the number of partitions of n as the sum of integers that are not multiples of 25. But if n be written in the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots,$$

where the a's are zeros or positive integers, then

$$n\sigma_1(n) = \prod_p \frac{p^{a_p}(p^{1+a_p}-1)}{p-1}, \qquad p = 2, 3, 5, \ldots. \tag{5.2.2}$$

But

$$\frac{p^{a_p}(p^{1+a_p}-1)}{p-1} \equiv 0 \pmod 5 \tag{5.2.3}$$

if

$$a_p \geq 1, \quad p = 5,$$

or

$$a_p \equiv 1 \pmod 2, \qquad p \equiv 4 \pmod 5,$$

or

$$a_p \equiv 3 \pmod 4, \qquad p \equiv 2 \text{ or } 3 \pmod 5,$$

or

$$a_p \equiv 4 \pmod 5, \qquad p \equiv 1 \pmod 5,$$

and for no other values. Suppose now that

$$\begin{cases} t_n &= 0, \qquad \tau(n) \equiv 0 \pmod 5, \\ t_n &= 1, \qquad \tau(n) \not\equiv 0 \pmod 5. \end{cases} \tag{5.2.4}$$

Then it follows from (5.2.3) that

$$\sum_{n=1}^{\infty} \frac{t_n}{n^s} = \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3, \tag{5.2.5}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1}{1-p^{-2s}},$$

with p being a prime of the form $5k-1$, and

$$\prod\nolimits_2 = \prod_p \frac{1-p^{-3s}}{(1-p^{-s})(1-p^{-4s})},$$

with p being a prime of the form $5k \pm 2$, and

$$\prod\nolimits_3 = \prod_p \frac{1-p^{-4s}}{(1-p^{-s})(1-p^{-5s})},$$

with p being a prime of the form $5k+1$.

It is easy to prove from (5.2.5) that

$$\sum_{k=1}^{n} t_k = o(n). \tag{5.2.6}$$

It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{Cn}{(\log n)^{1/4}}, \tag{5.2.7}$$

and

$$\sum_{k=1}^{n} t_k = C \int_1^n \frac{dx}{(\log x)^{1/4}} + O\left(\frac{n}{(\log n)^r}\right), \tag{5.2.8}$$

where C is a constant and r is any positive number.

The proof of (5.2.6) is quite elementary and very similar to that for showing that $\pi(x) = o(x)$,[4] with $\pi(x)$ being the number of primes not exceeding x. The result (5.2.6) can be stated roughly in other words that $\tau(n)$ and $\lambda(n)$ are divisible by 5 for almost all values of n, while (5.2.7) and (5.2.8) give a lot more information.

5.3 The Congruence $p(25n+24) \equiv 0 \pmod{25}$

Modulus 25

It is easily seen from (5.1.5) that

$$\begin{aligned} Q^3 - R^2 &= 2(Q^2 - PR) - (Q - P^2) + Q(Q-1)^2 - (R-P)^2 \\ &= 2(Q^2 - PR) - (Q - P^2) + 25J. \end{aligned} \tag{5.3.1}$$

But[5]

$$Q^2 - PR = 1008 \sum_{n=1}^{\infty} n\sigma_5(n) q^n; \tag{5.3.2}$$

and it is obvious that

$$(q;q)_\infty^{24} = \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} + 25J. \tag{5.3.3}$$

Now remembering that

$$\sigma_5(n) - \sigma_1(n) \equiv 0 \pmod{5}, \tag{5.3.4}$$

it follows from (5.1.7) and (5.3.1)–(5.3.3) that

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty} \left\{2n\sigma_5(n) - n\sigma_1(n)\right\} q^n + 25J. \tag{5.3.5}$$

[By extracting those terms with exponents that are multiples of 5 and by employing the congruence $p(5n-1) \equiv 0 \pmod{5}$,] we easily deduce that

$$(q;q)_\infty^5 \sum_{n=1}^{\infty} p(5n-1) q^n = \sum_{n=1}^{\infty} \left\{10n\sigma_5(n) - 5n\sigma_1(n)\right\} q^n + 25J,$$

and hence [by (5.3.4)] that

[4] See Landau's *Primzahlen* [204, pp. 641–669].
[5] See [275, Table II].

$$(q^5;q^5)_\infty \sum_{n=1}^{\infty} p(5n-1)q^n = 5\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 25J. \tag{5.3.6}$$

Since the coefficient of q^{5n} is a multiple of 25, it follows that

$$p(25n-1) \equiv 0 \pmod{25}. \tag{5.3.7}$$

It also follows from (5.3.6) that

$$\begin{aligned} &p(5n-1) - p(5n-26) - p(5n-51) + p(5n-126) \\ &+ p(5n-176) - \cdots - 5n\sigma_1(n) \equiv 0 \pmod{25}, \end{aligned}$$

where $1, 26, 51, 126, \ldots$ are the same as in (5.1.12).

5.4 Congruences Modulo 5^k

It is easy to see [by Fermat's little theorem] that

$$n\sigma_9(n) - 2n\sigma_5(n) + n\sigma_1(n) \equiv 0 \pmod{25}. \tag{5.4.1}$$

It follows from this and (5.3.3) and (5.3.5) that

$$\tau(n) - n\sigma_9(n) \equiv 0 \pmod{25}. \tag{5.4.2}$$

It appears that if k be any positive integer, it is possible to find two integers a and b such that

$$\tau(n) - n^a\sigma_b(n) \equiv 0 \pmod{5^k}, \tag{5.4.3}$$

if n is not a multiple of 5. Thus for instance

$$\tau(n) - n^{41}\sigma_{29}(n) \equiv 0 \pmod{125}, \tag{5.4.4}$$

if n is not a multiple of 5. I have not yet proved these results. If n is a multiple of 5, then

$$\tau(n) - 4830\tau\left(\frac{n}{5}\right) + 5^{11}\tau\left(\frac{n}{25}\right) = 0$$

in virtue of (5.7.6), with $\tau(x)$ being considered as 0 if x is not an integer.

It also appears that the coefficient of q^n in the left-hand side of (5.3.5) can be exactly determined in terms of the real divisors of n. Thus

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2}, \tag{5.4.5}$$

[where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol]. The allied function is given by

$$\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty} = 1 - 5\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1-q^n}. \tag{5.4.6}$$

It follows from (5.4.5) that

$$(q;q)_\infty^5 \sum_{n=1}^{\infty} p(5n-1)q^n = 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}$$

and hence that[6]

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{5.4.7}$$

5.5 Congruences Modulo 7

Modulus 7

Since[7]

$$Q^2 = 1 + 480\sum_{n=1}^{\infty}\frac{n^7q^n}{1-q^n}, \tag{5.5.1}$$

it is easy to see that

$$Q^2 = P + 7J; \qquad R = 1 + 7J; \tag{5.5.2}$$

and so

$$(Q^3 - R^2)^2 = P^3 - 2PQ + R + 7J. \tag{5.5.3}$$

But[8]

$$\begin{cases} PQ - R = 720\displaystyle\sum_{n=1}^{\infty} n\sigma_3(n)q^n, \\ P^3 - 3PQ + 2R = -1728\displaystyle\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n; \end{cases} \tag{5.5.4}$$

and it is obvious that

$$(q;q)_\infty^{48} = \frac{(q^{49};q^{49})_\infty}{(q;q)_\infty} + 7J. \tag{5.5.5}$$

It follows from all these that

$$q^2\frac{(q^{49};q^{49})_\infty}{(q;q)_\infty} = \sum_{n=1}^{\infty}\left\{n^2\sigma_1(n) - n\sigma_3(n)\right\}q^n + 7J. \tag{5.5.6}$$

In other words,

$$(q^{49};q^{49})_\infty\sum_{n=0}^{\infty} p(n)q^{n+2} = \sum_{n=1}^{\infty}\left\{n^2\sigma_1(n) - n\sigma_3(n)\right\}q^n + 7J. \tag{5.5.7}$$

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[6] For a direct proof of this result see [276].

[7] See [275, Table I].

[8] See [275, Tables II and III, resp.].

It follows that

$$p(7n-2) \equiv 0 \pmod 7, \tag{5.5.8}$$

and

$$\begin{aligned} &p(n-2) - p(n-51) - p(n-100) + p(n-247) \\ &+ p(n-345) - \cdots + n\sigma_3(n) - n^2\sigma_1(n) \equiv 0 \pmod 7, \end{aligned} \tag{5.5.9}$$

where $2, 51, 100, 247, \ldots$ are the numbers of the form $\frac{1}{2}(7\nu+1)(21\nu+4)$ and $\frac{1}{2}(7\nu-1)(21\nu-4)$.

The number of values of n not exceeding 200 for which $p(n) \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 7$ is 50, 33, 22, 28, 23, 23, 21, respectively, and the least value of n for which $p(n) \equiv 6 \pmod 7$ is 73. It appears that $p(n) \equiv 0 \pmod 7$ for about $\frac{1}{4}$ of the values of n, while $p(n) \equiv 1, 2, 3, 4, 5, 6 \pmod 7$ for about $\frac{1}{8}$ of the values of n each.

5.6 Congruences Modulo 7, Continued

It follows from (5.5.2) that

$$Q^3 - R^2 = PQ - R + 7J. \tag{5.6.1}$$

It is easy to see from this and (5.5.4) that

$$\tau(n) - n\sigma_3(n) \equiv 0 \pmod 7. \tag{5.6.2}$$

Now if $n = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots$, then

$$n\sigma_3(n) = \prod_p p^{a_p} \frac{p^{3(1+a_p)} - 1}{p^3 - 1}, \qquad p = 2, 3, 5, 7, \ldots. \tag{5.6.3}$$

But

$$p^{a_p} \frac{p^{3(1+a_p)} - 1}{p^3 - 1} \equiv 0 \pmod 7, \tag{5.6.4}$$

if

$$a_p \equiv 6 \pmod 7, \qquad p \equiv 1, 2, \text{ or } 4 \pmod 7,$$

or

$$a_p \equiv 1 \pmod 2, \qquad p \equiv 3, 5, \text{ or } 6 \pmod 7,$$

or

$$a_p \geq 1, \qquad p = 7.$$

Suppose now that

$$\begin{aligned} t_n &= 1, \qquad \tau(n) \not\equiv 0 \pmod 7, \\ t_n &= 0, \qquad \tau(n) \equiv 0 \pmod 7. \end{aligned}$$

Then it follows from (5.6.4) that

$$\sum_{n=1}^{\infty} \frac{t_n}{n^s} = \prod\nolimits_1 \prod\nolimits_2, \tag{5.6.5}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1-p^{-6s}}{(1-p^{-s})(1-p^{-7s})},$$

with p being a prime of the form $7k+1$, $7k+2$, $7k+4$, and

$$\prod\nolimits_2 = \prod_p \frac{1}{1-p^{-2s}},$$

with p being a prime of the form $7k+3$, $7k+5$, $7k+6$. It is easy to prove from (5.6.5) by quite elementary methods that

$$\sum_{k=1}^{n} t_k = o(n). \tag{5.6.6}$$

It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{Cn}{(\log n)^{1/2}}; \tag{5.6.7}$$

and

$$\sum_{k=1}^{n} t_k = C\int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right), \tag{5.6.8}$$

where r is any positive number and

$$\begin{aligned} C = {} & \frac{6^{1/2}}{7^{3/4}} \frac{1-2^{-6}}{1-2^{-7}} \frac{1-11^{-6}}{1-11^{-7}} \frac{1-23^{-6}}{1-23^{-7}} \frac{1-29^{-6}}{1-29^{-7}} \cdots \\ & \times \frac{1}{\{(1-3^{-2})(1-5^{-2})(1-13^{-2})(1-17^{-2})(1-19^{-2})\cdots\}^{1/2}}, \end{aligned}$$

where $2, 11, 23, \ldots$ are primes of the form $7k+1$, $7k+2$, and $7k+4$, while $3, 5, 13, \ldots$ are primes of the form $7k+3$, $7k+5$, and $7k+6$. Thus we see that $\tau(n)$ is divisible by 7 for almost all values of n; and at the same time the number of values of n for which $\tau(n)$ is divisible by 7 is far greater than that for which $\tau(n)$ is divisible by 5.

Now if

$$\sum_{n=1}^{\infty} \lambda(n) q^n = q^2 \frac{(q^{49};q^{49})_\infty}{(q;q)_\infty},$$

so that $\lambda(n+2)$ is the number of partitions of n as the sum of integers which are not multiples of 49, it is clear from (5.5.6) that

$$\lambda(n) - n^2\sigma_1(n) + n\sigma_3(n) \equiv 0 \pmod{7}. \tag{5.6.9}$$

But it is easy to show that $n^2\sigma_1(n)$ and $n\sigma_3(n)$ are divisible by 7 for almost all values of n. It follows that $\lambda(n)$ is divisible by 7 for almost all values of n. It can even be shown that the number of values of j not exceeding n for which $\lambda(j)$ is *not* divisible by j is

$$O\left(\frac{n}{(\log n)^{1/6}}\right). \tag{5.6.10}$$

The index $\frac{1}{6}$ in (5.6.10) is easily obtained by considering $n^2\sigma_1(n)$ and $n\sigma_3(n)$ separately; but whether this is the right index or not can be known only by considering

$$n^2\sigma_1(n) - n\sigma_3(n)$$

taken together, which seems rather complicated to deal with.

5.7 Congruences Modulo 49

Modulus 49

We have

$$\begin{aligned}(Q^3 - R^2)^2 &= (3P^2Q^2 - 4PQR - 2Q^3 + 3R^2)\\ &\quad - 2(P^3 - 2PQ + R) + 2P(Q^2 - P)^2 - (1 + 2PQ)(R-1)^2\\ &\quad + \left\{Q(Q^2 - P) - R^2 + 1\right\}^2\\ &= (3P^2Q^2 - 4PQR - 2Q^3 + 3R^2) - 2(P^3 - 2PQ + R) + 49J\end{aligned}$$

in virtue of (5.5.2). But[9]

$$\begin{cases} Q^3 - R^2 = 1728\displaystyle\sum_{n=1}^{\infty}\tau(n)q^n,\\ 3Q^3 + 2R^2 - 5PQR = 1584\displaystyle\sum_{n=1}^{\infty}n\sigma_9(n)q^n,\\ 5Q^3 + 4R^2 - 18PQR + 9P^2Q^2 = 8640\displaystyle\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n;\end{cases} \tag{5.7.1}$$

and it is obvious that

$$(q;q)_\infty^{48} = \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} + 49J. \tag{5.7.2}$$

Now remembering that

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[9] See [275, Equation (44), Table II, Table III, resp.].

$$\sigma_7(n) - \sigma_1(n) \equiv 0 \pmod 7, \quad \sigma_9(n) - \sigma_3(n) \equiv 0 \pmod 7, \tag{5.7.3}$$

it follows from the above equations and (5.5.4) that

$$\begin{aligned} q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} = \sum_{n=1}^{\infty} \bigl\{ & 2n\sigma_9(n) - 4n^2\sigma_7(n) + 2n\sigma_3(n) \\ & -2n^2\sigma_1(n) + 2\tau(n) \bigr\} q^n + 49J. \end{aligned} \tag{5.7.4}$$

From this [and (5.6.2)] we deduce that

$$(q;q)_\infty^7 \sum_{n=0}^{\infty} p(7n+5) q^{n+1} = \sum_{n=1}^{\infty} \{28n\sigma_3(n) + 2\tau(7n)\} q^n + 49J. \tag{5.7.5}$$

I have stated in my previous paper that[10]

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}, \tag{5.7.6}$$

where p assumes all prime values. This has since been proved by Mr Mordell.[11] Now by actual calculation we find that

$$\tau(7) \equiv 14 \pmod{49}.$$

It follows from this and (5.7.6) that

$$\tau(7n) - 14\tau(n) \equiv 0 \pmod{49}.$$

It is easy to see from this and (5.7.5) [and (5.6.2)] that

$$(q^7;q^7)_\infty \sum_{n=0}^{\infty} p(7n+5) q^{n+1} = 7 \sum_{n=1}^{\infty} n\sigma_3(n) q^n + 49J. \tag{5.7.7}$$

Now if

$$n \equiv 3, 5, 6 \pmod 7,$$

then n must contain an odd power of a prime p of the form $7k+3$, $7k+5$, or $7k+6$ as a divisor, since all perfect squares are of the form $7k$, $7k+1$, $7k+2$, or $7k+4$; and so $\sigma_3(n)$ is divisible by p^3+1, which is divisible by 7. Also it is obvious that if n is a multiple of 7 then $n\sigma_3(n)$ is also divisible by 7. It follows that if

$$n \equiv 0, 3, 5, 6 \pmod 7,$$

then

[10] [275, eq. (101)].

[11] On Mr Ramanujan's empirical expansions of modular functions, *Proc. Cambridge Philos. Soc.* 19 (1917), 117–124. A simpler proof is given in Hardy's lectures [166].

$$n\sigma_3(n) \equiv 0 \pmod 7.$$

It is easy to see from this and (5.7.7) that

$$p(49n-2),\ p(49n-9),\ p(49n-16),\ p(49n-30) \equiv 0 \pmod{49}. \tag{5.7.8}$$

It also follows from (5.7.7) that

$$\begin{aligned} &p(7n-2) - p(7n-51) - p(7n-100) + p(7n-247) \\ &+ p(7n-345) - \cdots - 7n\sigma_3(n) \equiv 0 \pmod{49}, \end{aligned}$$

where $2, 51, 100, 247, \ldots$ are the same as in (5.5.9).

5.8 Congruences Modulo 49, Continued

It appears that

$$q(q;q)_\infty^3(q^7;q^7)_\infty^3 + 8q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} = \sum_{n=1}^{\infty}\left(\frac{n}{7}\right)q^n\frac{1+q^n}{(1-q^n)^3} \tag{5.8.1}$$

[where $\left(\frac{n}{7}\right)$ denotes the Legendre symbol], while the allied function is given by

$$49q(q;q)_\infty^3(q^7;q^7)_\infty^3 + 8\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty} = 8 - 7\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{n^2q^n}{1-q^n}. \tag{5.8.2}$$

Now remembering that

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$$

and picking out the terms $q^7, q^{14}, q^{21}, \ldots$ from both sides in (5.8.1), we obtain

$$-7q(q;q)_\infty^3(q^7;q^7)_\infty^3 + 8(q;q)_\infty^7\sum_{n=1}^{\infty}p(7n-2)q^n = 49\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)q^n\frac{1+q^n}{(1-q^n)^3},$$

with the series in the right-hand side being the same as that in (5.8.1). It follows from this and (5.8.1) that[12]

[12] For a direct proof of this see §. [Ramanujan evidently intended to give a proof of (5.8.3) elsewhere in this manuscript. In his paper [276], (5.8.3) is stated without proof. This identity is also found on page 189 in Ramanujan's lost notebook [283], and in Chapter 6 we provide a proof of (5.8.3) along the lines of that sketched by Ramanujan in this manuscript. The proof, as well as other proofs of claims on page 189, is taken from a paper by Berndt, A.J. Yee, and J. Yi [70]. See the notes at the end of this chapter for references to further proofs of (5.8.3).]

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{5.8.3}$$

It also appears that if

$$\sum_{n=1}^{\infty} \lambda(n)q^n = q(q;q)_\infty^3(q^7;q^7)_\infty^3,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{1}{1+7^{1-s}} \prod\nolimits_1 \prod\nolimits_2, \tag{5.8.4}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1}{1-p^{2-2s}},$$

with p being a prime of the form $7k+3$, $7k+5$, or $7k+6$, and

$$\prod\nolimits_2 = \prod_p \frac{1}{1+(2p-a^2)p^{-s}+p^{2-2s}},$$

with p being a prime of the form $7k+1$, $7k+2$, or $7k+4$ and a and b being integers such that $4p = a^2+7b^2$. Thus $\lambda(n)$ can be completely ascertained. It follows from this and (5.8.1) and (5.8.2) that the coefficients of q^n in

$$\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}, \qquad q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}$$

can be completely ascertained.

Now it is easy to see that

$$3n^9 - 2n^3 \equiv 0, 1, \text{ or } -1 \pmod{49},$$

according as $n \equiv 0 \pmod 7$, $n \equiv 1, 2, 4 \pmod 7$, or $n \equiv 3, 5, 6 \pmod 7$. Also the coefficient of q^n in $q(1+q)/(1-q)^3$ is n^2. Hence the right side in (5.8.1) can be written as

$$\sum_{n=1}^{\infty} \left\{3n^2\sigma_7(n) - 2n^2\sigma_1(n)\right\} q^n + 49J. \tag{5.8.5}$$

It follows from this, (5.7.3), (5.7.4), and (5.8.1) that

$$\tau(n) - 3\lambda(n) + n\sigma_9(n) + n\sigma_3(n) \equiv 0 \pmod{49}, \tag{5.8.6}$$

where $\lambda(n)$ is the same as in (5.8.4). From the formulae (5.8.4) and (5.8.6) all the residues of $\tau(n)$ for modulus 49 can be completely ascertained.

5.9 The Congruence $p(11n+6) \equiv 0 \pmod{11}$

Modulus 11

In this case we start with the series[13]

$$\begin{cases} 1 - 264\sum_{n=1}^{\infty} \dfrac{n^9 q^n}{1-q^n} = QR, \\ 691 + 65520\sum_{n=1}^{\infty} \dfrac{n^{11} q^n}{1-q^n} = 441Q^3 + 250R^2. \end{cases} \tag{5.9.1}$$

It follows that

$$QR = 1 + 11J; \qquad Q^3 - 3R^2 = -2P + 11J. \tag{5.9.2}$$

It is easy to see from this that

$$\begin{aligned}(Q^3 - R^2)^5 &= (Q^3 - 3R^2)^5 - Q(Q^3 - 3R^2)^3 - R(Q^3 - 3R^2)^2 - 5QR + 11J \\ &= P^5 - 3P^3Q - 4P^2R - 5QR + 11J.\end{aligned}$$

But[14]

$$\begin{cases} P^5 - 10P^3Q + 20P^2R - 15PQ^2 + 4QR = -20736\sum_{n=1}^{\infty} n^4\sigma_1(n)q^n, \\ P^3Q - 3P^2R + 3PQ^2 - QR = 3456\sum_{n=1}^{\infty} n^3\sigma_3(n)q^n, \\ P^2R - 2PQ^2 + QR = -1728\sum_{n=1}^{\infty} n^2\sigma_5(n)q^n, \\ PQ^2 - QR = 720\sum_{n=1}^{\infty} n\sigma_7(n)q^n; \end{cases} \tag{5.9.3}$$

and it is obvious that

$$(q;q)_\infty^{120} = \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} + 11J. \tag{5.9.4}$$

It is easy to see from all these that

$$\begin{aligned} &q^5 \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} \\ &\quad = \sum_{n=1}^{\infty} \left\{-n^4\sigma_1(n) + 3n^3\sigma_3(n) + 3n^2\sigma_5(n) - 5n\sigma_7(n)\right\} q^n + 11J. \end{aligned} \tag{5.9.5}$$

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[13] See [275, Table I].
[14] See [275, Table III, Table II].

It follows from this that

$$p(11n-5) \equiv 0 \pmod{11}; \tag{5.9.6}$$

and

$$\begin{aligned} &p(n-5) - p(n-126) - p(n-247) + p(n-610) + p(n-852) \\ &- \cdots + n^4\sigma_1(n) - 3n^3\sigma_3(n) - 3n^2\sigma_5(n) + 5n\sigma_7(n) \equiv 0 \pmod{11}, \end{aligned} \tag{5.9.7}$$

where $5, 126, 247, 610, \ldots$ are numbers of the form $\frac{1}{2}(11\nu+2)(33\nu+5)$ and $\frac{1}{2}(11\nu-2)(33\nu-5)$. It is only to prove the general result (5.9.7) we require all the details in (5.9.3). But we don't require all these details in order to prove (5.9.6) and the proof can be very much simplified as follows: we have[15]

$$q\frac{dP}{dq} = \frac{P^2-Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ-R}{3}, \qquad q\frac{dR}{dq} = \frac{PR-Q^2}{2}. \tag{5.9.8}$$

Now using (5.9.2) and (5.9.8) we can show that[16]

$$(Q^3-R^2)^5 = q\frac{dJ}{dq} + 11J.$$

It follows from this and (5.9.4) that

$$q^5\frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} = q\frac{dJ}{dq} + 11J. \tag{5.9.9}$$

Since the coefficient of q^{11n} in the right-hand side is a multiple of 11, it follows that

$$p(11n-5) \equiv 0 \pmod{11}.$$

The number of values of n not exceeding 200 for which $p(n) \equiv 0$, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 $\pmod{11}$ is 77, 23, 24, 14, 15, 14, 5, 12, 8, 8, 0, respectively. Even though these values seem to be very irregular, it appears from the residues of $p(n)$ for moduli 5 and 7 and also from the next section that $p(n) \equiv 0 \pmod{11}$ for about $\frac{1}{6}$ of the values of n, while $p(n) \equiv 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{11}$ for about $\frac{1}{12}$ of the values of n each.

5.10 Congruences Modulo 11, Continued

Mr H.B.C. Darling observed the remarkable fact (before I began to write this paper) that $p(n)$ is divisible by 11 for 45 values of n not exceeding 100. This can be explained by the formula (5.9.7) and the congruency of

[15] See [275, Equation (30)].

[16] As mentioned in the beginning, the J's are not the same functions.

$$n^4\sigma_1(n) - 3n^3\sigma_3(n) - 3n^2\sigma_5(n) + 5n\sigma_7(n) \tag{5.10.1}$$

for modulus 11. It can be shown by quite elementary methods that (5.10.1) is divisible by 11 for almost all values of n. [A proof of this fact is sketched in Section 19.] It can even be shown that the number of values of n not exceeding n for which (5.10.1) is *not* divisible by 11 is

$$O\left(\frac{n}{(\log n)^{1/10}}\right) \tag{5.10.2}$$

by considering the divisibility of the four terms in (5.10.1) separately; but a better result can be found only by considering all the four terms in (5.10.1) taken together. The same remarks apply to the function $\lambda(n)$ defined by

$$\sum_{n=1}^{\infty} \lambda(n)q^n = q^5 \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty}; \tag{5.10.3}$$

so that $\lambda(n+5)$ is the number of partitions of n as the sum of integers which are not multiples of 121; that is to say, $\lambda(n)$ is divisible by 11 for almost all values of n; and the number of values of $\lambda(n)$ not divisible by 11 is of the form (5.10.2). It appears from (5.10.3) that the number of values of n for which $p(n) \equiv 0 \pmod{11}$ cannot be so high as 45% if n exceeds 120. Thus the number of values of p divisible by 11 is

$$\begin{aligned}
45\%, &\qquad 0 < n \le 40,\\
45\%, &\qquad 40 < n \le 80,\\
45\%, &\qquad 80 < n \le 120,\\
35\%, &\qquad 120 < n \le 160,\\
22\tfrac{1}{2}\%, &\qquad 160 < n \le 200.
\end{aligned}$$

It is also very remarkable that in the table of the first 200 values of $p(n)$, there is not a single value of $p(n)$ of the form $11k-1$. This is probably due to such a high percentage of the values of $p(n)$ divisible by 11 in the beginning.

I have not yet investigated completely the residues of $\tau(n)$ for modulus 11. But it appears that if

$$\sum_{n=1}^{\infty} \lambda(n)q^n = q(q;q)_\infty^2 (q^{11};q^{11})_\infty^2,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{1}{1-11^{-s}} \prod_p \frac{1}{1-\lambda(p)p^{-s}+p^{1-2s}}, \tag{5.10.4}$$

p assuming all prime values except 11, and that $\lambda(p)$ can be determined also. If that is so then the residues of $\tau(n)$ for modulus 11 can also be ascertained, since it is easily seen that

$$\tau(n) - \lambda(n) \equiv 0 \pmod{11}. \tag{5.10.5}$$

Again it is easy to show by using (5.7.6) [and the values $\tau(2) = -24$, $\tau(3) = 252$, $\tau(5) = 4830$, $\tau(7) = -16744$, and $\tau(11) = 534612$, which can be found in a table in Ramanujan's paper [275], [281, p. 153]] that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} &= \frac{1}{1+2^{1-s}+2^{1-2s}} \frac{1}{(1-3^{3-s})^2} \frac{1}{(1-5^{2-s})(1-5^{4-s})} \\ &\quad \times \frac{1}{(1+7^{2-s})(1-7^{4-s})} \frac{1}{1-11^{-s}} \cdots + 11j,\end{aligned} \tag{5.10.6}$$

where j is a Dirichlet series of the form

$$\sum \frac{a_n}{n^s},$$

with a_n being an integer.

From this we can deduce a number of results such as

$$\tau(2^{4\lambda-1}n) \equiv 0 \pmod{11} \tag{5.10.7}$$

if n is an odd integer;

$$\tau(3^{11\lambda-1}n) \equiv 0 \pmod{11} \tag{5.10.8}$$

if n is not a multiple of 3;

$$\tau(5^{5\lambda-1}n) \equiv 0 \pmod{11} \tag{5.10.9}$$

if n is not a multiple of 5;

$$\tau(7^{10\lambda-1}n) \equiv 0 \pmod{11} \tag{5.10.10}$$

if n is not a multiple of 7;

$$\tau(11^{\lambda}n) - \tau(n) \equiv 0 \pmod{11} \tag{5.10.11}$$

and so on. [The five congruences above can be established by expanding the appropriate factors in (5.10.6) in geometric series. For example, consider

$$\begin{aligned}\frac{1}{1+2^{1-s}+2^{1-2s}} &= -\frac{i}{2^{1-s}+1-i} + \frac{i}{2^{1-s}+1+i} \\ &= -\frac{i}{1-i} \sum_{n=0}^{\infty} \left(\frac{2^{1-s}}{i-1}\right)^n + \frac{i}{1+i} \sum_{n=0}^{\infty} \left(\frac{2^{1-s}}{-i-1}\right)^n \\ &= i \sum_{n=0}^{\infty} 2^{n(1-s)} e^{-3\pi i(n+1)/4} - i \sum_{n=0}^{\infty} 2^{n(1-s)} e^{3\pi i(n+1)/4}.\end{aligned}$$

Since $\sin\{3\pi(n+1)/4\} = 0$ if and only if $n \equiv -1 \pmod 4$, the assertion (5.10.7) follows from (5.10.6).]

Even though (5.10.7)–(5.10.10) are very analogous to one another, further equations are not necessarily quite similar to these; sometimes there is more than one equation and sometimes there are equations of the form

$$\tau(19n) \equiv 0 \pmod{11} \tag{5.10.12}$$

if n is not a multiple of 19, and

$$\tau(29n) \equiv 0 \pmod{11} \tag{5.10.13}$$

if n is not a multiple of 29.

It is very likely that the primes $19, 29, \dots$ occurring in equations like (5.10.12) and (5.10.13) are such that the sum of their reciprocals is a *divergent* series. If this assertion is true then $\tau(n)$ is divisible by 11 for almost all values of n, which is easily seen from (5.10.2).

5.11 Divisibility by 2 or 3

Moduli 2 and 3

[It will be convenient to introduce Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$, defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} \tag{5.11.1}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5.11.2}$$

where the product representations are easy consequences of Jacobi's triple product identity.]

Before we proceed to consider higher moduli, we shall see what the analogous formulae are in the cases of moduli 2 and 3. It is easy to see that [by (5.11.2)]

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} + 2J = \psi(q) + 2J. \tag{5.11.3}$$

It follows that

$$p(n) - p(n-4) - p(n-8) + p(n-20) + p(n-28) - \cdots \tag{5.11.4}$$

is odd or even according as n is a triangular number or not, where $4, 8, 20, \dots$ are numbers of the form $2\nu(3\nu+1)$ and $2\nu(3\nu-1)$.

$p(n)$ is odd for 110 values of n not exceeding 200 and even for 90 values of n in the same range. Thus $p(n)$ seems to be odd for more values of n than those for which $p(n)$ is even.

If

$$\sum_{n=0}^{\infty} \lambda(n) q^n = \frac{(q^4; q^4)_\infty}{(q; q)_\infty},$$

so that $\lambda(n)$ is the number of partitions of n as the sum of integers which are not multiples of 4 then [by (5.11.3) and (5.11.2)] $\lambda(n)$ is odd or even according as n is a triangular number or not.

Again we have

$$\frac{(q^9; q^9)_\infty}{(q; q)_\infty} = \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} + 3J. \tag{5.11.5}$$

But it can be shown [59] that

$$q\frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty} = \sum_{n=1}^{\infty} \chi_0(n) \frac{q^n}{1 + q^n + q^{2n}} \tag{5.11.6}$$

[where $\chi_0(n)$ is the principal character modulo 3]. But the right-hand side in (5.11.6) is of the form

$$\sum_{n=1}^{\infty} \chi_0(n) \frac{q^n}{(1 - q^n)^2} + 3J;$$

and the coefficient of q^{3n+1} in the above series is $\sigma_1(3n + 1)$. It follows from this and (5.11.5) and (5.11.6) that

$$\frac{(q^9; q^9)_\infty}{(q; q)_\infty} = \sum_{n=0}^{\infty} \sigma_1(3n + 1) q^n + 3J. \tag{5.11.7}$$

From this we easily deduce that

$$\begin{aligned} &p(n) - p(n - 9) - p(n - 18) + p(n - 45) + p(n - 63) \\ &\quad - p(n - 108) - \cdots - \sigma_1(3n + 1) \equiv 0 \pmod 3, \end{aligned} \tag{5.11.8}$$

where $9, 18, 45, \ldots$ are numbers of the form $\frac{9}{2}\nu(3\nu + 1)$ and $\frac{9}{2}\nu(3\nu - 1)$.

The number of values of n not exceeding 200 for which $p(n) \equiv 0, 1, 2 \pmod 3$ is 66, 68, 66 respectively. Thus it appears that $p(n) \equiv 0, 1, 2 \pmod 3$ for about $\frac{1}{3}$ of the number of values of n each.

It follows from (5.11.7) that if

$$\sum_{n=0}^{\infty} \lambda(n) q^n = \frac{(q^9; q^9)_\infty}{(q; q)_\infty},$$

so that $\lambda(n)$ is the number of partitions of n as the sum of integers which are not multiples of 9, then

$$\lambda(n) - \sigma_1(3n + 1) \equiv 0 \pmod 3.$$

Again the left-hand side of (5.11.6) is of the form

$$q(q;q)_\infty^{24} + 3J, \tag{5.11.9}$$

while the right-hand side of (5.11.6) is of the form

$$\sum_{n=1}^{\infty} \frac{n^2 q^n}{(1-q^n)^2} + 3J.$$

It follows that

$$\tau(n) - n\sigma_1(n) \equiv 0 \pmod 3. \tag{5.11.10}$$

Suppose now that

$$\begin{cases} t_n = 0, & \lambda(n) \equiv 0 \pmod 3, \\ t_n = 1, & \lambda(n) \not\equiv 0 \pmod 3, \end{cases}$$

and that

$$\begin{cases} T_n = 0, & \tau(n) \equiv 0 \pmod 3, \\ T_n = 1, & \tau(n) \not\equiv 0 \pmod 3. \end{cases}$$

Then we can easily deduce from (5.11.9), (5.11.10), and (5.2.2) that

$$\sum_{n=0}^{\infty} \frac{t_n}{(3n+1)^s} = \sum_{n=0}^{\infty} \frac{T_n}{n^s} = \prod\nolimits_1 \prod\nolimits_2,$$

where

$$\prod\nolimits_1 = \prod_p \frac{1}{1-p^{-2s}},$$

where p assumes prime values of the form $3k-1$ and

$$\prod\nolimits_2 = \prod_p \frac{1+p^{-s}}{1-p^{-3s}},$$

where p assumes prime values of the form $3k+1$. We easily deduce from this that

$$\begin{cases} \displaystyle\sum_{k=1}^{n} t_k = o(n), \\ \displaystyle\sum_{k=1}^{n} T_k = o(n). \end{cases}$$

In other words, $\lambda(n)$ and $\tau(n)$ are divisible by 3 for almost all values of n. We can show by transcendental methods that

$$\begin{cases} \displaystyle\sum_{k=1}^{n} t_k = C\int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right), \\ \displaystyle\sum_{k=1}^{n} T_k = \frac{C}{3}\int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right), \end{cases} \tag{5.11.11}$$

where r is any positive number and

$$C = \frac{2^{1/2}}{3^{1/4}} \frac{1-7^{-2}}{1-7^{-3}} \frac{1-13^{-2}}{1-13^{-3}} \frac{1-19^{-2}}{1-19^{-3}} \cdots \frac{1}{\{(1-2^{-2})(1-5^{-2})(1-11^{-2})\cdots\}^{1/2}};$$

in both cases, $2, 5, 11, \ldots$ are primes of the form $3k-1$, and $7, 13, 19, \ldots$ are primes of the form $3k+1$.

5.12 Divisibility of $\tau(n)$

Further properties of $\tau(n)$

It is easy to see [from (5.11.2)] that

$$(q;q)_\infty^{24} = \frac{(q^2;q^2)_\infty^8}{(q;q^2)_\infty^8} + 32J = \psi^8(q) + 32J.$$

But [55, p. 139, Example (ii)]

$$q\psi^8(q) = \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^{2n}},$$

and

$$\sum_{n=1}^{\infty} n^4 q^n = \frac{q}{1-q^2} + 16J,$$

and

$$\sum_{n=1}^{\infty} n^8 q^n = \frac{q}{1-q^2} + 32J$$

[since

$$\sum_{n=1}^{\infty} n^4 q^n \equiv 1\cdot q + 0\cdot q^2 + 1\cdot q^3 + 0\cdot q^4 + \cdots = \frac{q}{1-q^2} \pmod{16},$$

since $n^4 \equiv 0, 1 \pmod{16}$, according as n is even or odd, and

$$\sum_{n=1}^{\infty} n^8 q^n \equiv 1\cdot q + 0\cdot q^2 + 1\cdot q^3 + 0\cdot q^4 + \cdots = \frac{q}{1-q^2} \pmod{32},$$

since $n^8 \equiv 0, 1 \pmod{32}$, according as n is even or odd.] It is easy to see from all these that

$$\tau(n) - n^3\sigma_1(n) \equiv 0 \pmod{16}; \quad \tau(n) - n^3\sigma_5(n) \equiv 0 \pmod{32}. \tag{5.12.1}$$

Again we have

$$(q;q)_\infty^{24} = \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3} + 27J.$$

But it can be shown that [58, p. 143, Theorem 8.7]

$$q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3} = \sum_{n=1}^\infty \frac{n^2q^n}{1+q^n+q^{2n}}. \tag{5.12.2}$$

Now it is easy to see that

$$\sum_{n=1}^\infty n^3q^n = \frac{q}{1+q+q^2} + 9J$$

and

$$\sum_{n=1}^\infty n^9q^n = \frac{q}{1+q+q^2} + 27J$$

[since

$$\begin{aligned}\sum_{n=1}^\infty n^3q^n &\equiv 1\cdot q - 1\cdot q^2 + 0\cdot q^3 + 1\cdot q^4 - 1\cdot q^5 + 0\cdot q^6 + \cdots \\ &= \frac{q-q^2}{1-q^3} = \frac{q}{1+q+q^2} \pmod 9,\end{aligned}$$

since $n^3 \equiv 0, 1, -1 \pmod 9$, according as $n \equiv 0, 1, -1 \pmod 3$, and

$$\sum_{n=1}^\infty n^9q^n \equiv 1\cdot q - 1\cdot q^2 + 0\cdot q^3 + 1\cdot q^4 - 1\cdot q^5 + 0\cdot q^6 + \cdots = \frac{q}{1+q+q^2} \pmod{27},$$

since $n^9 \equiv 0, 1, -1 \pmod{27}$, according as $n \equiv 0, 1, -1 \pmod 3$.] It follows that

$$\begin{cases} \tau(n) - n^2\sigma_1(n) \equiv 0 \pmod 9, \\ \tau(n) - n^2\sigma_7(n) \equiv 0 \pmod{27}. \end{cases} \tag{5.12.3}$$

It is easy to deduce from (5.2.1), (5.4.2), (5.12.1), and (5.12.3) that

$$\begin{cases} \tau(n) - n\sigma_1(n) \equiv 0 \pmod{30}, \\ \tau(n) - n^2\sigma_1(n) \equiv 0 \pmod{36}, \\ \tau(n) - n^3\sigma_1(n) \equiv 0 \pmod{48}, \\ \tau(n) - n^5\sigma_1(n) \equiv 0 \pmod{120}, \end{cases} \tag{5.12.4}$$

$$\begin{cases} \tau(n) - n\sigma_3(n) \equiv 0 \pmod{42}, \\ \tau(n) - n^2\sigma_3(n) \equiv 0 \pmod{60}, \\ \tau(n) - n^4\sigma_3(n) \equiv 0 \pmod{168}, \end{cases} \tag{5.12.5}$$

$$\begin{cases} \tau(n) - n^3\sigma_5(n) \equiv 0 \pmod{288}, \\ \tau(n) - n^2\sigma_7(n) \equiv 0 \pmod{540}, \\ \tau(n) - n\sigma_9(n) \equiv 0 \pmod{1050}. \end{cases} \tag{5.12.6}$$

Again it easily follows from the second equation in (5.9.1) that

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{691}. \tag{5.12.7}$$

It is easy to deduce from this that $\tau(n)$ is divisible by 691 for almost all values of n, and by transcendental methods that the number of values of n not exceeding n for which $\tau(n)$ is *not* divisible by 691 is of the form

$$C\int_1^n \frac{dx}{(\log x)^{1/690}} + O\left(\frac{n}{(\log n)^r}\right), \tag{5.12.8}$$

where C is a constant and r is any positive number.

It is easy to prove that

$$q(-q;-q)_\infty^{24} = q(q;q)_\infty^{24} + 48q^2(q^2;q^2)_\infty^{24} + 2^{12}q^4(q^4;q^4)_\infty^{24}. \tag{5.12.9}$$

[To prove (5.12.9), set, after Ramanujan,

$$f(-q) := (q;q)_\infty.$$

Thus, (5.12.9) can be written in the equivalent formulation

$$qf^{24}(q) = qf^{24}(-q) + 48q^2f^{24}(-q^2) + 2^{12}q^4f^{24}(-q^4). \tag{5.12.10}$$

To prove (5.12.10), we use the catalogue of evaluations for f found in Entry 12 of Chapter 17 in Ramanujan's second notebook [55, p. 124], in particular,

$$f(q) = \sqrt{z}2^{-1/6}\left\{x(1-x)/q\right\}^{1/24}, \quad f(-q) = \sqrt{z}2^{-1/6}(1-x)^{1/6}(x/q)^{1/24}, \tag{5.12.11}$$
$$f(-q^2) = \sqrt{z}2^{-1/3}\left\{x(1-x)/q\right\}^{1/12}, \quad f(-q^4) = \sqrt{z}2^{-2/3}(1-x)^{1/24}(x/q)^{1/6},$$

where $x = k^2$, with k being the modulus, and $z = (2/\pi)K$, with K being the complete elliptic integral of the first kind. Using these evaluations in (5.12.10), we easily verify its truth.] From this it is easy to deduce that

$$\tau(2n) + 24\tau(n) + 2^{11}\tau(\tfrac{1}{2}n) = 0, \tag{5.12.12}$$

where n is any integer and $\tau(x) = 0$ if x is not an integer.

[Recall that φ and ψ are defined in (5.11.1) and (5.11.2), respectively.] Again it is easy to prove that

$$q\psi^8(q)\varphi^{16}(-q) = qf^{24}(-q).$$

[To prove this identity, use (5.12.11) and the evaluations [55, p. 123, Entry 11(i), p. 122, Entry 10(ii)]

$$\psi(q) = \sqrt{\tfrac{1}{2}z}(x/q)^{1/8} \qquad \text{and} \qquad \varphi(-q) = \sqrt{z}(1-x)^{1/4}.\Big] \tag{5.12.13}$$

But [by the binomial theorem]

$$\varphi^{16}(-q) = -4\varphi^4(-q) + 16\varphi^2(-q) - 11 + 256J.$$

Hence

$$\begin{aligned} &qf^{24}(-q) \\ &\quad = 4\left\{1-\varphi^4(-q)\right\} q\psi^8(q) - 16\left\{1-\varphi^2(-q)\right\} q\psi^8(q) + q\psi^8(q) + 256J \\ &\quad = 4\left\{1-\varphi^4(-q)\right\} q\psi^4(q^2) - 16\left\{1-\varphi^2(-q)\right\} q\psi^4(q^2) + q\psi^8(q) + 256J. \end{aligned}$$

But

$$q\psi^8(q) = \sum_{n=0}^{\infty} \frac{n^3 q^n}{1-q^{2n}}, \tag{5.12.14}$$

$$q\psi^4(q^2) = \sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{4n+2}}, \tag{5.12.15}$$

$$q\psi^4(q^2)\varphi^4(-q) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3 q^n}{1-q^{2n}}, \tag{5.12.16}$$

$$\begin{aligned} q\psi^4(q^2)\varphi^2(-q) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2 q^n}{1+q^{2n}} \qquad (5.12.17) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)^2 q^{2n+1}}{1-q^{4n+2}} - \sum_{n=1}^{\infty} \frac{(2n)^2 q^{2n}}{1+q^{4n}} + 16J. \end{aligned}$$

[The identities (5.12.14) and (5.12.15) are, respectively, Examples (ii) and (iii) in Section 17 of Chapter 17 in Ramanujan's second notebook [55, p. 139].

By Entry 11(iii) in Chapter 17 of Ramanujan's second notebook [55, p. 123],

$$\psi(q^2) = \frac{1}{2}\sqrt{z}(x/q)^{1/4}. \tag{5.12.18}$$

It follows from (5.12.13) and (5.12.18) that

$$q\psi^4(q^2)\varphi^4(-q) = \frac{1}{16} z^4 x(1-x). \tag{5.12.19}$$

On the other hand, by Entries 14(ii), (ix) in Chapter 17 of the second notebook [55, p. 130],

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3 q^n}{1-q^{2n}} &= \sum_{n=1}^{\infty} (-1)^{n-1} n^3 \left(\frac{q^n}{1+q^n} + \frac{q^{2n}}{1-q^{2n}} \right) \\ &= \frac{1}{16} \left(1 + 16 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3 q^n}{1+q^n} - 1 + 16 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3 q^{2n}}{1-q^{2n}} \right) \\ &= \frac{1}{16} z^4 x(1-x). \qquad (5.12.20) \end{aligned}$$

The equality (5.12.16) is now a trivial consequence of (5.12.19) and (5.12.20).

To prove (5.12.17), first observe, by (5.12.13) and (5.12.18), that

$$q\psi^4(q^2)\varphi^2(-q) = \frac{1}{16}z^3x\sqrt{1-x}. \tag{5.12.21}$$

Next,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2q^n}{1+q^{2n}} &= -\sum_{n=1}^{\infty}\frac{4n^2q^{2n}}{1+q^{4n}} + \sum_{n=1}^{\infty}\frac{(2n+1)^2q^{2n+1}}{1+q^{4n+2}} \\ &= -8\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{1+q^{4n}} + \sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}} \\ &= -8\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{1+q^{4n}} + \frac{1}{16}z^3x, \end{aligned} \tag{5.12.22}$$

by Entry 17(ii) in Chapter 17 of Ramanujan's second notebook [55, p. 138]. To evaluate the sum on the far right side of (5.12.22), we apply the process of duplication [55, p. 125] to Entry 17(ii) cited above. Accordingly,

$$\begin{aligned}-8\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{1+q^{4n}} &= -\frac{1}{2}\left(\frac{1}{2}z(1+\sqrt{1-x})\right)^3\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2 \\ &= -\frac{1}{16}z^3x(1-\sqrt{1-x}), \end{aligned} \tag{5.12.23}$$

after simplification. Putting (5.12.23) into (5.12.22), we readily find that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2q^n}{1+q^{2n}} = \frac{1}{16}z^3x\sqrt{1-x}. \tag{5.12.24}$$

Combining (5.12.21) and (5.12.23), we complete the proof of the first part of (5.12.17).

To prove the second part of (5.12.17), it clearly suffices to prove that

$$S := \sum_{n=0}^{\infty}\frac{(2n+1)^2q^{2n+1}}{1+q^{4n+2}} \equiv \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2q^{2n+1}}{1-q^{4n+2}} =: T \pmod{16}. \tag{5.12.25}$$

Now,

$$\begin{aligned}S &= \sum_{n=0}^{\infty}\frac{(2n+1)^2q^{2n+1}}{1-q^{4n+2}} - 2\sum_{n=0}^{\infty}\frac{(2n+1)^2q^{6n+3}}{1-q^{8n+4}} \\ &= T + 2\sum_{n=0}^{\infty}\frac{(4n+3)^2q^{4n+3}}{1-q^{8n+6}} - 2\sum_{n=0}^{\infty}\frac{(2n+1)^2q^{6n+3}}{1-q^{8n+4}} \\ &\equiv T + 2\sum_{n=0}^{\infty}\frac{q^{4n+3}}{1-q^{8n+6}} - 2\sum_{n=0}^{\infty}\frac{q^{6n+3}}{1-q^{8n+4}} \pmod{16}\end{aligned}$$

$$= T + 2\sum_{n=0}^{\infty} \frac{q^{6n+3}}{1-q^{8n+4}} \pmod{16} - 2\sum_{n=0}^{\infty} \frac{q^{6n+3}}{1-q^{8n+4}} \pmod{16}$$
$$= T \pmod{16},$$

where in the antepenultimate line above we expanded the summands of the first series in geometric series and then reversed the order of summation. This completes the proof of (5.12.25), and hence the proof of the second equality of (5.12.17).]

It follows from all these that

$$q(q;q)_\infty^{24} = -3\sum_{n=0}^{\infty} \frac{(2n+1)^3 q^{2n+1}}{1-q^{4n+2}} + 5\sum_{n=1}^{\infty} \frac{(2n)^3 q^{2n}}{1-q^{4n}} - 12\sum_{n=0}^{\infty} \frac{(2n+1) q^{2n+1}}{1-q^{4n+2}}$$
$$+ 16\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)^2 q^{2n+1}}{1-q^{4n+2}} - 16\sum_{n=1}^{\infty} \frac{(2n)^2 q^{2n}}{1+q^{4n}} + 256J.$$

Now equating only the odd powers of q we obtain

$$\sum_{n=0}^{\infty} \tau(2n+1) q^{2n+1} = -3\sum_{n=0}^{\infty} \frac{(2n+1)^3 q^{2n+1}}{1-q^{4n+2}} + 16\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)^2 q^{2n+1}}{1-q^{4n+2}}$$
$$- 12\sum_{n=0}^{\infty} \frac{(2n+1) q^{2n+1}}{1-q^{4n+2}} + 256J.$$

But if n be of the form $4k+1$ then it is easy to see that

$$n^{11} + 3n^3 - 16n^2 + 12n \equiv 0 \pmod{256}.$$

Changing n to $-n$ in this formula, we see that if n be of the form $4k-1$ then

$$n^{11} + 3n^3 + 16n^2 + 12n \equiv 0 \pmod{256}.$$

It follows that

$$\sum_{n=0}^{\infty} \tau(2n+1) q^{2n+1} = \sum_{n=0}^{\infty} \frac{(2n+1)^{11} q^{2n+1}}{1-q^{4n+2}} + 256J.$$

In other words,

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{256} \tag{5.12.26}$$

for all *odd* values of n, while the formula (5.12.12) combined with this enables us to find the residues of $\tau(n)$ for modulus 2^{11} for *even* values of n. Thus

$$\tau(n) + 24\sigma_{11}(n) \equiv 0 \pmod{2048}$$

for all values of n.

It follows from (5.12.7) and (5.12.26) that

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{176896}$$

for all odd values of n.

5.13 Congruences Modulo 13

Modulus 13

In this case we start with the second series in (5.9.1) and the series

$$1 - 24\sum_{n=1}^{\infty} \frac{n^{13}q^n}{1-q^n} = Q^2R. \tag{5.13.1}$$

It follows from these that

$$Q^3 - 3R^2 = -2 + 13J; \qquad Q^2R = P + 13J. \tag{5.13.2}$$

Hence we have

$$\begin{aligned}(Q^3 - R^2)^7 &= -2(R^2-1)^7 + 13J \\ &= -5R^6(3R^2-2)^4 - 2R^4(3R^2-2)^3 + 6R^4(3R^2-2)^2 \\ &\quad - 6R^2(3R^2-2)^2 - 6R^2(3R^2-2) - 2(R^2-1) + 13J \\ &= -5P^6 - 2P^4Q + 6P^3R - 6P^2Q^2 - 6PQR - (Q^3-R^2) + 13J.\end{aligned} \tag{5.13.3}$$

But[17]

$$\begin{aligned}&5(P^6 - 15P^4Q + 40P^3R - 45P^2Q^2 + 24PQR) \\ &-(9Q^3 + 16R^2) = -248832\sum_{n=1}^{\infty} n^5\sigma_1(n)q^n, \\ &7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR) + (3Q^3 + 4R^2) \\ &= 41472\sum_{n=1}^{\infty} n^4\sigma_3(n)q^n, \\ &2(P^3R - 3P^2Q^2 + 3PQR) - (Q^3 + R^2) = -5184\sum_{n=1}^{\infty} n^3\sigma_5(n)q^n, \\ &9(PQ - R)^2 + 5(Q^3 - R^2) = 8640\sum_{n=1}^{\infty} n^3\sigma_7(n)q^n, \\ &5PQR - (3Q^3 + 2R^2) = -1584\sum_{n=1}^{\infty} n\sigma_9(n)q^n, \\ &Q^3 - R^2 = 1728\sum_{n=1}^{\infty} \tau(n)q^n;\end{aligned} \tag{5.13.4}$$

and it is obvious that

[17] See [275], where not all these equalities are given, but where the same methods can be employed to provide proofs.

$$(q;q)_\infty^{168} = \frac{(q^{169};q^{169})_\infty}{(q;q)_\infty} + 13J. \tag{5.13.5}$$

It is easy to see from all these that

$$\begin{aligned} q^7\frac{(q^{169};q^{169})_\infty}{(q;q)_\infty} = (q^{169};q^{169})_\infty \sum_{n=0}^\infty p(n)q^{n+7} = \sum_{n=1}^\infty \{n^5\sigma_1(n) - 4n^4\sigma_3(n) \\ -3n^3\sigma_5(n) + 6n^2\sigma_7(n) - 3n\sigma_9(n) + 3\tau(n)\}\, q^n + 13J. \end{aligned} \tag{5.13.6}$$

It is easy to see by actual calculation that $\tau(13) \equiv 8 \pmod{13}$ in virtue of (5.7.6) and hence

$$\tau(13n) - 8\tau(n) \equiv 0 \pmod{13}. \tag{5.13.7}$$

It follows from this and (5.13.6) that

$$\sum_{n=1}^\infty p(13n-7)q^n \ (q^{13};q^{13})_\infty = 11\sum_{n=1}^\infty \tau(n)q^n + 13J. \tag{5.13.8}$$

It is not necessary to know all the details above in order to prove (5.13.8). The proof can be very much simplified as follows; using (5.9.8) and (5.13.2) we can show that

$$(Q^3-R^2)^7 = q\frac{dJ}{dq} + 3(Q^3-R^2) + 13J. \tag{5.13.9}$$

It follows from this that

$$q^7\frac{(q^{169};q^{169})_\infty}{(q;q)_\infty} = q\frac{dJ}{dq} + 3\sum_{n=1}^\infty \tau(n)q^n + 13J. \tag{5.13.10}$$

From this we easily deduce (5.13.8).

Again picking out the terms q^{13}, $q^{26}, q^{39}, \dots$ in (5.13.8), we obtain [using the congruence $\tau(13n) \equiv 8\tau(n) \pmod{13}$]

$$\sum_{n=1}^\infty p(13^2n-7)q^n \ (q;q)_\infty = 10\sum_{n=1}^\infty \tau(n)q^n + 13J. \tag{5.13.11}$$

It follows from (5.13.6) that if

$$\sum_{n=1}^\infty \lambda(n)q^n = q^7\frac{(q^{169};q^{169})_\infty}{(q;q)_\infty},$$

so that $\lambda(n+7)$ is the number of partitions of n as the sum of integers which are not multiples of 169, then

$$\begin{aligned} &\lambda(n) - n^5\sigma_1(n) + 4n^4\sigma_3(n) + 3n^3\sigma_5(n) \\ &-6n^2\sigma_7(n) + 3n\sigma_9(n) - 3\tau(n) \equiv 0 \pmod{13}. \end{aligned} \tag{5.13.12}$$

The results analogous to (5.10.7)–(5.10.13) in the case of modulus 13 are

$$\tau(5^{12\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 5;

$$\tau(7n) \equiv 0 \pmod{13}$$

if n is not a multiple of 7;

$$\tau(11n) \equiv 0 \pmod{13}$$

if n is not a multiple of 11;

$$\tau(13n) - 8\tau(n) \equiv 0 \pmod{13}$$

if n is any integer;

$$\tau(19^{4\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 19;

$$\tau(23^{3\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 23;

$$\tau(29^{6\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 29; and so on.

5.14 Congruences for $p(n)$ Modulo 13

The formulae (5.13.8) and (5.13.11) can be written as

$$\sum_{n=0}^{\infty} p(13n+6)q^n = 11(q;q)_\infty^{11} + 13J; \tag{5.14.1}$$

and

$$\sum_{n=0}^{\infty} p(13^2n+162)q^n = 23(q;q)_\infty^{23} + 13J. \tag{5.14.2}$$

Since I began to write this paper I have found by a different method that if λ be any positive odd integer then

$$\sum_{n=0}^{\infty} p\left(13^\lambda n + \frac{11\cdot 13^\lambda+1}{24}\right) q^n = -2^{(5\lambda-3)/2}(q;q)_\infty^{11} + 13J; \tag{5.14.3}$$

and if λ be any positive even integer then

$$\sum_{n=0}^{\infty} p\left(13^\lambda n + \frac{23 \cdot 13^\lambda + 1}{24}\right) q^n = -2^{(5\lambda-2)/2}(q;q)_\infty^{23} + 13J. \tag{5.14.4}$$

I shall reserve the discussion of these results to another paper.

A number of results such as the following can be deduced from (5.14.3) and (5.14.4). [Note that

$$(q;q)_\infty^{11} = 1 - 11q + 44q^2 - 55q^3 - 110q^4 + 374q^5 - 143q^6 + \cdots$$

and

$$(q;q)_\infty^{23} = 1 - 23q + 230q^2 - 1265q^3 + 3795q^4 - 3519q^5 - 16445q^6 + \cdots .]$$

If λ be any positive odd integer then

$$\left\{\begin{array}{ll} p\left(\frac{11 \cdot 13^\lambda + 1}{24}\right) + 2^{(5\lambda-3)/2}, & p\left(\frac{35 \cdot 13^\lambda + 1}{24}\right) + 2^{(5\lambda-1)/2}, \\ p\left(\frac{59 \cdot 13^\lambda + 1}{24}\right) - 2^{(5\lambda+3)/2}, & p\left(\frac{83 \cdot 13^\lambda + 1}{24}\right) - 2^{(5\lambda+1)/2}, \\ p\left(\frac{107 \cdot 13^\lambda + 1}{24}\right) - 2^{(5\lambda+7)/2}, & p\left(\frac{131 \cdot 13^\lambda + 1}{24}\right) - 2^{(5\lambda+1)/2}, \\ p\left(\frac{155 \cdot 13^\lambda + 1}{24}\right), & \end{array}\right. \tag{5.14.5}$$

and so on are all divisible by 13; and if λ be any positive even integer then

$$\left\{\begin{array}{ll} p\left(\frac{23 \cdot 13^\lambda + 1}{24}\right) + 2^{(5\lambda-2)/2}, & p\left(\frac{47 \cdot 13^\lambda + 1}{24}\right) + 2^{(5\lambda+6)/2}, \\ p\left(\frac{71 \cdot 13^\lambda + 1}{24}\right) - 2^{(5\lambda+2)/2}, & p\left(\frac{95 \cdot 13^\lambda + 1}{24}\right) - 2^{(5\lambda+2)/2}, \\ p\left(\frac{119 \cdot 13^\lambda + 1}{24}\right) - 2^{(5\lambda-2)/2}, & p\left(\frac{143 \cdot 13^\lambda + 1}{24}\right) + 2^{(5\lambda+2)/2}, \\ p\left(\frac{167 \cdot 13^\lambda + 1}{24}\right), & \end{array}\right. \tag{5.14.6}$$

and so on are all divisible by 13. In other words, if n is fixed and $\lambda + n$ is an even integer then the residue of

$$p\left(\frac{13^\lambda(12n-1)+1}{24}\right) \tag{5.14.7}$$

for modulus 13 can be completely ascertained.

General Theory

5.15 Congruences to Further Prime Moduli

Modulus ϖ
where ϖ is a prime greater than 3

We start with the two series

$$v_{\varpi-1} + (-1)^{(\varpi-1)/2} 2(\varpi-1)\delta_{\varpi-1} \sum_{n=1}^{\infty} \frac{n^{\varpi-2} q^n}{1-q^n} = \sum K'_{\ell,m} Q^{\ell} R^m, \quad (5.15.1)$$

where $K'_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 1;$$

and

$$v_{\varpi+1} + (-1)^{(\varpi+1)/2} 2(\varpi+1)\delta_{\varpi+1} \sum_{n=1}^{\infty} \frac{n^{\varpi} q^n}{1-q^n} = \sum K_{\ell,m} Q^{\ell} R^m, \quad (5.15.2)$$

where $K_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi + 1.$$

In both series, v_s and δ_s are the numerator and the denominator of B_s in its lowest terms, where

$$B_2 = \frac{1}{6}, \quad B_4 = \frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad \ldots$$

are the Bernoulli numbers. Now by von Staudt's Theorem

$$\delta_{\varpi-1} \equiv 0 \pmod{\varpi},$$

and also we have

$$n^{\varpi} - n \equiv 0 \pmod{\varpi}.$$

And so the left-hand side in (5.15.1) is of the form

$$c' + \varpi J, \quad (5.15.3)$$

where c' is a constant integer, while that in (5.15.2) is of the form

$$k + cP + \varpi J, \quad (5.15.4)$$

where c and k are constant integers.

It appears that k can be taken as zero always. This involves the assertion that

$$6v_{\varpi+1} + (-1)^{(\varpi+1)/2}\frac{\varpi+1}{2}\delta_{\varpi+1} \equiv 0 \pmod{\varpi}. \tag{5.15.5}$$

I have not yet proved this result but in every particular case this can actually be found to be true. Thus (5.15.4) can be replaced by

$$cP + \varpi J. \tag{5.15.6}$$

Now using (5.15.3), (5.15.6), and (5.9.8), we can show in particular cases that

$$(Q^3 - R^2)^{(\varpi^2-1)/24} = q\frac{dJ}{dq} + (Q^3 - R^2)\sum k_{\ell,m}Q^\ell R^m + \varpi J, \tag{5.15.7}$$

where $k_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 13.$$

But it is obvious that

$$(q;q)_\infty^{\varpi^2-1} = \frac{(q^{\varpi^2};q^{\varpi^2})_\infty}{(q;q)_\infty} + \varpi J. \tag{5.15.8}$$

It follows from (5.15.7) and (5.15.8) that

$$q^{(\varpi^2-1)/24}\frac{(q^{\varpi^2};q^{\varpi^2})_\infty}{(q;q)_\infty} = q\frac{dJ}{dq} + (Q^3 - R^2)\sum k_{\ell,m}Q^\ell R^m + \varpi J, \tag{5.15.9}$$

where the remark about the summation in (5.15.7) applies here also. From this we can always deduce in every particular case that

$$\begin{aligned}\sum_{n=1}^{\infty} p\left(n\varpi + \varpi\left[\frac{\varpi}{24}\right] - \frac{\varpi^2-1}{24}\right) q^{n+[\varpi/24]}\,(q^\varpi;q^\varpi)_\infty \\ = (Q^3 - R^2)^{1+[\varpi/24]}\sum k_{\ell,m}Q^\ell R^m + \varpi J,\end{aligned} \tag{5.15.10}$$

where $k_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 13 \tag{5.15.11}$$

and $[t]$ denotes as usual the greatest integer in t.

Even though all these results are very difficult to prove in general, they can be easily proved when $\varpi \le 23$.

[The condition (5.15.11) should be replaced by

$$4\ell + 6m = \varpi - 13 - 12\left[\frac{\varpi}{24}\right]. \tag{5.15.12}$$

It is understandable that Ramanujan had missed the last term in (5.15.12), since he likely had calculated examples only for $\varpi \le 23$.]

5.16 Congruences for $p(n)$ Modulo 17, 19, 23, 29, or 31

Moduli 17, 19, and 23

In these cases we can easily prove that

$$\sum_{n=1}^{\infty} p(17n-12)q^n \ (q^{17};q^{17})_\infty = 7\sum_{n=1}^{\infty} \tau_2(n)q^n + 17J, \tag{5.16.1}$$

where

$$\sum_{n=1}^{\infty} \tau_2(n)q^n = Qq(q;q)_\infty^{24}; \tag{5.16.2}$$

$$\sum_{n=1}^{\infty} p(19n-15)q^n \ (q^{19};q^{19})_\infty = 5\sum_{n=1}^{\infty} \tau_3(n)q^n + 19J, \tag{5.16.3}$$

where

$$\sum_{n=1}^{\infty} \tau_3(n)q^n = Rq(q;q)_\infty^{24};$$

and

$$\sum_{n=1}^{\infty} p(23n-22)q^n \ (q^{23};q^{23})_\infty = \sum_{n=1}^{\infty} \tau_5(n)q^n + 23J, \tag{5.16.4}$$

where

$$\sum_{n=1}^{\infty} \tau_5(n)q^n = QRq(q;q)_\infty^{24}. \tag{5.16.5}$$

I have stated without proof in my previous paper[18] that

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\tau_2(n)}{n^s} = \prod_p \frac{1}{1-\tau_2(p)p^{-s}+p^{15-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^s} = \prod_p \frac{1}{1-\tau_3(p)p^{-s}+p^{17-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_4(n)}{n^s} = \prod_p \frac{1}{1-\tau_4(p)p^{-s}+p^{19-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_5(n)}{n^s} = \prod_p \frac{1}{1-\tau_5(p)p^{-s}+p^{21-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_7(n)}{n^s} = \prod_p \frac{1}{1-\tau_7(p)p^{-s}+p^{25-2s}}, \end{cases} \tag{5.16.6}$$

[18] See [275, Equation (108)].

where

$$\sum_{n=1}^{\infty} \tau_4(n)q^n = Q^2 q(q;q)_\infty^{24}$$

and

$$\sum_{n=1}^{\infty} \tau_7(n)q^n = Q^2 Rq(q;q)_\infty^{24},$$

and p assumes all prime values. All these seem to be capable of proof as the case of $\tau(n)$ by Mordell's method.[19]

Now using (5.16.6) we can deduce from (5.16.1), (5.16.3), and (5.16.4) that

$$\sum_{n=1}^{\infty} p(n17^2 - 12)q^n \ (q;q)_\infty = c_2 \sum_{n=1}^{\infty} \tau_2(n)q^n + 17J, \tag{5.16.7}$$

$$\sum_{n=1}^{\infty} p(n19^2 - 15)q^n \ (q;q)_\infty = c_3 \sum_{n=1}^{\infty} \tau_3(n)q^n + 19J, \tag{5.16.8}$$

and

$$\sum_{n=1}^{\infty} p(n23^2 - 22)q^n \ (q;q)_\infty = c_5 \sum_{n=1}^{\infty} \tau_5(n)q^n + 23J, \tag{5.16.9}$$

where c_2, c_3, and c_5 are constants.

I have found that there are formulae quite analogous to those for modulus 13 even in these cases. I shall reserve the discussion of these as well as those for higher primes to another paper; but I shall consider in the II part of this paper the analogous formulae for the smaller primes 5, 7, and 11.

The corresponding formulae for primes greater than 23 are not quite analogous. For instance in the cases of

Moduli 29 and 31

we have

$$\sum_{n=1}^{\infty} p(29n - 6)q^{n+1} \ (q^{29};q^{29})_\infty = 8 \sum_{n=1}^{\infty} \Omega_2(n)q^n + 29J, \tag{5.16.10}$$

where

$$\sum_{n=1}^{\infty} \Omega_2(n)q^n = Qq^2(q;q)_\infty^{48};$$

and

$$\sum_{n=1}^{\infty} p(31n - 9)q^{n+1} \ (q^{31};q^{31})_\infty = 10 \sum_{n=1}^{\infty} \Omega_3(n)q^n + 31J, \tag{5.16.11}$$

where

[19] loc. cit.

$$\sum_{n=1}^{\infty} \Omega_3(n)q^n = Rq^2(q;q)_\infty^{48}.$$

The functions

$$\sum_{n=1}^{\infty} \frac{\Omega_2(n)}{n^s}, \qquad \sum_{n=1}^{\infty} \frac{\Omega_3(n)}{n^s}$$

are obviously not capable of a single product as in (5.16.6); but they are, as a matter of fact, the differences of two such products.

5.17 Divisibility of $\tau(n)$ by 23

I have not yet investigated the residues of $\tau(n)$ for other moduli besides what was stated before, but the case 23 seems to be (comparatively) simple. For it appears that if

$$\sum_{n=1}^{\infty} \lambda(n)q^n = q(q;q)_\infty(q^{23};q^{23})_\infty,$$

so that

$$\tau(n) - \lambda(n) \equiv 0 \pmod{23}, \tag{5.17.1}$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{1}{1-23^{-s}} \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3, \tag{5.17.2}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1}{1-p^{-2s}},$$

with p assuming all prime values of the form[20]

$$p \equiv 5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22 \pmod{23} \tag{5.17.3}$$

and

$$\prod\nolimits_2 = \prod_p \frac{1}{1+p^{-s}+p^{-2s}},$$

with p assuming all prime values of the form[21]

$$p \equiv 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 \pmod{23}, \tag{5.17.4}$$

except of the form $23a^2+b^2$, and

$$\prod\nolimits_3 = \prod_p \frac{1}{(1-p^{-s})^2},$$

[20] This can be written as $p^{11} \equiv -1 \pmod{23}$.
[21] This can be written as $p^{11} \equiv 1 \pmod{23}$.

with p assuming all primes of the form $23a^2+b^2$. Thus $\lambda(n)$ can be completely determined, and consequently the residues of $\tau(n)$ for modulus 23 can be completely ascertained.

Suppose now that

$$\begin{cases} t_n = 0, & \tau(n) \equiv 0 \pmod{23}; \\ t_n = 1, & \tau(n) \not\equiv 0 \pmod{23}. \end{cases} \tag{5.17.5}$$

Then it is easy to see from ( ) that

$$\sum_{n=1}^{\infty} \frac{t_n}{n^s} = \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3, \tag{5.17.6}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1}{1-p^{-2s}},$$

where p assumes all primes of the form (5.17.3),

$$\prod\nolimits_2 = \prod_p \frac{1+p^{-s}}{1-p^{-3s}},$$

where p assumes all primes of the form (5.17.4) except those of the form $23a^2+b^2$, and

$$\prod\nolimits_3 = \prod_p \frac{1-p^{-22s}}{(1-p^{-s})(1-p^{-23s})},$$

with p assuming all primes of the form $23a^2+b^2$.

It is easy to prove from (5.17.6) by quite elementary methods that

$$\sum_{k=1}^{n} t_k = o(n); \tag{5.17.7}$$

and by transcendental methods that

$$\sum_{k=1}^{n} t_k = C\int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right), \tag{5.17.8}$$

where r is any positive number and

$$\begin{aligned} C = {} & \frac{66^{1/2}}{23^{3/4}} \frac{1-2^{-2}}{1-2^{-3}} \frac{1-3^{-2}}{1-3^{-3}} \frac{1-13^{-2}}{1-13^{-3}} \frac{1-29^{-2}}{1-29^{-3}} \cdots \\ & \times \frac{1}{\{(1-5^{-2})(1-7^{-2})(1-11^{-2})(1-17^{-2})\cdots\}^{1/2}} \\ & \times \frac{1-59^{-22}}{1-59^{-23}} \frac{1-101^{-22}}{1-101^{-23}} \frac{1-167^{-22}}{1-167^{-23}} \cdots, \end{aligned}$$

with $2, 3, 13, \ldots$ being primes of the form (5.17.4) except those of the form $23a^2 + b^2$, and $5, 7, 11, 17, \ldots$ being primes of the form (5.17.3), and $59, 101, 167, \ldots$ being those of the form $23a^2 + b^2$. Thus we see that $\tau(n)$ is almost always divisible by 23.

We have also shown that among the values of $\tau(n)$, multiples of 3, 7, and 23 are more or less equally numerous, while the multiples of 5 are less numerous than these and multiples of 2 are the most numerous.

Since

$$\begin{aligned}(1-p^{-s})(1-p^{11-s}) &= (1-p^{-2s}) - (p^{11}+1)(p^{-s}-p^{-2s}) \\ &= (1-p^{-s})^2 - (p^{11}-1)(p^{-s}-p^{-2s}),\end{aligned}$$

it is easy to see from (5.17.2) and (5.12.7) that if the prime divisors of n are of the form (5.17.3) or of the form $23a^2 + b^2$ then[22]

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{15893}, \tag{5.17.9}$$

where $15893 = 23 \cdot 691$. If, in addition to the restrictions on the values of n in (5.17.9), we impose the restriction that n is odd also, then it follows from (5.12.26) that

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{4068608},$$

with $4068608 = 23 \cdot 256 \cdot 691$.

5.18 The Congruence $p(121n-5) \equiv 0 \pmod{121}$

Modulus 121

The case of modulus ϖ^2 seems to be much more complicated than the case of modulus ϖ even though the method is practically the same as may be seen from the case of modulus 49. I shall now consider the case of modulus 121.

It is easy to show by using (5.9.2) that

$$\begin{aligned}(Q^3-R^2)^5 &= P(Q^3-3R^2)(3P^3-PQ+4R) + 4QR(4P^3Q-3P^2R+2QR) \\ &\quad - 26P^5 + 23P^3Q + 16P^2R - 22PQ^2 + 9QR + 121J.\end{aligned} \tag{5.18.1}$$

From this we can deduce that

$$\begin{aligned}q^5\frac{(q^{11};q^{11})_\infty^{11}}{(q;q)_\infty} = \sum_{n=1}^{\infty}\Big[&n^4\left\{a_1\sigma_1(n)+b_1\sigma_{11}(n)\right\} + n^3\left\{a_2\sigma_3(n)+b_2\sigma_{13}(n)\right\} \\ &+ n^2\left\{a_3\sigma_5(n)+b_3\sigma_{15}(n)\right\} + n\left\{a_4\sigma_7(n)+b_4\sigma_{17}(n)\right\} \\ &+ c_1n^2\tau_2(n) + c_2n\tau_3(n) + c_3\tau_4(n)\Big]\,q^n + 121J,\end{aligned} \tag{5.18.2}$$

[22] Some may be of one form and some may be of the other form.

where the a's, b's, and c's are constant integers and $\tau_2(n)$, $\tau_3(n)$, and $\tau_4(n)$ are the same as in (5.16.6). But it is easy to show that

$$\left.\begin{cases} \tau_2(n) - n\sigma_3(n) \\ \tau_3(n) - n\sigma_5(n) \\ \tau_4(n) - n\sigma_7(n) \end{cases} \right. \equiv 0 \pmod{11}. \tag{5.18.3}$$

It is easy to see from (5.16.6) that

$$\tau_4(11n) - \tau_4(11)\tau_4(n) \equiv 0 \pmod{121}, \tag{5.18.4}$$

and by actual calculation we find that

$$\tau_4(11) \equiv 0 \pmod{11}. \tag{5.18.5}$$

It is also obvious that

$$\sigma_{17}(n) - \sigma_7(n) \equiv 0 \pmod{11}. \tag{5.18.6}$$

Now remembering (5.18.3)–(5.18.6) and picking out the terms $q^{11}, q^{22}, q^{33}, \dots$ in ( ) we obtain

$$\sum_{n=1}^{\infty} p(11n-5)q^n \; (q^{11};q^{11})_\infty = 11 \sum_{n=1}^{\infty} n\sigma_7(n)q^n + 121J. \tag{5.18.7}$$

It follows from this that

$$p(121n-5) \equiv 0 \pmod{121}, \tag{5.18.8}$$

and

$$\begin{aligned} &p(11n-5) - p(11n-126) - p(11n-247) \\ &\quad + p(11n-610) + \cdots - 11n\sigma_7(n) \equiv 0 \pmod{121}. \end{aligned} \tag{5.18.9}$$

5.19 Divisibility of $\tau(n)$ for Almost All Values of n

In concluding the first part of this paper I shall consider the numbers which are the divisors of $\tau(n)$ for almost all values of n.

Suppose that $\varpi_1, \varpi_2, \varpi_3, \dots$ are an infinity of primes such that

$$\sum_{n=1}^{\infty} \frac{1}{\varpi_n} \tag{5.19.1}$$

is a *divergent* series and also suppose that $a_2, a_3, a_5, a_7, \dots$ assume some or all of the positive integers (including zero) but that $a_{\varpi_1}, a_{\varpi_2}, a_{\varpi_3}, \dots$ *never*

assume the value unity. Then it is easy to show that the number of numbers of the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots \tag{5.19.2}$$

not exceeding n is of the form

$$o(n). \tag{5.19.3}$$

In particular, if a_ϖ *never* assumes the value unity for all prime values of ϖ of the form

$$\varpi \equiv c \pmod{k}, \tag{5.19.4}$$

where c and k are any two integers which are prime to each other, then the number of numbers of the form (5.19.2) is of the form

$$o(n) \tag{5.19.5}$$

and more accurately is of the form

$$O\left(\frac{n}{(\log n)^{1/(k-1)}}\right), \tag{5.19.6}$$

where k is the same as in (5.19.4).

Thus for example if s be an odd positive integer, the number of values of n not exceeding n for which $\sigma_s(n)$ is *not* divisible by k, where k is any positive integer, is of the form

$$o(n) \tag{5.19.7}$$

and more accurately is of the form

$$O\left(\frac{n}{(\log n)^{1/(k-1)}}\right). \tag{5.19.8}$$

For if n be written in the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots$$

then we have

$$\sigma_s(n) = \prod_p \frac{p^{s(1+a_p)} - 1}{p^s - 1}, \qquad p = 2, 3, 5, 7, 11, \ldots.$$

Since s is odd, $\sigma_s(n)$ is divisible by k at any rate when $a_p = 1$ for all values of p of the form

$$p \equiv -1 \pmod{k},$$

and hence the results stated follow. Thus we see that if s is odd, $\sigma_s(n)$ is divisible by any given integer for almost all values of n.

It follows from all these and the formulae in Sections 4, 8, 12, and 17 that

$$\tau(n) \equiv 0 \pmod{2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 691} \tag{5.19.9}$$

for almost all values of n.

It appears that $\tau(n)$ is almost always divisible by any power of 2, 3, and 5. It also appears from Section 9 that there are reasons to suppose that $\tau(n)$ is almost always divisible by 11 also. But I have no evidence at present to say anything about the other powers of 7 and other primes one way or the other.

Among the values of $\tau(n)$, multiples of 2, 3, 5, 7, and 23 are very numerous from the beginning, but multiples of 691 begin at a very late stage. For instance, $\tau(n)$ is divisible by 23 for 132 values of n not exceeding 200, while the first value of n for which $\tau(n)$ is divisible by 691 is 1381, and this is the only such value of n among the first 5000 values.

II

5.20 The Congruence $p(5n+4) \equiv 0 \pmod 5$, Revisited

Moduli 5 and 25

In this second part we shall use J_1, J_2, J_3 and G_1, G_2, G_3 to denote functions of q with integral powers of q as well as integral coefficients. These are the same functions in the same section, unlike J. We shall also use J in the same sense as in the first part.

We start with Euler's identity

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \tag{5.20.1}$$

and Jacobi's identity

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{5.20.2}$$

It is easy to see from (5.20.1) that

$$\frac{(q^{1/5};q^{1/5})_\infty}{(q^5;q^5)_\infty} = J_1 - q^{1/5} + q^{2/5} J_2. \tag{5.20.3}$$

Now cubing both sides, we obtain

$$\begin{aligned}\frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/10}}{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{5n(n+1)/2}} &= (J_1^3 - 3J_2^2 q) - q^{1/5}(3J_1^2 - J_2^3 q) \\ &\quad + 3J_1 q^{2/5}(1+J_1J_2) - q^{3/5}(1+6J_1J_2) + 3J_2 q^{4/5}(1+J_1J_2).\end{aligned} \tag{5.20.4}$$

But it is easy to see that

$$\frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/10}}{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{5n(n+1)/2}} = G_1 + q^{1/5}G_2 + 5q^{3/5}. \tag{5.20.5}$$

Hence

$$J_1(1+J_1J_2)=0, \qquad 1+6J_1J_2=-5, \qquad J_2(1+J_1J_2)=0. \tag{5.20.6}$$

These three equations give one and the same relation between J_1 and J_2, viz.

$$J_1J_2=-1.$$

Using this we obtain

$$\frac{(q^5;q^5)_\infty}{(q^{1/5};q^{1/5})_\infty}=\frac{1}{J_1-q^{1/5}+q^{2/5}J_2}=\frac{(J_1^4+3J_2q)+q^{1/5}(J_1^3+2J_2^2q)}{J_1^5-11q+q^2J_2^5} \\ +\frac{q^{2/5}(2J_1^2+J_2^3q)+q^{3/5}(3J_1+J_2^4q)+5q^{4/5}}{J_1^5-11q+q^2J_2^5} \tag{5.20.7}$$

by rationalizing the denominator $J_1-q^{1/5}+q^{2/5}J_2$. It follows from (5.20.7) that

$$\sum_{n=0}^{\infty}p(5n+4)q^n\ (q^5;q^5)_\infty=\frac{5}{J_1^5-11q+q^2J_2^5}. \tag{5.20.8}$$

But we see from (5.20.3) that

$$\frac{(\omega q^{1/5};\omega q^{1/5})_\infty}{(q^5;q^5)_\infty}=J_1-\omega q^{1/5}+\omega^2q^{2/5}J_2, \tag{5.20.9}$$

where $\omega^5=1$. Now writing the five values of ω in (20.21) and multiplying them together we obtain

$$\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}=J_1^5-11q+q^2J_2^5. \tag{5.20.10}$$

It follows from this and (5.20.8) that

$$\sum_{n=0}^{\infty}p(5n+4)q^n=5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{5.20.11}$$

It follows that

$$p(5n-1)\equiv 0 \pmod 5. \tag{5.20.12}$$

Again the right-hand side in (5.20.11) is of the form

$$5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty}+25J.$$

It follows from this and (5.20.12) that the coefficients of q^4, q^9, $q^{14}, \ldots$ in this are all multiples of 25 and consequently the coefficient of q^{5n-1} in the left-hand side of (5.20.11) is a multiple of 25. In other words,

$$p(25n-1)\equiv 0 \pmod{25}. \tag{5.20.13}$$

It follows also from (5.20.11) that

$$\sum_{n=0}^{\infty}p(5n+4)q^n=5(q;q)_\infty^{19}+125J.$$

5.21 The Congruence $p(25n+24) \equiv 0 \pmod{25}$, Revisited

Modulus 125

Changing q to $q^{1/5}$ in (5.20.11) and arguing as before, using (5.20.7) and (5.20.10) we find that

$$\sum_{n=0}^{\infty} p(25n+24)q^n = 5^2 \cdot 63 \frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^7} + 5^5 \cdot 52q \frac{(q^5;q^5)_\infty^{12}}{(q;q)_\infty^{13}} + 5^7 \cdot 63q^2 \frac{(q^5;q^5)_\infty^{18}}{(q;q)_\infty^{19}} + 5^{10} \cdot 6q^3 \frac{(q^5;q^5)_\infty^{24}}{(q;q)_\infty^{25}} + 5^{12} q^4 \frac{(q^5;q^5)_\infty^{30}}{(q;q)_\infty^{31}}. \tag{5.21.1}$$

Now

$$\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^7} = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} (q^5;q^5)_\infty^4 + 5J \quad \text{etc.}, \tag{5.21.2}$$

and the coefficients of q^{5n-1}, q^{5n-2}, q^{5n-3} in $\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$ are easily seen to be zero or multiples of 5. It follows that the coefficients of q^{5n-1}, q^{5n-2}, q^{5n-3} in the left-hand side of (5.21.1) are multiples of 125. In other words,

$$p(125n-1), \ p(125n-26), \ p(125n-51) \equiv 0 \pmod{125}. \tag{5.21.3}$$

It is also easy to see from (5.21.1) that

$$\sum_{n=0}^{\infty} p(25n+24)q^n = 75(q;q)_\infty^{23} + 125J. \tag{5.21.4}$$

The right-hand side in (5.21.4) can be written in the form

$$75 \frac{(q;q)_\infty^{48}}{(q^{25};q^{25})_\infty} + 125J. \tag{5.21.5}$$

But it is easy to show that

$$(Q^3 - R^2)^2 = -2 \sum_{n=1}^{\infty} (n^3 - n)\sigma_1(n) q^n + 5J. \tag{5.21.6}$$

[To prove (5.21.6), we need Ramanujan's formula [275, Table III], [281, p. 142],

$$6912 \sum_{n=1}^{\infty} n^3 \sigma_1(n) q^n = 6P^2Q - 8PR + 3Q^2 - P^4.$$

Using this formula together with (5.1.7) and (5.1.5), we can readily prove that

$$2\sum_{n=1}^{\infty}(n^3-n)\sigma_1(n)q^n = -1+2P^2-P^4+5J.$$

On the other hand, from (5.1.5) and (5.1.6),

$$(Q^3-R^2)^2 = 1-2P^2+P^4+5J.$$

The last two equalities yield (5.21.6).] It follows that

$$\sum_{n=0}^{\infty} p(25n+24)q^{n+2}\,(q^{25};q^{25})_\infty = 25\sum_{n=1}^{\infty}(n^3-n)\sigma_1(n)q^n + 125J. \quad (5.21.7)$$

In other words,

$$\begin{aligned} &p(25n-26)-p(25n-651)-p(25n-1276)\\ &+p(25n-3151)+\cdots-25(n^3-n)\sigma_1(n)\equiv 0 \pmod{125}. \end{aligned} \quad (5.21.8)$$

$p(199)$ is the coefficient of q^7 in (5.21.2).

$$\begin{aligned} p(199) &= 5^2\cdot 63\cdot 12195+5^2\cdot 52\cdot 60541+5^7\cdot 63\cdot 66862\\ &\quad +5^{10}\cdot 6\cdot 29575+5^{12}\cdot 6448 = 3646072432125. \end{aligned}$$

5.22 Congruences for $p(n)$ Modulo Higher Powers of 5

Moduli $5^4, 5^5, \ldots$

Changing again q to $q^{1/5}$ in (5.21.1) and arguing as before using (5.20.7) and (5.20.10) we can show that

$$\sum_{n=0}^{\infty} p(125n+99)q^n = \sum_{r=1}^{25} a_r \frac{(q^5;q^5)_\infty^{6r-1}}{(q;q)_\infty^{6r}}, \quad (5.22.1)$$

where the a's are positive integers such that $a_1 = p(99) = 5^3\cdot 1353839$ and $a_2, a_3, a_4, \ldots$ contain higher powers of 5 than a_1 as factors. It is easy to see from this that

$$\sum_{n=0}^{\infty} p(125n+99)q^n = 4\cdot 5^3(q;q)_\infty^{19}+5^4J. \quad (5.22.2)$$

In this way arguing as before, we can show that if λ be any positive odd integer, then

$$\sum_{n=0}^{\infty} p\left(\frac{19\cdot 5^\lambda+1}{24}+5^\lambda n\right)q^n = \sum_{\nu=1}^{5^{\lambda-1}} a_\nu \frac{(q^5;q^5)_\infty^{6\nu-1}}{(q;q)_\infty^{6\nu}}, \quad (5.22.3)$$

where the a's are positive integers such that $a_2, a_3, a_4, \ldots$ contain higher powers of 5 than a_1 as factors; and if λ be a positive even integer then

$$\sum_{n=0}^{\infty} p\left(\frac{23 \cdot 5^\lambda + 1}{24} + 5^\lambda n\right) q^n = \sum_{\nu=1}^{5^{\lambda-1}} a_\nu \frac{(q^5;q^5)_\infty^{6\nu}}{(q;q)_\infty^{6\nu+1}}, \tag{5.22.4}$$

where the a's have the same properties as before. We deduce from (5.22.3) and (5.22.4) that if λ is a positive odd integer then

$$\sum_{n=0}^{\infty} p\left(\frac{19 \cdot 5^\lambda + 1}{24} + 5^\lambda n\right) q^n = c_\lambda \cdot 5^\lambda (q;q)_\infty^{19} + 5^{\lambda+1} J, \tag{5.22.5}$$

and if λ is a positive even integer then

$$\sum_{n=0}^{\infty} p\left(\frac{23 \cdot 5^\lambda + 1}{24} + 5^\lambda n\right) q^n = c_\lambda \cdot 5^\lambda (q;q)_\infty^{23} + 5^{\lambda+1} J, \tag{5.22.6}$$

where c_λ in both cases is a constant.

We easily deduce from these that if λ is an odd integer greater than 1, then

$$\left.\begin{array}{l} p\left(5^\lambda n - \dfrac{5^{\lambda-1}-1}{24}\right) \\ p\left(5^\lambda n - \dfrac{5^{\lambda+1}-1}{24}\right) \\ p\left(5^\lambda n - \dfrac{49 \cdot 5^{\lambda-1}-1}{24}\right) \end{array}\right\} \equiv 0 \pmod{5^\lambda}, \tag{5.22.7}$$

and if λ is a positive even integer, then

$$p\left(5^\lambda n - \frac{5^\lambda - 1}{24}\right) \equiv 0 \pmod{5^\lambda}. \tag{5.22.8}$$

(λ may also be 1 in the second congruence of (5.22.7).)

5.23 Congruences for $p(n)$ Modulo Higher Powers of 5, Continued

We have seen that we can take $c_1 = 1$, $c_2 = -2$, $c_3 = 4$ in (5.22.5) and (5.22.6). It appears from Section 22 that c_λ may probably be some simple function such as $(-2)^\lambda$. If we calculate a few more values of c_λ, we can definitely know what it is. Then we can make use of the formulae (5.22.5) and (5.22.6) to determine completely the residues of

$$p\left(5^\lambda n - \frac{5^{\lambda+1}-1}{24}\right)$$

for odd values of λ and those of

$$p\left(5^\lambda n - \frac{5^\lambda - 1}{24}\right)$$

for even values of λ for modulus $5^{\lambda+1}$. [To determine these residues, we need the expansions

$$\begin{aligned}(q;q)_\infty^{19} = 1 &- 19q + 152q^2 - 627q^3 + 1140q^4 + 988q^5 - 9063q^6 \\ &+ 14212q^7 + 7410q^8 - 44270q^9 + 22781q^{10} + 38114q^{11} \\ &+ 36176q^{12} - 137256q^{13} - 154850q^{14} + 480605q^{15} + \cdots\end{aligned}$$

and

$$\begin{aligned}(q;q)_\infty^{23} = 1 &- 23q + 230q^2 - 1265q^3 + 3795q^4 - 3519q^5 - 16445q^6 \\ &+ 64285q^7 - 64515q^8 - 120175q^9 + 354706q^{10} - 123763q^{11} \\ &- 407560q^{12} - 48530q^{13} + 817190q^{14} + 1464341q^{15} + \cdots\end{aligned}$$

in, respectively, (5.22.5) and (5.22.6).] Thus for instance it follows immediately from (5.22.5) and (5.22.6) that if λ is an odd integer, then

$$\begin{aligned}
&p\left(5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 5^\lambda c_\lambda, && p\left(2\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 5^\lambda c_\lambda,\\
&p\left(3\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^\lambda c_\lambda, && p\left(4\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) + 2\cdot 5^\lambda c_\lambda,\\
&p\left(5\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right), && p\left(6\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^\lambda c_\lambda,\\
&p\left(7\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^\lambda c_\lambda, && p\left(8\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^\lambda c_\lambda,\\
&p\left(9\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right), && p\left(10\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right),\\
&p\left(11\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 5^\lambda c_\lambda, && p\left(12\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) + 5^\lambda c_\lambda,\\
&p\left(13\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) - 5^\lambda c_\lambda, && p\left(14\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right) + 5^\lambda c_\lambda,\\
&p\left(15\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right), && p\left(16\cdot 5^\lambda - \frac{5^{\lambda+1}-1}{24}\right),
\end{aligned}$$

and so on are all multiples of $5^{\lambda+1}$; and if λ is an even integer, then

$$p\left(5^\lambda - \frac{5^\lambda - 1}{24}\right) - 5^\lambda c_\lambda, \qquad p\left(2\cdot 5^\lambda - \frac{5^\lambda - 1}{24}\right) - 2\cdot 5^\lambda c_\lambda,$$

$$p\left(3\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), \qquad p\left(4\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),$$
$$p\left(5\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), \qquad p\left(6\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 5^\lambda c_\lambda,$$
$$p\left(7\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), \qquad p\left(8\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),$$
$$p\left(9\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), \qquad p\left(10\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),$$
$$p\left(11\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 5^\lambda c_\lambda, \qquad p\left(12\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 2\cdot 5^\lambda c_\lambda,$$
$$p\left(13\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), \qquad p\left(14\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),$$
$$p\left(15\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), \qquad p\left(16\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 5^\lambda c_\lambda,$$

and so on are all multiples of $5^{\lambda+1}$.

5.24 The Congruence $p(7n+5) \equiv 0 \pmod 7$

Moduli 7 and 49

It is easy to see from (5.20.1) that

$$\frac{(q^{1/7};q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3. \tag{5.24.1}$$

Now cubing both sides we obtain

$$\begin{aligned}
&\frac{\sum_{n=0}^\infty (-1)^n(2n+1)q^{n(n+1)/14}}{\sum_{n=0}^\infty (-1)^n(2n+1)q^{7n(n+1)/2}} \\
&= (J_1^3 + 3J_2^2J_3q - 6J_1J_3q) + q^{1/7}(3J_1^2J_2 - 6J_2J_3q + J_3^2q^2) \\
&\quad + 3q^{2/7}(J_1J_2^2 - J_1^2 + J_3q) + q^{3/7}(J_2^3 - 6J_1J_2 + 3J_1J_3^2q) \\
&\quad + 3q^{4/7}(J_1 - J_2^2 + J_2J_3^2q) + 3q^{5/7}(J_2 + J_1^2J_3 - J_3^2q) + q^{6/7}(6J_1J_2J_3 - 1).
\end{aligned}$$

But it is easy to see that

$$\frac{\sum_{n=0}^\infty (-1)^n(2n+1)q^{n(n+1)/14}}{\sum_{n=0}^\infty (-1)^n(2n+1)q^{7n(n+1)/2}} = G_1 + q^{1/7}G_2 + q^{3/7}G_3 - 7q^{6/7}.$$

Hence

$$\begin{cases} J_1J_2^2 - J_1^2 + J_3q = 0, \\ J_1 - J_2^2 + J_2J_3^2q = 0, \\ J_2 + J_1^2J_3 - J_3^2q = 0, \\ \qquad 6J_1J_2J_3 - 1 = -7. \end{cases} \tag{5.24.2}$$

All these four equations give the two independent relations

$$J_1 J_2 J_3 = -1, \qquad \frac{J_1^2}{J_3} + \frac{J_2}{J_3^2} = q. \tag{5.24.3}$$

Now write (5.24.1) in the form

$$\frac{(\omega q^{1/7}; \omega q^{1/7})_\infty}{(q^7; q^7)_\infty} = J_1 + \omega q^{1/7} J_2 - \omega^2 q^{2/7} + \omega^5 q^{5/7} J_3, \tag{5.24.4}$$

where $\omega^7 = 1$. Again writing the seven values of ω in (5.24.4) and multiplying them together and using (5.24.3) we can show that

$$J_1^7 + J_2^7 q + J_3^7 q^5 = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} + 14q \frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} + 57q^2, \tag{5.24.5}$$

$$J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q, \tag{5.24.6}$$

$$J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 5q. \tag{5.24.7}$$

Again taking the reciprocals of both sides in (5.24.1) and rationalizing the denominator, as we also did in Section 20, we can show that

$$\sum_{n=0}^\infty p(7n+5) q^n = 7 \frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{5.24.8}$$

$$7^2 \cdot 2546, \quad 7^4 \cdot 48934, \quad 7^5 \cdot 1418989, \quad 7^7 \cdot 2488800.$$

$$\{p(47)q^3 + \cdots\}(q^{49}; q^{49})_\infty = 7 \sum_{n=1}^\infty \left\{22n^4 \sigma_0(n) - 21n^2 \sigma_1(n) - \tau(n)\right\} q^n + 7^3 J.$$

5.25 Commentary

In the following commentary, the section numbers correspond to the sections with the same numbers in Ramanujan's manuscript. However, the designation, Section 0, for the first batch of Ramanujan's insertions is due to Ono and the second author.

K.G. Ramanathan [270] also observed that $\tau(n)$ is even unless n is an odd square. For an extensive discussion of this result, see a paper by M.R. Murty, V.K. Murty, and T.N. Shorey [236], and for a generalization to the coefficients of other modular forms, see another paper by the Murty brothers [235].

The congruences $\tau(7n-r) \equiv 0 \pmod 7$, $r = 0, 1, 2, 4$, were evidently first proved in print by J.R. Wilton [341]. Hardy, in his book *Ramanujan* [166, pp. 165–166], also gives a proof, as does Ramanathan [271].

The congruences $\tau(23n-r) \equiv 0 \pmod{23}$, where r is a quadratic residue modulo 23, were also first established by Wilton [341]. D.H. Lehmer [205] proved the following interesting theorem about congruences for $\tau(n)$ modulo 23.

Theorem 5.25.1. *Let n be a positive integer, and let $p_1, p_2, \ldots, p_t$ be those prime factors of n (if any) that are not of the form u^2+23v^2, but are quadratic residues of 23. Define n_1 by*

$$n =: n_1 \prod_{i=1}^{t} p_i^{\alpha_i},$$

where $p_i^{\alpha_i} \| n$. Then

$$\tau(n) \equiv \sigma_{11}(n_1) 2^t 3^{-t/2} \prod_{i-1}^{t} \sin \frac{2\pi}{3}(1+\alpha_i) \pmod{23}.$$

According to Lehmer, this result is equivalent to a theorem of Wilton [341].

5.1 The Congruence $p(5n+4) \equiv 0 \pmod 5$

Without the insertions, the beginning of the paper actually starts with the definitions of the Eisenstein series P, Q, and R, which are denoted by L, M, and N, respectively, in Ramanujan's notebooks [282]. Since the remainder of this section was extracted for [280] with additional details supplied by Hardy, we have not added more details here. However, it seems appropriate to provide an introduction to congruences for the partition function in arithmetic progressions, since a large portion of the manuscript focuses on this topic.

In this manuscript Ramanujan proves his well-known congruences for $p(n)$, namely,

$$\begin{cases} p(5n+4) \equiv 0 \pmod 5, \\ p(7n+5) \equiv 0 \pmod 7, \\ p(11n+6) \equiv 0 \pmod{11}. \end{cases} \tag{5.1.13}$$

These congruences are the first cases of the infinite families

$$p(5^k n + \delta_{5,k}) \equiv 0 \pmod{5^k}, \tag{5.1.14}$$

$$p(7^k n + \delta_{7,k}) \equiv 0 \pmod{7^{[k/2]+1}}, \tag{5.1.15}$$

$$p(11^k n + \delta_{11,k}) \equiv 0 \pmod{11^k},$$

where $\delta_{p,k} :\equiv 1/24 \pmod{p^k}$. The literature on these congruences is extensive, and there are now many proofs and approaches to them, especially for (5.1.14) and (5.1.15), e.g., [17], [23], [97], [127], [137], [143], [146], [153], [154], [175],

[176], [177], [181], [194], [239], [255], [267], and [336]. In particular, A. Folsom, Z. Kent, and Ono [138] have devised a proof of all three general congruences that does not depend on modular equations of any sort.

These congruences are indeed surprising, for they appear to be examples of a very rare and isolated phenomenon. In fact, Ramanujan [279], [281, p. 230] remarked that "It appears that there are no equally simple properties for any moduli involving primes other than these three." Over 80 years later, in 2003, S. Ahlgren and M. Boylan [5] confirmed Ramanujan's suspicions and proved that the congruences (5.1.13) are indeed the only ones when the prime modulus of the arithmetic progression matches the prime modulus of the congruence.

In view of Ramanujan's claim, it is natural to ask about the frequency of congruences for $p(n)$ and the possibility of finding new ones. In this direction, Ono has made great progress [254], [256] towards quantifying the rarity of such congruences. Before Ono's work, A.O.L. Atkin and J.N. O'Brien [24], [27] had found a few congruences for $p(n)$. For instance, Atkin proved that

$$p(17303n + 237) \equiv 0 \pmod{13}.$$

It is reasonable to conclude that such congruences are quite rare, but not so rare that one cannot find infinitely many such congruences. Indeed, Ono [258] has found infinitely many classes of such congruences. In particular, he proved that for any prime $\ell \geq 5$, there exist infinitely many congruences of the form $p(An + B) \equiv 0 \pmod{\ell}$. Ahlgren [3] extended Ono's result by showing that each prime ℓ could be replaced by any prime power ℓ^k. A delightful account of their work and other recent work in the theory of partitions can be found in [7]. R. Weaver [338] has explicitly determined over 30,000 examples to illustrate Ono's theorem. For example,

$$p(11864749n + 56062) \equiv 0 \pmod{13}.$$

5.2 Divisibility of $\tau(n)$ by 5

The congruence $\tau(n) \equiv n\sigma(n) \pmod{5}$ was established by Wilton [340], and is also proved in Hardy's book [166, pp. 166–167]. This congruence was generalized by R.P. Bambah and S. Chowla [38], [113, pp. 676–681], who proved that if n is not a multiple of 5, then

$$\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3}.$$

The asymptotic formula (5.2.7) can be proved using the method devised by E. Landau in his paper [203] and book [204, Section 183] in determining an asymptotic formula for the number of integers $\leq x$ that can be represented as a sum of two squares. Alternatively, one can appeal to a general Tauberian theorem, such as that proved by H. Delange [121]. However, as first pointed

out by G.K. Stanley [325], the claim (5.2.8) is false. Indeed, using the ideas of Landau [204, Sections 176–183], one can establish an asymptotic formula of the shape

$$\sum_{n\leq x} t_n = C\frac{x}{(\log x)^{1/4}}\left(1+\sum_{n=1}^{r-1}\frac{c_n}{(\log x)^n}+O\left(\frac{1}{(\log x)^r}\right)\right), \tag{5.2.9}$$

where $r \geq 2$ is an integer and the numbers c_n are constants, $1 \leq n \leq r-1$. However, generally, these constants are not equal to those that would be obtained by successive integrations by parts in (5.2.8). Ramanujan made a similar error in his first letter to Hardy [281, p. xxiv], [68, p. 24] when he claimed that the number of integers $\leq x$ that can be represented as a sum of two squares is asymptotic to a constant times a similar integral. See either Landau's paper [203] or Hardy's book [166, pp. 60–63] for a correct asymptotic formula with a proof. For a more detailed examination of Ramanujan's mistake and what led him to it, see the paper by P. Moree and J. Cazaran [230]. In Sections 5.6, 5.11, and 5.17, Ramanujan records similar asymptotic formulas, and, in contrast to the asymptotic formula in this section, calculates the leading coefficients in each case.

Moree [228] has carefully and systematically examined all of Ramanujan's incorrect asymptotic formulas and has calculated the leading coefficient C in each case; see also his paper with H.J.J. te Riele [231]. Set

$$\begin{aligned} D := &\prod_{p\equiv 1 \pmod 5}\frac{1-p^{-4}}{1-p^{-5}}\prod_{p\equiv \pm 2 \pmod 5}\frac{1-p^{-3}}{(1-p^{-2})^{1/2}(1-p^{-4})^{3/4}}\\ &\times\prod_{p\equiv 4 \pmod 5}\frac{1}{\sqrt{1-p^{-2}}}. \end{aligned} \tag{5.2.10}$$

Then Moree showed that the constant C in (5.2.9) is given by

$$C = \frac{4}{5\Gamma(\frac{3}{4})}\frac{\sqrt{\pi}}{\left(2\sqrt{5}\log\left(\frac{3+\sqrt{5}}{2}\right)\right)^{1/4}}D,$$

where D is defined by (5.2.10). According to Ramanujan's asymptotic formula (5.2.8), the constant c_2 is equal to $\frac{1}{4}$. Moree [228] numerically calculated c_2 and found that $c_2 = 0.1501\ldots$. In his paper [229], Moree considers Ramanujan's asymptotic formulas in a broader setting. In particular, he calculates the constant term in the expansion about $s = 1$ of the logarithmic derivative of many Dirichlet series with singularities (in most cases).

Stanley [325] attempted to correct Ramanujan's work on the divisibility of $\tau(n)$ by 5 but unfortunately made a large number of errors, which nullified her attempts at correction. For a complete discussion of Stanley's mistakes, see the last portion of Moree's interesting paper [228].

R.A. Rankin [291] verified that the leading term, including the coefficient, is correct in each of the asymptotic expansions cited by Ramanujan. Previously, in his dissertation, Rushforth [305] had verified the leading terms for the cases in which $\tau(n)$ is divisible by 3 or 7.

5.4 Congruences Modulo 5^k

The congruence (5.4.2) was first proved in print by Wilton [340] and later by Bambah [33].

Rankin [289, p. 5] pointed out that Ramanujan's conjecture (5.4.3) is false for $k \geq 4$. Observe that 443 is prime and that its powers are congruent to ± 1, $\pm 443 \pmod{5^4}$. From Watson's [337] table of values for $\tau(n), \tau(443) \equiv -58 \pmod{5^4}$. Hence, no integers a and b exist for which (5.4.3) holds with $n = 443$ and $k \geq 4$.

However, congruence (5.4.4) is true and is implied by each of two congruences due to D.B. Lahiri [202, Equations (13.10), (13.11)], namely,

$$\begin{aligned}19008\tau(n) \equiv &-691\sigma_{13}(n) + 27300\sigma_{11}(n) - 691\,\{33(4n-11)\sigma_9(n)\\ &+10(63n+400)\sigma_7(n) - 21(860n-463)\sigma_5(n)\\ &+60(252n^2-226n+5)\sigma_3(n) + (2520n-431)\sigma(n)\big\}\\ &\pmod{2^{10}\cdot 3^5\cdot 5^3\cdot 7\cdot 691}\end{aligned}$$

and

$$\begin{aligned}38016\tau(n) \equiv &-691\sigma_{13}(n) + 54600\sigma_{11}(n) - 691\,\{33(8n+19)\sigma_9(n)\\ &-30(179n-200)\sigma_7(n) + 3(2400n^2-5880n-12299)\sigma_5(n)\\ &+60(120n^2+3039n-1030)\sigma_3(n)\\ &-(151200n^2-75600n+6301)\sigma(n)\big\} \pmod{2^{11}\cdot 3^6\cdot 5^3\cdot 7\cdot 691},\end{aligned}$$

where in each case, n is coprime to the modulus. Moreover, (5.4.4) is equivalent to the congruence

$$\begin{aligned}\tau(n) \equiv 101n\sigma_9(n) + 5^2\,\{&n\sigma_3(n) + (n-1)\sigma(n)\\ &-\tfrac{1}{5}(n\sigma_9(n) - n^2\sigma_7(n))\big\} \pmod{5^3},\end{aligned}$$

if $(n,5) = 1$, which was stated without proof by Bambah and Chowla [36], [113, pp. 617–619].

The equality below (5.4.4) is a special instance of the relation

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}), \qquad n \geq 1, \tag{5.4.8}$$

where p is a prime, which, along with (5.7.6), was first proved by L.J. Mordell [227] after Ramanujan had made these conjectures in his paper [275, Section 18], [281, p. 153].

Proofs of either of the famous equalities (5.4.5) or (5.4.6) (or both) have been given by, in chronological order, Ramanujan [276], [281, pp. 210–213], Darling [120], Mordell [227], H. Rademacher and H.S. Zuckerman [267], [266, pp. 186–202], Chowla [111], [113, pp. 611–612], D. Kruyswijk [201], W.N. Bailey [31], [32], J.M. Dobbie [125], N.J. Fine [137], M.D. Hirschhorn [172], [177], [176], S. Raghavan [269], H.H. Chan [97], and A. Milas [224]. These proofs are quite varied. Some authors use q-series; some, such as Rademacher, Zuckerman, and Raghavan, use the theory of modular forms; Chan's proof uses a variant of one of Ramanujan's trigonometric series identities in [275]. Milas uses vertex operator algebras and obtains a remarkable generalization of (5.4.6).

As indicated by Ramanujan, (5.4.6) is a companion to (5.4.5). Bailey, Chan, Darling, Mordell, and Raghavan in the aforementioned papers have also given proofs of (5.4.6). In contrast to (5.4.5), equality (5.4.6) can be found in Ramanujan's notebooks [55, p. 257, Entry 9(i)].

5.5 Congruences Modulo 7

Since this section was also extracted by Hardy for [280], we have not added details here.

5.6 Congruences Modulo 7, Continued

The congruence (5.6.2) was established by Ramanathan [271], H. Gupta [164], and Bambah [34].

The comments made in Section 5.2 about Ramanujan's asymptotic formulas have analogues here. Although the asymptotic formula (5.6.7) is correct, Ramanujan's stronger claim (5.6.8) is false, since the constants obtained by integrating by parts in (5.6.8) do not generally match those obtained in a proper asymptotic expansion of $\sum_{k=1}^{n} t_k$. In particular, in the notation of (5.2.9), from (5.6.8), Ramanujan claims that $c_2 = \frac{1}{2}$. Moree [228] numerically calculated c_2 and found it to be $0.3841\dots$.

5.7 Congruences Modulo 49

The content of this section can be found with more detail in Rushforth's paper [306]. (In the second equality of (5.7.1), Rushforth [306, Equation (7.3)] wrote $-2R^2$ for $2R^2$, and in (5.7.4) [306, penultimate equality on p. 407] Rushforth wrote $-2n\sigma_3(n)$ for $2n\sigma_3(n)$.)

5.8 Congruences Modulo 49, Continued

Ramanujan's proof of (5.8.1) can be found in his paper [276], [281], while other proofs of (5.8.1) have been given by Mordell [227], Rademacher and Zuckerman [267], Fine [137], F.G. Garvan [143], O. Kolberg [197], Raghavan [269], and Chan [97]. Further identities akin to (5.8.1) and (5.8.2) have been established by Rademacher [264], [266, pp. 252–279]. These authors then continue to prove (5.8.3). A systematic development of several new identities of the type (5.8.3) has been given by Chan, H. Hahn, R.P. Lewis, and S.L. Tan [100].

Equality (5.8.4) is true, and its truth is equivalent to the assertion that $\eta^3(z)\eta^3(7z)$ is a Hecke eigenform with complex multiplication in $S_3(\Gamma_0(7), \chi_{-7})$, where $S_k(\Gamma_0(N), \chi)$ denotes the complex vector space of cusp forms of weight k with respect to the congruence subgroup $\Gamma_0(N)$ with Nebentypus character χ [293] [195, p. 130]. (For future reference, we note that the notation $S_k(\Gamma_0(N))$, with χ absent, simply means that the character χ is trivial.) Here the character χ_{-7} denotes the usual Kronecker character for the field $\mathbb{Q}(\sqrt{-7})$. That this form is an eigenform follows immediately from the fact that this space is one-dimensional [118]. To deduce (5.8.4) in a more elementary fashion, first notice that Jacobi's identity

$$q(q^8;q^8)^3_\infty = \eta^3(8z) = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}$$

implies that

$$\eta^3(8z)\eta^3(56z) = \sum_{x,y=0}^{\infty}(-1)^{x+y}(2x+1)(2y+1)q^{(2x+1)^2+7(2y+1)^2}.$$

The proof of Ramanujan's claim now follows after a straightforward computation.

The claims regarding the Euler product expansions Π_1 and Π_2 were established by H.H. Chan, S. Cooper, and W.-C. Liaw [99], who gave two different proofs. These authors also generalized their results by obtaining formulas analogous to Π_1 and Π_2 for $\eta^3(a\tau)\eta^3(b\tau)$ when $a+b=8$.

In regard to the congruence (5.8.6), we remark that O. Kolberg [199] proved the beautiful congruence

$$\tau(n) \equiv n\sigma_9(n) \pmod{49}, \qquad \text{if } \left(\frac{n}{7}\right) = -1.$$

5.9 The Congruence $p(11n+6) \equiv 0 \pmod{11}$

This proof is given in more detail in [280]. However, the proof can be simplified using a further idea of Ramanujan introduced in Section 13; see our commentary below.

5.10 Congruences Modulo 11, Continued

Equality (5.10.4) is true, and its truth is equivalent to the assertion that $\eta^2(z)\eta^2(11z)$ is an eigenform of the Hecke operators acting on $S_2(\Gamma_0(11))$ [139, p. 432]. That this form is an eigenform follows immediately from the fact that this space is one-dimensional.

Some of Ramanujan's congruences for $\tau(n)$ are immediate consequences of its multiplicative properties. For instance, Ramanujan [275], [281, p. 153, Equation (103)] conjectured and Mordell [227] proved that if m and n are relatively prime integers, then

$$\tau(mn) = \tau(m)\tau(n). \tag{5.10.14}$$

For example, the congruences (5.10.12) and (5.10.13) follow easily, since $\tau(19) = 10661420 \equiv 0 \pmod{11}$ and $\tau(29) = 128406630 \equiv 0 \pmod{11}$. Other congruences follow from (5.4.8) or from (5.4.8) and (5.10.14) together.

In 1969, P. Deligne [122] proved Serre's conjecture [311] on the existence of ℓ-adic Galois representations ρ_ℓ attached to modular forms on $\Gamma_0(N)$. Then, in 1972, Swinnerton-Dyer [326] determined the possible images of $\tilde{\rho}_\ell$, the reduction $\pmod{\ell}$ of ρ_ℓ, and showed that "small" images imply certain congruences for the coefficients of modular forms.

However, in general, the Galois groups that occur are mostly nonabelian (and even nonsolvable). In these cases there cannot be a congruence of the form

$$\tau(p) \equiv c \pmod{\ell},$$

where the primes p constitute those primes in any given arithmetic progression containing infinitely many primes [326, Lemma 7]. In particular, for $\tau(p)$, such congruences exist only for the primes $\ell \in \{2, 3, 5, 7, 23, 691\}$. Although this implies that there are no further Ramanujan-type congruences for $\tau(p)$, it turns out that for every positive integer M, there is a positive Frobenian set of primes with positive density for which [314]

$$\tau(p) \equiv 0 \pmod{M}.$$

This follows from the Chebotarev density theorem and the existence of Galois representations. As a consequence, it follows that

$$\tau(n) \equiv 0 \pmod{M}$$

for almost every positive integer n.

The existence of these representations and their study has been at the forefront of arithmetic geometry ever since Serre formulated his original conjectures. Every congruence for $\tau(n)$ involving divisor functions, and the congruence

$$\tau(n) \equiv 0 \pmod{23}$$

for $\left(\frac{n}{23}\right) = -1$, follows from this theory. For more details, readers should consult [122], [310], [311], [326], [327], [328], [329].

5.11 Divisibility by 2 or 3

Ramanujan's speculation that $p(n)$ is odd more often than it is even is not substantiated by more extensive calculations. Indeed, it is a long outstanding conjecture that asymptotically $p(n)$ is equally often even and odd. In Sections 5.1, 5.5, 5.9, and 5.11, based on a table of values for $p(n), 1 \leq n \leq 200$, computed by P.A. MacMahon, Ramanujan offers conjectures on the distribution of $p(n)$ modulo 5, 7, 11, and 3, respectively. We examine these conjectures in detail.

If $D(r, M)$ denotes the proportion of integers n for which $p(n) \equiv r \pmod{M}$ (assuming that such densities exist), Ramanujan conjectured (in Sections 5.11, 5.11, 5.1, 5.5, and 5.9, respectively) that

$$D(0,2) < D(1,2),$$

$$D(i,3) = \tfrac{1}{3}, \quad \text{for } 0 \leq i \leq 2,$$

$$D(i,5) = \begin{cases} \frac{1}{3}, & \text{if } i = 0, \\ \frac{1}{6}, & \text{if } 1 \leq i \leq 4, \end{cases}$$

$$D(i,7) = \begin{cases} \frac{1}{4}, & \text{if } i = 0, \\ \frac{1}{8}, & \text{if } 1 \leq i \leq 6, \end{cases}$$

$$D(i,11) = \begin{cases} \frac{1}{6}, & \text{if } i = 0, \\ \frac{1}{12}, & \text{if } 1 \leq i \leq 10. \end{cases}$$

From elementary considerations, we show that Ramanujan's conjectures for $D(i, M), M = 5, 7, 11$, are unlikely to be true. Remove the values $n = 5k + 4, 7k + 5, 11k + 6$, from consideration when $M = 5, 7, 11$, respectively. Assuming that the remaining values of $p(n)$ are distributed randomly among the M residue classes in each of these three cases, we would expect that

$$D(i, M) = \begin{cases} \dfrac{2M-1}{M^2}, & \text{if } i = 0, \\ \dfrac{M-1}{M^2}, & \text{if } 1 \leq i \leq M. \end{cases}$$

In particular, we expect that $D(0,5) = \frac{9}{25}$ and $D(i,5) = \frac{4}{25}$, $1 \leq i \leq 4$, in contrast to Ramanujan's conjectures. Similar discrepancies exist for $M = 7, 11$.

Let $\delta(r, M)$ denote the proportion of integers $n \leq 100000$ for which $p(n) \equiv r \pmod{M}$. Here are some values of $\delta(r, M)$ for $M \in \{2, 3, 5, 7, 11, 13\}$.

r	$\delta(r,2)$	$\delta(r,3)$	$\delta(r,5)$	$\delta(r,7)$	$\delta(r,11)$	$\delta(r,13)$
0	0.498	0.333	0.362	0.272	0.174	0.080
1	0.502	0.332	0.158	0.121	0.083	0.078
2	$*$	0.334	0.161	0.122	0.083	0.076
3	$*$	$*$	0.160	0.122	0.082	0.077
4	$*$	$*$	0.158	0.122	0.084	0.077
5	$*$	$*$	$*$	0.120	0.083	0.076
6	$*$	$*$	$*$	0.120	0.083	0.075
7	$*$	$*$	$*$	$*$	0.081	0.077
8	$*$	$*$	$*$	$*$	0.082	0.076
9	$*$	$*$	$*$	$*$	0.081	0.078
10	$*$	$*$	$*$	$*$	0.082	0.075
11	$*$	$*$	$*$	$*$	$*$	0.076
12	$*$	$*$	$*$	$*$	$*$	0.077

As this data suggests, if the densities $\delta(r,M)$ are well defined, then Ramanujan's conjectures are mostly incorrect. The data suggests that he may be correct when $M=3$, but not for any other values. At present, very little is known about the densities $\delta(r,M)$ apart from lower bounds for $\delta(0,M)$ for those M possessing congruences of the sort discussed in the commentary for Section 5.1. Several papers have been written obtaining lower bounds for the number of times $p(n)$ is odd or even. These authors include S. Ahlgren [1], [2], J.-L. Nicolas, I.Z. Ruzsa, A. Sárközy, and Serre [251], Nicolas and Sárközy [252], Berndt, Yee, and Zaharescu [71],[72], Nicolas [247], [248], and Ono [260]. Currently, the best results are

$$\#\{n \leq X \;:\; p(n) \equiv 0 \pmod 2\} \geq 0.28\sqrt{X}(\log\log X)^{1/2}, \tag{5.11.12}$$

$$\#\{n \leq X \;:\; p(n) \equiv 1 \pmod 2\} \gg_K \frac{\sqrt{X}(\log\log X)^K}{\log X}, \tag{5.11.13}$$

$$\#\{n \leq X \;:\; p(n) \not\equiv 0 \pmod M\} \gg \frac{\sqrt{X}}{\log X},$$

where K is any positive number. The first two results are due to Nicolas [248], while the last is due to Ahlgren [2]. We remark that Serre [251] proved the more general result

$$\frac{\#\{n \leq X \;:\; p(an+b) \equiv 0 \pmod 2\}}{\sqrt{X}} \to \infty,$$

as $X \to \infty$, for any pair of positive integers a, b. On the other hand, Ono [258] has shown that if $M \geq 5$ is prime, then

$$\#\{n \leq X \;:\; p(n) \equiv 0 \pmod M\} \gg_M X.$$

The methods of Ahlgren, Ono, and Serre are based on the theory of modular forms. However, working in the ring of formal power series in one variable

over the field of two elements $\mathbb{Z}/2\mathbb{Z}$, Berndt, Yee, and A. Zaharescu [71], [72] developed new elementary methods for deriving lower bounds for both the number of even values and the number of odd values taken by a variety of partition functions, many of which cannot be approached through the theory of modular forms.

In a similar direction, M. Newman [240] conjectured that every positive integer M has the property that each residue class $m \pmod{M}$ has infinitely many integers n for which $p(n) \equiv m \pmod{M}$. Atkin [24], Kolberg [198], and Newman [240] verified this conjecture for each $M \in \{2, 5, 7, 13\}$, Hirschhorn and M.V. Subbarao [182] verified it for $M = 16$, and Hirschhorn [174] proved it for $M = 12, 40$. Because of the validity of (5.14.3) and (5.14.4), Ramanujan had also proved this conjecture when $M = 13$. Motivated by Ramanujan's work, Ono [258] proved Newman's conjecture for every prime $M < 1000$, with the exception of $M = 3$. He also found a simple criterion for verifying Newman's conjecture for any prime $M \geq 5$. Carefully studying the filtration of certain modular forms related to the partition function, Ahlgren and Boylan [5] established Newman's conjecture for every prime modulus M, except for $M = 3$. Then in a sequel to [5], they [6] strengthened their former result by proving Newman's conjecture for every prime power $\ell^j, j \geq 1$. Moreover, they established a quantitative result, namely, for each prime $\ell \geq 5$,

$$\#\{0 \leq n \leq X : p(n) \equiv r \pmod{\ell^j}\} \gg_{r,\ell^j} \begin{cases} \sqrt{X}/\log X, & \text{if } r \not\equiv 0 \pmod{\ell^j}, \\ X, & \text{if } r \equiv 0 \pmod{\ell^j}, \end{cases} \tag{5.11.14}$$

with the case $j = 1$ being proved in [5] and the general case being established in [6].

The equality (5.11.6) can be found in a fragment published with Ramanujan's lost notebook [283, p. 354, Equation (1.42)]. A proof may be found in Berndt's paper [59, Entry 21] or in Chapter 18 of [15].

The congruence (5.11.10) has been proved several times in the literature. Most frequently, it is given in the equivalent formulation

$$\tau(n) \equiv \begin{cases} \sigma(n) \pmod 3, & \text{if } (3, n) = 1, \\ 0 \pmod 3, & \text{if } 3 \mid n. \end{cases}$$

For proofs, see papers by D.P. Banerji [42], Bambah and Chowla [35], [113, pp. 622–623], Gupta [163], and Bambah, Chowla, Gupta, and Lahiri [41], [113, pp. 627–630]. Bambah and Chowla [38], [113, pp. 676–681] proved the generalization

$$\tau(n) \equiv (n^2 + k)\sigma_7(n) \pmod{3^4}, \qquad (3, n) = 1,$$

where $k = 0$ if $n \equiv 1 \pmod 3$, and $k = 9$ if $n \equiv 2 \pmod 3$.

The asymptotic formulas in (5.11.11) need to be corrected in the same manner that the asymptotic formulas in Sections 5.2 and 5.6 need to be recast. The constant c_2 in (5.2.9) according to Ramanujan's formula (5.11.11) should be equal to $\frac{1}{2}$. Moree [228] determined that $c_2 = 0.2325\ldots.$

5.12 Divisibility of $\tau(n)$

The sums $\sum_{n=1}^{\infty} n^a q^n$, where a is a positive integer, can be explicitly evaluated in terms of Eulerian polynomials [54, p. 113, Entry 4].

Bambah, Chowla, and Gupta [40], [113, pp. 631–632] and Bambah, Chowla, Gupta, and Lahiri [41], [113, pp. 627–630] proved the congruence

$$\tau(n) \equiv \sigma(n) \pmod{8}, \qquad \text{if } n \text{ is odd},$$

which, in fact, is implied by the first congruence in (5.12.1).

We have been unable to find the identity (5.12.2) in the literature prior to the work of Ramanujan. On page 257 in his second notebook [282], Ramanujan actually offers a general formula for

$$S_{2r} := \sum_{n=1}^{\infty} \frac{n^{2r} q^n}{1 + q^n + q^{2n}},$$

which was first proved by Berndt, S. Bhargava, and Garvan [61], [58, p. 143]. The values of $S_{2r}, 1 \leq r \leq 4$, are explicitly given by Ramanujan. The formula for S_2 is given without proof in an equivalent form in a paper by J.M. Borwein and P.B. Borwein [79], and this equivalent formula is proved in [80] by the Borweins and Garvan. A particularly simple proof of (5.12.2), based on an identity of Fine, has been given by S.H. Son [318, Lemma 2.6].

A proof of the first congruence in (5.12.3) was given by Bambah and Chowla [37], [113, pp. 633–634]. The second congruence in (5.12.3) was established in another paper by the same authors [38], [113, pp. 676–681].

Bambah and Chowla [37], [113, pp. 633–634] proved the second congruence in (5.12.4).

The first proof in print of (5.12.7) was evidently given by Wilton [340]. Later proofs were found by Watson [334] and Lehmer [206].

As with corresponding results in Sections 5.2, 5.6, and 5.11, the asymptotic formula (5.12.8) needs to be corrected. Watson [334] established that the first-order term in (5.12.8) is correct, but as we previously pointed out, the result (5.12.8) is not correct in general. The coefficient c_2 in (5.2.9), according to Ramanujan, should be equal to $1/690$, but in fact, from Moree's work [228], $c_2 = 0.0006\ldots$. As Moree [228] indicates, the constant C can be explicitly written down, but it is very complicated. Numerically, $C = -0.5717\ldots$.

Bambah and Chowla [39], [113, pp. 644–651] gave the first published proof of (5.12.26).

The congruence below (5.12.26) is false, in general. For example, it is false for $n = 1, 3, 4, 5$. However, if we require that n be twice an odd number, then the result is true. To see this, use (5.12.26) in (5.12.12) to deduce immediately that for odd n,

$$\tau(2n) + 24\sigma_{11}(n) \equiv 0 \pmod{2048}. \tag{5.12.1}$$

For odd n, write $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, p_i prime, $p_i \neq p_j$ for $i \neq j$, and $\alpha_i > 0$ for $1 \leq i,\, j \leq r$. Then, if $k > 0$ and n is odd,

$$\sigma_k(2n) = \prod_{i=1}^{r} \frac{p_i^{k(\alpha_i+1)} - 1}{p_i^k - 1} \cdot \frac{2^{2k} - 1}{2^k - 1} = \sigma_k(n)(2^k + 1) \equiv \sigma_k(n) \pmod{2^k}.$$

Hence, from (5.12.1) and our calculation above, we conclude that for n odd,

$$\tau(2n) + 24\sigma_{11}(2n) \equiv 0 \pmod{2048}.$$

5.13 Congruences Modulo 13

In Sections 5.2, 5.6, 5.10, 5.11, and 5.13, Ramanujan considers the t-regular partition functions $\lambda(n)$ whose generating function is given by

$$\sum_{n=0}^{\infty} \lambda(n) q^n = \sum_{n=0}^{\infty} b_t(n) q^n := \frac{(q^t;q^t)_\infty}{(q;q)_\infty}.$$

The dependence of λ on t is always clear from the context. For instance, in Section 5.2, he considers the case $t = 25$. In this case he shows that $\lambda(n)$ is almost always a multiple of 5. A paper by B. Gordon and Ono [157] makes considerable progress in describing this phenomenon for all t. Let $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of t. By [157, Theorem 1], if p_i is a prime for which $p_i^{a_i} \geq \sqrt{t}$, then for every positive integer k, almost every integer n has the property that $b_t(n)$ is a multiple of p_i^k. This theorem immediately implies all of Ramanujan's claims of this sort for the functions $\lambda(n)$.

Equality (5.13.8) is the first of a series of remarkable equalities, the remainder of which are addressed in the next section. It was first proved in print by Zuckerman [353] and W.H. Simons [317]. The details of Ramanujan's proof of (5.13.8) are adequate, but since those for (5.13.9) were not supplied by Ramanujan, we do so here. We follow the presentation of Rushforth [305].

If the form of (5.13.9) is correct, then it must be true that the right side of (5.13.3) can be written in the form

$$a\vartheta P^5 + b\vartheta P^3 Q + c\vartheta P^2 R + d\vartheta P Q^2 + e\vartheta Q R + f(Q^3 - R^2) + 13J, \quad (5.13.14)$$

for certain constants a, b, c, d, e, f, where $\vartheta = q\dfrac{d}{dq}$. Using the differential equations (5.9.8), we see that (5.13.14) can be written in the form

$$\begin{aligned} &\{-5aP^4 - 3bP^2Q - 2cPR - dQ^2\}(P^2 - Q) \\ &+\{-4bP^3 + 5dPQ - 4eR\}(PQ - R) \\ &+\{-6cP^2 - 6eQ\}(PR - Q^2) + f(Q^3 - R^2) + 13J, \end{aligned} \quad (5.13.15)$$

where upon applying the differential operator ϑ in (5.13.14), we multiplied by -12 and remembered that we are taking congruences modulo 13. If we apply the distributive law throughout (5.13.15), collect like terms, and equate coefficients of corresponding terms in (5.13.3), we obtain a set of congruence

relations from which the values of a, b, c, d, e, and f can be obtained. More precisely, we find that

$$a = b = e = 1, \qquad \text{and} \qquad c = d = f = 3. \tag{5.13.16}$$

Hence, from (5.13.3) and (5.13.14)–(5.13.16),

$$\begin{aligned}-5P^6 &- 2P^4Q + 6P^3R - 6P^2Q^2 - 6PQR - (Q^3 - R^2) + 13J \\ &= \vartheta P^5 + \vartheta P^3 Q + 3\vartheta P^2 R + 3\vartheta PQ^2 + \vartheta QR + 3(Q^3 - R^2) + 13J \\ &= q\frac{dJ}{dq} + 3(Q^3 - R^2) + 13J,\end{aligned}$$

which is what we sought to prove.

Now from (5.13.3), (5.13.9), (5.13.5), and (5.13.4), it follows that

$$\frac{q^7(q^{169}; q^{169})_\infty}{(q;q)_\infty} = -q\frac{dJ}{dq} + 3\sum_{n=1}^{\infty} \tau(n) q^n + 13J. \tag{5.13.17}$$

The coefficient of $q^{13n}, n \geq 1$, in (5.13.17) is clearly a multiple of 13, and so extracting those terms arising from q^{13n} in (5.13.17) and replacing q^{13} by q, we deduce (5.13.8). Extracting those terms in (5.13.8) where the powers are multiples of 13, replacing q^{13} by q, and using (5.13.7), we deduce that

$$(q;q)_\infty \sum_{n=1}^{\infty} p(n \cdot 13^2 - 7) q^n = 10 \sum_{n=1}^{\infty} \tau(n) q^n + 13J. \tag{5.13.18}$$

Lastly, we can clearly rewrite (5.13.8) and (5.13.18) in the respective forms

$$\sum_{n=0}^{\infty} p(13n + 6) q^n = 11(q;q)_\infty^{11} + 13J$$

and

$$\sum_{n=0}^{\infty} p(13^2 n + 162) q^n = 23(q;q)_\infty^{23} + 13J,$$

which are (5.14.1) and (5.14.2), respectively, in the next section.

We remarked briefly in our commentary on Section 5.9 that Ramanujan's proof of $p(11n+6) \equiv 0 \pmod{11}$, extracted by Hardy in [280], can be somewhat simplified using the ideas in the present section. In particular, Ramanujan and Hardy showed that [280], [281, p. 237, Equation (5.2)]

$$P^5 - 3P^3Q - 4P^2R + 6QR = q\frac{dJ}{dq} + 11J. \tag{5.13.19}$$

Using the differential equations (5.9.8), we easily find that

$$\begin{aligned}
P^5 - 3P^3Q - 4P^2R + 6QR &= \{P^3 + 3PQ + 5R\}(P^2 - Q) - 5P^2(PQ - R) \\
&\quad - 3P(PR - Q^2) + 11J \\
&= \{P^3 + 3PQ + 5R\}\vartheta P - 4P^2\vartheta Q + 5P\vartheta R + 11J \\
&= 3\vartheta P^4 - 4\vartheta P^2 Q + 5\vartheta PR + 11J \\
&= q\frac{dJ}{dq} + 11J,
\end{aligned}$$

which provides an easier proof of (5.13.19) than the one given in [280].

5.14 Congruences for $p(n)$ Modulo 13

The claims (5.14.1)–(5.14.6) are among the most fascinating results in the unpublished manuscript. For example, these results indicate that

$$\sum_{n=0}^{\infty} p(13n+6)q^{24n+11} \equiv 11\eta^{11}(24z) \pmod{13}, \qquad q = e^{2\pi i z}.$$

Newman [239] proved some of these claims. However, Ono [258] first showed that this phenomenon also holds with respect to other moduli. In particular, if $m \geq 5$ is prime and k is a positive integer, then

$$\sum_{n=0}^{\infty} p\left(\frac{m^k n + 1}{24}\right) q^n$$

is the reduction modulo m of a holomorphic cusp form with weight $(m^2 - m - 1)/2$. This implies that results like (5.14.1)–(5.14.6) exist for every prime $m \geq 5$, not just for $m = 13$. Moreover, using the theory of Hecke operators of half-integral weight, the Shimura correspondence, and the theory of Galois representations, Ono [258] proved that for every prime $m \geq 5$, there are integers $0 \leq b < a$ for which

$$p(an + b) \equiv 0 \pmod{m}$$

for every nonnegative integer n.

5.15 Congruences to Further Prime Moduli

In Section 5.15 Ramanujan gives a brief description of the method he employs to obtain generating functions of the type

$$\sum p(\varpi n + b_{\varpi})q^n \pmod{\varpi}, \tag{5.15.13}$$

where $\varpi > 3$ is prime. Let B_k denote the kth Bernoulli number in contemporary notation. Note that Ramanujan's convention for Bernoulli numbers is different from the contemporary one in which the Bernoulli numbers B_k are defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k, \qquad |x| < 2\pi. \tag{5.15.14}$$

Ramanujan's claims about the generating function (5.15.13) have been proved by Ahlgren and Boylan [5]. We briefly describe their result, although we do not give all details. As usual, set $q = e^{2\pi i z}$, $\operatorname{Im} z > 0$. Let $\ell \geq 5$ denote a prime and let j be a positive integer. Define $\beta_{\ell,j}$ to be the unique positive integer such that

$$24\beta_{\ell,j} \equiv 1 \pmod{\ell^j}.$$

Ahlgren and Boylan then define a positive integer $k_{\ell,j}$ in terms of ℓ, j, and $\beta_{\ell,j}$, which is too complicated to relate here. Lastly, M_k denotes the space of all holomorphic modular forms of weight k on the full modular group $\mathbb{SL}_2(\mathbb{Z})$. We now state their theorem.

Theorem 5.15.1. *If $\ell \geq 5$ is a prime and j is a positive integer, then there exists a modular form $F_{\ell,j}(z) \in M_{k_{\ell,j}}$ with integral coefficients such that*

$$\sum_{n=0}^{\infty} p(\ell^j n + \beta_{\ell,j}) q^n \equiv (q;q)_{\infty}^{(24\beta_{\ell,j}-1)/\ell^j} \cdot F_{\ell,j}(z) \pmod{\ell^j}. \tag{5.15.15}$$

As Ramanujan claims, (5.15.3) and (5.15.4) are easy deductions from Fermat's little theorem and the von Staudt–Claussen theorem. However, the claim that $k = 0$ in (5.15.4) is not entirely clear. In fact, this is another of the claims in the manuscript that Ramanujan admits still require proof.

Proposition 5.15.1. *Ramanujan's claim that $k \equiv 0 \pmod{\varpi}$ in (5.15.4) is true.*

Proof. A simple calculation verifies Ramanujan's assertion that the truth of (5.15.5) implies that $k \equiv 0 \pmod{\varpi}$. In particular, it suffices to prove that

$$12 v_{\varpi+1} + (-1)^{(\varpi+1)/2} \delta_{\varpi+1} \equiv 0 \pmod{\varpi}. \tag{5.15.16}$$

Using the well-known Voronoï congruences [184, p. 237, Proposition 15.2.3], we find, for every integer a coprime to ϖ, that

$$(a^2 - 1) v_{\varpi+1} \equiv a \cdot (-1)^{(\varpi-1)/2} \delta_{\varpi+1} \sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] \pmod{\varpi},$$

since the sign of B_{2k} is $(-1)^{k+1}$ for every positive integer k. Therefore we find that

$$\left(\frac{a^2-1}{a}\right) v_{\varpi+1} + (-1)^{(\varpi+1)/2}\delta_{\varpi+1} \sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] \equiv 0 \,(\mathrm{mod}\, \varpi). \tag{5.15.17}$$

In view of (5.15.16) and (5.15.17) it suffices to prove that for each integer a coprime to ϖ,

$$\sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] \equiv \frac{a^2-1}{12a} \,(\mathrm{mod}\, \varpi). \tag{5.15.18}$$

We now prove (5.15.18) by examining Dedekind sums. If k is a positive integer, and h is coprime to k, then the Dedekind sum $s(h,k)$ is defined by

$$s(h,k) := \sum_{j=1}^{k-1} \frac{j}{k} \left(\frac{hj}{k} - \left[\frac{hj}{k}\right] - \frac{1}{2}\right).$$

It is easy to verify that

$$12a\varpi s(a,\varpi) = \frac{12a^2}{\varpi} \cdot \frac{\varpi(\varpi-1)(2\varpi-1)}{6} - 12a \sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] - 6a \cdot \frac{\varpi(\varpi-1)}{2}.$$

However, by [22, p. 64, Theorem 3.8], it is known that

$$12a\varpi s(a,\varpi) \equiv a^2 + 1 \,(\mathrm{mod}\, \varpi),$$

and so we find that

$$a^2 + 1 \equiv 2a^2 - 12a \sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] \,(\mathrm{mod}\, \varpi).$$

This is (5.15.18), and this completes the proof of Ramanujan's claim. □

A proof of the previous proposition has also been given by Swinnerton-Dyer [326, Theorem 2(i)].

Let

$$\varDelta(z) = q(q;q)_\infty^{24}, \qquad q = e^{2\pi i z},$$

and let

$$\vartheta = q\frac{d}{dq}.$$

In (5.15.7), Ramanujan remarks that if $p > 3$ is an odd prime, then the following theorem is true.

Theorem 5.15.2. *The function*

$$\varDelta(z)^{(p^2-1)/24} \equiv \vartheta(J) + F \,(\mathrm{mod}\, p),$$

where J and F are modular forms over $\mathbb{SL}_2(\mathbb{Z})$ and the weight of F is $p-1$.

The first proof of this theorem arose from correspondence of H.H. Chan with Serre and was communicated to the second author by Chan in 2000 [98]. Since the proof of Chan and Serre has never been published, we present it here. The proof of Theorem 5.15.2 is a consequence of the following theorem of Serre.

Theorem 5.15.3. *Let $f(q)$ be a modular form of weight k over $\mathbb{SL}_2(\mathbb{Z})$, with integral coefficients. If $k \equiv 0 \pmod{p-1}$ and $k < (p-1)^2$, then*

$$f(q) \equiv \vartheta(J) + F \pmod{p},$$

where J and F are modular forms on $\mathbb{SL}_2(\mathbb{Z})$ and the weight of F is $p-1$.

Before we give a proof of Theorem 5.15.3, we need a few definitions. Let M_k be the space of modular forms of weight k over $\mathbb{SL}_2(\mathbb{Z})$ with p-integral coefficients. If $f \in M_k$ then the reduction $\widetilde{f}$ of f modulo p belongs to the algebra of formal power series $\mathbf{F}_p[[q]]$, with $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$. We denote the set $\{\widetilde{f} \mid f \in M_k\}$ by $\widetilde{M}_k$. The space M_k is stable under the action of the Hecke operators T_l and T_p. By reduction modulo p, we see that T_l and T_p act on $\widetilde{M}_k$. Since $T_p \equiv U \pmod{p}$, where

$$U\left(\sum_{n=0}^{\infty} a_n q^n\right) = \sum_{n=0}^{\infty} a_{pn} q^n, \tag{5.15.19}$$

we conclude that U is an operator on the space $\widetilde{M}_k$, and therefore also on the space

$$\widetilde{M}^{[\alpha]} = \bigcup_{k\in[\alpha]} \widetilde{M}_k, \qquad [\alpha] \in \mathbb{Z}/(p-1)\mathbb{Z}.$$

In fact, U "contracts" $\widetilde{M}_k$. More precisely, we have the following theorem of Serre [312, Theorem 6].

Theorem 5.15.4. (i) *If $k > p+1$, then U maps $\widetilde{M}_k$ to $\widetilde{M}_{k'}$, where $k' < k$.*
(ii) *The restriction of U on $\widetilde{M}_{p-1}$ is bijective.*

If $f \in M_k$, then it may happen that there exists some $k' < k$ such that $\widetilde{f} = \widetilde{g}$ with $g \in M_{k'}$. The smallest integer $k' \leq k$ such that $\widetilde{f} = \widetilde{g}$ is called the filtration of f, denoted by $w(f)$.

The following lemma of Serre [312, Lemma 2] is the key to the proof of Theorem 5.15.3.

Lemma 5.15.1. (i) *We have $w(U(f)) \leq p + (w(f)-1)/p$.*
(ii) *If $w(f) = p-1$, then $w(U(f)) = p-1$.*

We now give Serre's proof of Theorem 5.15.3.

Proof. Let f be the function satisfying the hypothesis of Theorem 5.15.3. Consider $U(f)$. Its weight is congruent to 0 modulo $(p-1)$ and by Lemma 5.15.1 (a), the filtration $w(U(f))$ is at most $p+(k-1)/p < 2(p-1)$, since $k < (p-1)^2$. Hence $w(U(f))$ is equal to 0 or $p-1$. In both cases, by Theorem 5.15.4(ii), we may write $U(f)$ as $U(f')$ with f' of weight $p-1$. The form $h = f - f'$ is such that $U(h) = 0$; hence $h = \vartheta(g)$ with $g = \vartheta^{p-2}(h)$. This completes the proof of Theorem 5.15.3. □

From the congruence

$$\Delta^{(l^2-1)/24}(z) \equiv (q^{\ell^2};q^{\ell^2})_\infty \sum_{n=0}^{\infty} p(n-(l^2-1)/24)q^n \pmod{\ell}$$

and the operator identity (5.15.19), Ramanujan deduced that

$$U(\Delta^{(l^2-1)/24}(z)) \equiv \prod_{n=1}^{\infty}(1-q^{ln}) \sum_{n=0}^{\infty} p(ln-(l^2-1)/24)q^n \pmod{l}.$$

Since $\Delta^{(l^2-1)/24}(z)$ is congruent to a modular form of weight $l-1$ modulo l by Theorem 5.15.2, and since $U \equiv T_l \pmod{l}$, we conclude that the following corollary holds.

Corollary 5.15.1. *The congruence*

$$(q^\ell;q^\ell)_\infty \sum_{n=0}^{\infty} p\left(ln + \frac{1-l^2}{24}\right) q^n \equiv F(z) \pmod{l},$$

where $F(z)$ is a cusp form of weight $l-1$ on $\mathbb{SL}_2(\mathbb{Z})$, is valid.

Corollary 5.15.1 can be found in Ramanujan's manuscript as (5.15.10). It was rediscovered and proved by K.S. Chua [117] without using Theorem 5.15.2.

If we apply U to both sides of Corollary 5.15.1 using (5.15.19), we immediately obtain the next corollary.

Corollary 5.15.2. *The congruence*

$$(q;q)_\infty \sum_{n=0}^{\infty} p\left(l^2 n + \frac{1-l^2}{24}\right) q^n \equiv F(z) \pmod{l},$$

where $F(z)$ is a cusp form of weight $l-1$ on $\mathbf{SL}_2(\mathbb{Z})$, is valid.

In general, the following theorem is valid.

Theorem 5.15.5. *Let*

$$\delta_{l,k} = \begin{cases} \dfrac{1-l^k}{24}, & \text{if } k \text{ is even,} \\ \dfrac{1-l^{k+1}}{24}, & \text{if } k \text{ is odd,} \end{cases} \tag{5.15.20}$$

and

$$\epsilon = \begin{cases} l, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even.} \end{cases} \tag{5.15.21}$$

Then the congruence

$$(q^\epsilon; q^\epsilon)_\infty \sum_{n=0}^{\infty} p\left(l^k n + \delta_{l,k}\right) q^n \equiv F(z) \pmod{l},$$

where $F(z)$ is a cusp form of weight $l-1$ on $\mathbb{SL}_2(\mathbb{Z})$, holds.

Proof. We proceed by induction. It is clear that if the conclusion in Theorem 5.15.5 is true for an odd positive integer k, then by exactly the same argument that we used to deduce Corollary 5.15.2 from Corollary 5.15.1, we deduce that the result is true for $k+1$.

Suppose the conclusion holds for an even positive integer k, that is,

$$(q; q)_\infty \sum_{n=0}^{\infty} p\left(l^k n + \delta_{l,k}\right) q^n \equiv F(z) \pmod{l}, \tag{5.15.22}$$

where $F(z)$ is a form of weight $l-1$. Multiplying both sides of (5.15.22) by $\Delta^{(l^2-1)/24}$, we find that the right-hand side is a modular form $F\Delta^{(l^2-1)/24}$ with weight $< (l-1)^2$ for $l > 5$. Hence, by Theorem 5.15.3, we conclude that

$$(q^{\ell^2}; q^{\ell^2})_\infty \sum_{n=0}^{\infty} p\left(l^k n + \delta_{l,k}\right) q^n \equiv G(z) + \vartheta(J) \pmod{l}, \tag{5.15.23}$$

where $G(z)$ is a modular form of weight $l-1$. Applying U from (5.15.19) to both sides of (5.15.23) and using (5.15.19), we conclude that the statement holds for $k+1$. $\square$

Theorem 5.15.5 was originally conjectured by Chua [117], who proved that if the result holds for an odd integer k, then it implies that the result is valid for $k+1$. The method above of passing from even k to $k+1$ can be found in J. Lehner's paper [208]. We mention here that Corollary 5.15.1 was verified by Weaver [338] for $5 \leq l \leq 31$, but the general case was not mentioned there.

In Section 5.15, Ramanujan asserted that in particular cases, he was able to show the following result (see (5.15.7)).

Proposition 5.15.2. *We have*

$$(Q^3 - R^2)^{(\varpi^2-1)/24} \equiv q\frac{dJ}{dq} + F(z) \pmod{\varpi},$$

where $F(z)$ is a cusp form of weight $\varpi - 1$ on $\mathbb{SL}_2(\mathbb{Z})$.

As a corollary, he deduced (see (5.15.10)) the following observation. A proof of this proposition has also been established by Ahlgren and Boylan [5].

Proposition 5.15.3. *The congruence*

$$(q^{\varpi};q^{\varpi})_\infty \sum_{n=0}^{\infty} p\left(\varpi n + \frac{1-\varpi^2}{24}\right) q^n \equiv F(z) \pmod{\varpi},$$

where $F(z)$ is a cusp form of weight $\varpi - 1$ on $\mathbb{SL}_2(\mathbb{Z})$, holds.

Since the space of cusp forms on $\mathbb{SL}_2(\mathbb{Z})$ of weight $\varpi - 1$ for $\varpi \leq 11$ is trivial, Proposition 5.15.3 immediately implies the congruences

$$p(5n+4) \equiv 0 \pmod{5}, \qquad p(7n+5) \equiv 0 \pmod{7}, \qquad p(11n+6) \equiv 0 \pmod{11}.$$

A proof of Proposition 5.15.2 for general ϖ has been given by Serre using [312, Lemma 2]; see also Theorem 5.15.2. Proposition 5.15.3 was recently rediscovered and proved by Chua [117] without using Proposition 5.15.2.

The remainder of Section 5.15 is straightforward and follows from Ramanujan's collection of formulas involving the operator $q\dfrac{d}{dq}$.

5.16 Congruences for $p(n)$ Modulo 17, 19, 23, 29, or 31

In the next several pages, we provide Rushforth's [305] sketch of what may have been Ramanujan's proof of the identity

$$(q^{17};q^{17})_\infty \sum_{n=0}^{\infty} p(17n+5)q^n = 7\sum_{n=1}^{\infty} \tau_2(n)q^n + 17J, \tag{5.16.12}$$

where $\tau_2(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau_2(n)q^n := Qq(q;q)_\infty^{24}. \tag{5.16.13}$$

Recall that (5.16.12) is the same as (5.16.1). This proof is followed by sketches of Rushforth's [305] proofs of the corresponding identities for the moduli 19 and 23.

Far simpler proofs can now be given with the use of Corollary 5.15.2. In particular,

$$\begin{aligned}
S_{16}(\mathbb{SL}_2(\mathbb{Z})) &= \langle Q\varDelta \rangle, \\
S_{18}(\mathbb{SL}_2(\mathbb{Z})) &= \langle R\varDelta \rangle, \\
S_{22}(\mathbb{SL}_2(\mathbb{Z})) &= \langle QR\varDelta \rangle.
\end{aligned}$$

Therefore, as Ramanujan has indicated in (5.16.1), (5.16.3), and (5.16.4), respectively, for some constants c_1, c_2, and c_3,

$$(q^{17};q^{17})_\infty \sum_{n=1}^{\infty} p(17n-12)q^n \equiv c_1 Q\varDelta \pmod{17},$$
$$(q^{19};q^{19})_\infty \sum_{n=1}^{\infty} p(19n-15)q^n \equiv c_2 R\varDelta \pmod{19},$$
$$(q^{23};q^{23})_\infty \sum_{n=1}^{\infty} p(23n-22)q^n \equiv c_3 QR\varDelta \pmod{23}.$$

Although the proofs that follow proceed along the lines indicated by Ramanujan for the moduli 11 and 13, in particular, they are extraordinarily tedious. According to Rushforth [305], the $p(n)/\tau(n)$ manuscript was sent to Hardy a few months before Ramanujan's death. It seems inconceivable that Ramanujan would have had the patience to perform such laborious algebraic manipulations in the last year of his life. Either he had more efficient means of deriving his formulas, or he had proved these results earlier while lying in cold nursing homes in England. But with the ideas Ramanujan gave us, Rushforth's proofs seem as straightforward as one can expect.

To prove (5.16.12), it will be necessary to prove the formula

$$\begin{aligned}(Q^3-R^2)^{12} = &-P^8 - 4P^6Q + 6P^5R + 3P^4Q^2 + 4P^3QR + 6P^2Q^3 \\ &- 3P^2R^2 + 2PQ^2R + Q^4 + 3QR^2 + 17J. \qquad (5.16.14)\end{aligned}$$

The proof of (5.16.14) is rather tortuous. It will be convenient to introduce Ramanujan's basic functions [275]

$$\varPhi_{r,s}(q) := \sum_{m,n=1}^{\infty} m^r n^s q^{mn}, \qquad |q|<1, \qquad (5.16.15)$$

where r and s are nonnegative integers. From Ramanujan's tables [275], [281, p. 141, Table I, nos. 8, 9],

$$3617 + 16320\varPhi_{0,15}(q) = 1617Q^4 + 2000QR^2 \qquad (5.16.16)$$

and

$$43867 - 28728\varPhi_{0,17}(q) = 38367Q^3R + 5500R^3. \qquad (5.16.17)$$

Since, by Fermat's little theorem,

$$\varPhi_{0,1}(q) = \varPhi_{0,17}(q) + 17J$$

and

$$P(q) = 1 - 24\varPhi_{0,1}(q), \qquad (5.16.18)$$

we may deduce from (5.16.3) and (5.16.4), respectively, that

$$Q^4 = 3QR^2 - 2 + 17J \tag{5.16.19}$$

and

$$P = 7Q^3R - 6R^3 + 17J. \tag{5.16.20}$$

To obtain (5.16.14), a tedious amount of algebra is needed. We first expand $(Q^3 - R^2)^{12}$. We then use (5.16.19) to calculate several powers of Q, which we insert in the expansion of $(Q^3 - R^2)^{12}$. However, since (5.16.14) involves powers of P, we then use (5.16.20) to calculate various powers of P. Lastly, we shall find expressions for powers of 1 that are necessary in making the proper reductions of certain powers Q^rR^s.

First, expanding and reducing modulo 17, we find that

$$\begin{aligned}(Q^3 - R^2)^{12} = Q^{36} &+ 5Q^{33}R^2 - 2Q^{30}R^4 + Q^{27}R^6 + 2Q^{24}R^8 + 7Q^{21}R^{10}\\ &+ 6Q^{18}R^{12} + 7Q^{15}R^{14} + 2Q^{12}R^{16} + Q^9R^{18}\\ &- 2Q^6R^{20} + 5Q^3R^{22} + R^{24} + 17J.\end{aligned} \tag{5.16.21}$$

Second, we use (5.16.19) to calculate $Q^8, \ldots, Q^{36}$ in powers of $(QR^2)^n$, $1 \le n \le 9$. Accordingly,

$$\begin{aligned}
Q^4 &= 3QR^2 - 2 + 17J,\\
Q^8 &= -8Q^2R^4 + 5QR^2 + 4 + 17J,\\
Q^{12} &= -7Q^3R^6 - 3Q^2R^4 + 2QR^2 - 8 + 17J,\\
Q^{16} &= -4Q^4R^8 + 5Q^3R^6 - 5Q^2R^4 + 6QR^2 - 1 + 17J,\\
Q^{20} &= 5Q^5R^{10} + 6Q^4R^8 - 8Q^3R^6 - 6Q^2R^4 + 2QR^2 + 2 + 17J,\\
Q^{24} &= -2Q^6R^{12} + 8Q^5R^{10} - 2Q^4R^8 - 2Q^3R^6 + Q^2R^4 + 2QR^2 - 4 + 17J,\\
Q^{28} &= -6Q^7R^{14} - 6Q^6R^{12} - 5Q^5R^{10} - 2Q^4R^8 + 7Q^3R^6 + 4Q^2R^4 + QR^2\\
&\quad + 8 + 17J,\\
Q^{32} &= -Q^8R^{16} - 6Q^7R^{14} - 3Q^6R^{12} + 4Q^5R^{10} + 8Q^4R^8 - 2Q^3R^6 - 5Q^2R^4\\
&\quad + 5QR^2 + 1 + 17J,\\
Q^{36} &= -3Q^9R^{18} + Q^8R^{16} + 3Q^7R^{14} + Q^6R^{12} - Q^5R^{10} - 5Q^4R^8 + 6Q^3R^6\\
&\quad + 8Q^2R^4 - 7QR^2 - 2 + 17J.
\end{aligned}$$

Substituting these equalities in (5.16.21) but ignoring the last three terms in (5.16.21), we find that

$$\begin{aligned}(Q^3 - R^2)^{12} = &-5Q^9R^{18} + 2Q^7R^{14} + 5Q^6R^{12} + 6Q^5R^{10} - 8Q^3R^6\\ &- 2QR^2 - 2 - 2Q^6R^{20} + 5Q^3R^{22} + R^{24} + 17J.\end{aligned} \tag{5.16.22}$$

Third, we employ (5.16.20) to calculate the powers of P up to P^8. The first two equalities below are listed for completeness and later considerations. Thus, we find that

$$
\begin{aligned}
QR^2 &= QR^2,\\
Q^4 &= Q^4,\\
PQ^2R &= 7Q^5R^2 - 6Q^2R^4 + 17J,\\
P^2R^2 &= -2Q^6R^4 + Q^3R^6 + 2R^8 + 17J,\\
P^2Q^3 &= -2Q^9R^2 + Q^6R^4 + 2Q^3R^6 + 17J,\\
P^3QR &= 3Q^{10}R^4 + 2Q^7R^6 + 8Q^4R^8 + 5QR^{10} + 17J,\\
P^4Q^2 &= 4Q^{14}R^4 - 4Q^{11}R^6 - 7Q^8R^8 + 4Q^5R^{10} + 4Q^2R^{12} + 17J,\\
P^5R &= -6Q^{15}R^6 - Q^{12}R^8 - 8Q^9R^{10} + 2Q^6R^{12} + 4Q^3R^{14} - 7R^{16} + 17J,\\
P^6Q &= -8Q^{19}R^6 - 5Q^{16}R^8 + Q^{13}R^{10} - 6Q^{10}R^{12} - Q^7R^{14} - 5Q^4R^{16}\\
&\quad + 8QR^{18} + 17J,\\
P^8 &= -Q^{24}R^8 + 2Q^{21}R^{10} - 6Q^{18}R^{12} + 3Q^{15}R^{14} - 2Q^{12}R^{16} - 3Q^9R^{18}\\
&\quad - 6Q^6R^{20} - 2Q^3R^{22} - R^{24} + 17J.
\end{aligned}
$$

With the use of our formulas for powers of Q, all the terms in the expressions above that are of the form Q^rR^s, where $s \leq 2r$, may be reduced to terms in Q^9R^{18} and lower powers of QR^2 as we did above. The remaining terms may also be written in powers of QR^2 and terms in Q^6R^{20}, Q^3R^{22}, and R^{24}. To do this, we calculate the following powers of 1 from (5.16.19):

$$
\begin{aligned}
1 &= 8Q^4 - 7QR^2 + 17J,\\
1^2 &= -4Q^8 + 7Q^5R^2 - 2Q^2R^4 + 17J,\\
1^3 &= 2Q^{12} - Q^9R^2 + 3Q^6R^4 - 3Q^3R^6 + 17J,\\
1^4 &= -Q^{16} - 5Q^{13}R^2 - 3Q^{10}R^4 + 6Q^7R^6 + 4Q^4R^8 + 17J,\\
1^5 &= -8Q^{20} + Q^{17}R^2 - 6Q^{14}R^4 + Q^{11}R^6 + 7Q^8R^8 + 6Q^5R^{10} + 17J,\\
1^6 &= 4Q^{24} - 4Q^{21}R^2 - 4Q^{18}R^4 - Q^{15}R^6 - 2Q^{12}R^8 - Q^9R^{10} - 8Q^6R^{12} + 17J.
\end{aligned}
$$

Now each term that is not of the form Q^rR^s, $s \leq 2r$, is multiplied by the appropriate power of 1 to introduce the required terms Q^6R^{20} and Q^3R^{22}. For example, Q^4R^{16} and QR^{18} are multiplied by 1^2, and R^8 is multiplied by 1^6. Reducing, as before, we obtain the aggregate of the expressions $P^8, P^6Q, \dots$ in the required forms. Thus, we have

$$
\begin{aligned}
QR^2 &= QR^2,\\
Q^4 &= 3QR^2 - 2 + 17J,\\
PQ^2R &= -2Q^2R^4 + 3QR^2 + 17J,\\
P^2R^2 &= -6Q^9R^{18} - 4Q^8R^{16} + 3Q^7R^{14} + 2Q^6R^{12} + 7Q^5R^{10} - Q^4R^8\\
&\quad - 5Q^3R^6 + 4Q^2R^4 + Q^6R^{20} + 17J,\\
P^2Q^3 &= 4Q^3R^6 + 5Q^2R^4 - 8QR^2 + 17J,\\
P^3QR &= 7Q^9R^{18} - Q^8R^{16} + 5Q^7R^{14} - 8Q^6R^{12} + 6Q^5R^{10} + 7Q^4R^8
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 6Q^3R^6 - 5Q^2R^4 - 4Q^6R^{20} + 17J,\\
P^4Q^2 &= 6Q^9R^{18} + 4Q^8R^{16} - 3Q^7R^{14} - 2Q^6R^{12} + 4Q^5R^{10} - Q^4R^8\\
&\quad - 8Q^3R^6 + 2Q^2R^4 - Q^6R^{20} + 17J,\\
P^5R &= 3Q^9R^{18} + 8Q^8R^{16} - 2Q^7R^{14} - 6Q^6R^{12} - 5Q^5R^{10} + Q^4R^8\\
&\quad - 3Q^3R^6 + Q^6R^{20} + 4Q^3R^{22} + 17J,\\
P^6Q &= -7Q^9R^{18} - 6Q^8R^{16} + 6Q^7R^{14} - 8Q^6R^{12} - 8Q^4R^8 + 8Q^3R^6\\
&\quad - 2Q^6R^{20} + Q^3R^{22} + 17J,\\
P^8 &= -4Q^9R^{18} + 7Q^8R^{16} - 2Q^7R^{14} - 2Q^6R^{12} - 4Q^5R^{10} - 2Q^4R^8\\
&\quad - 6Q^6R^{20} - 2Q^3R^{22} - R^{24} + 17J.
\end{aligned}
$$

We now multiply this set of equations by $a, b, c, d, \ldots$, respectively, add, and equate coefficients on the right side with those on the right side of (5.16.22). We therefore obtain a set of thirteen consistent simultaneous linear congruence relations. Solving these, we obtain the solution set

$$
\begin{aligned}
a &= -1, \quad b = -4, \quad c = 6, \quad d = 3, \quad e = 4,\\
f &= 6, \quad g = -3, \quad h = 2, \quad i = 1, \quad j = 3.
\end{aligned}
$$

Thus, from the left side of (5.16.22) and the left side of the appropriate multiples of this last set of equations, we obtain the required relation

$$
\begin{aligned}
(Q^3 - R^2)^{12} &= -P^8 - 4P^6Q + 6P^5R + 3P^4Q^2 + 4P^3QR + 6P^2Q^3\\
&\quad - 3P^2R^2 + 2PQ^2R + Q^4 + 3QR^2 + 17J. \qquad (5.16.23)
\end{aligned}
$$

We now show that (5.16.23) can be written in the form

$$
(Q^3 - R^2)^{12} = q\frac{dJ}{dq} - 5Q(Q^3 - R^2) + 17J. \qquad (5.16.24)
$$

To that end, using the differential equations (5.9.8), we see that

$$
\begin{aligned}
&-P^8 - 4P^6Q + 6P^5R + 3P^4Q^2 + 4P^3QR + 6P^2Q^3\\
&-3P^2R^2 + 2PQ^2R + Q^4 + 3QR^2 + 17J\\
&\quad = \{-P^6 + P^4Q - 2P^2Q^2 - 4PQR + 2Q^3 + 7R^2\}(P^2 - Q)\\
&\qquad + \{-6P^5 + 6P^3Q - 8P^2R + 7PQ^2 - 5QR\}(PQ - R)\\
&\qquad + \{5P^2Q - PR - 8Q^2\}(PR - Q^2) - 5(Q^4 - QR^2) + 17J\\
&\quad = \{5P^6 - 5P^4Q - 7P^2Q^2 + 3PQR + 7Q^3 - R^2\}\vartheta P\\
&\qquad + \{-P^5 + P^3Q - 7P^2R + 4PQ^2 + 2QR\}\vartheta Q\\
&\qquad + \{-7P^2Q - 2PR + Q^2\}\vartheta R - 5(Q^4 - QR^2) + 17J\\
&\quad = 8\vartheta P^7 - \vartheta P^5Q - 8\vartheta P^3Q^2 - 7\vartheta P^2QR + 7\vartheta PQ^3
\end{aligned}
$$

$$\begin{aligned}&-\vartheta PR^2+\vartheta Q^2R-5(Q^4-QR^2)+17J\\&=q\frac{dJ}{dq}-5(Q^4-QR^2)+17J.\end{aligned}$$

Thus, (5.16.24) follows from (5.16.23) and the calculations above.

From (5.1.4) and the binomial theorem,

$$\begin{aligned}(Q^3-R^2)^{12}=1728^{12}q^{12}(q;q)_\infty^{288}&=-4q^{12}(q;q)_\infty^{288}+17J\\&=-4q^{12}\frac{(q^{289};q^{289})_\infty}{(q;q)_\infty}+17J\end{aligned}$$

and

$$Q^4-QR^2=Q\cdot 1728q(q;q)_\infty^{24}=-6\sum_{n=1}^{\infty}\tau_2(n)q^n+17J,$$

where $\tau_2(n)$ is defined by (5.16.13). Combining these last two calculations with (5.16.24), we find that

$$-4q^{12}\frac{(q^{289};q^{289})_\infty}{(q;q)_\infty}=q\frac{dJ}{dq}-4\sum_{n=1}^{\infty}\tau_2(n)q^n+17J. \tag{5.16.25}$$

We know that

$$\tau_2(17n)-\tau_2(17)\tau_2(n)\equiv 0 \pmod{17}, \tag{5.16.26}$$

and we can find by a direct calculation that

$$\tau_2(17)\equiv 7 \pmod{17}. \tag{5.16.27}$$

Therefore, by (5.16.26) and (5.16.27),

$$\tau_2(17n)-7\tau_2(n)\equiv 0 \pmod{17}. \tag{5.16.28}$$

Hence, choosing only those terms in q^{17n} from (5.16.25), using (5.16.28), and replacing q^{17} by q, we find that

$$-4(q^{17};q^{17})_\infty\sum_{n=0}^{\infty}p(17n+5)q^{n+1}=6\sum_{n=1}^{\infty}\tau_2(n)q^n+17J,$$

or

$$(q^{17};q^{17})_\infty\sum_{n=0}^{\infty}p(17n+5)q^{n+1}=7\sum_{n=1}^{\infty}\tau_2(n)q^n+17J. \tag{5.16.29}$$

Equating powers of q^{17} in (5.16.29) and replacing q^{17} by q, we furthermore find that

$$(q^{17};q^{17})_\infty \sum_{n=1}^{\infty} p(17^2 n - 12)q^n = 7\sum_{n=1}^{\infty} \tau_2(17n)q^n + 17J$$
$$= -2\sum_{n=1}^{\infty} \tau_2(n)q^n + 17J. \qquad (5.16.30)$$

The equalities (5.16.29) and (5.16.30) are (5.16.1) and (5.16.7), respectively, where we now have determined that c_2 is equal to -2.

Our next task is to provide Rushforth's sketch of his proof of Ramanujan's formula

$$(q^{19};q^{19})_\infty \sum_{n=1}^{\infty} p(19n-15)q^n = 5\sum_{n=1}^{\infty} \tau_3(n)q^n + 19J, \qquad (5.16.31)$$

where $\tau_3(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau_3(n)q^n := Rq(q;q)_\infty^{24}.$$

In order to accomplish this, we need to prove the formula

$$(Q^3-R^2)^{15} = P^9 - 5P^7Q + 6P^6R + 5P^5Q^2 + P^3Q^3 - P^2Q^2R$$
$$+ 2PQ^4 - PQR^2 - 9Q^3R + R^3 + 19J. \qquad (5.16.32)$$

First [275], [281, p. 141, no. 10],

$$174611 + 13200\Phi_{0,19}(q) = 53361Q^5 + 121250Q^2R^2. \qquad (5.16.33)$$

Using (5.16.17), (5.16.18), (5.16.33), and the elementary congruence

$$\Phi_{0,1}(q) = \Phi_{0,19}(q) + 19J,$$

we find that

$$-4 = 6Q^3R + 9R^3 + 19J, \qquad (5.16.34)$$
$$P = 9Q^5 - 8Q^2R^2 + 19J. \qquad (5.16.35)$$

Expanding $(Q^3-R^2)^{15}$, we find that

$$(Q^3-R^2)^{15} = Q^{45} + 4Q^{42}R^2 - 9Q^{39}R^4 + Q^{36}R^6 - 3Q^{33}R^8 - Q^{30}R^{10}$$
$$+ 8Q^{27}R^{12} + 6Q^{24}R^{14} - 6Q^{21}R^{16} - 8Q^{18}R^{18} + Q^{15}R^{20}$$
$$+ 3Q^{12}R^{22} - Q^9R^{24} + 9Q^6R^{26} - 4Q^3R^{28} - R^{30} + 19J. \qquad (5.16.36)$$

We now use (5.16.35) to deduce the representations

$$PQR^2 = 9Q^6R^2 - 8Q^3R^4 + 19J,$$

$$\begin{aligned}
PQ^4 &= 9Q^9 - 8Q^6R^2 + 19J,\\
P^2Q^2R &= 5Q^{12}R + 8Q^9R^3 + 7Q^6R^5 + 19J,\\
P^3R^2 &= 7Q^{15}R^2 - 6Q^{12}R^4 - Q^9R^6 + Q^6R^8 + 19J,\\
P^3Q^3 &= 7Q^{18} - 6Q^{15}R^2 - Q^{12}R^4 + Q^9R^6 + 19J,\\
P^4QR &= 6Q^{21}R + 4Q^{18}R^3 + Q^{15}R^5 - 2Q^{12}R^7 - 8Q^9R^9 + 19J,\\
P^5Q^2 &= -3Q^{27} + 7Q^{24}R^2 - 4Q^{21}R^4 - 7Q^{18}R^6 + Q^{15}R^8 + 7Q^{12}R^{10} + 19J,\\
P^6R &= -8Q^{30}R - 8Q^{27}R^3 + 3Q^{24}R^5 + 7Q^{21}R^7 + 8Q^{18}R^9 - 2Q^{15}R^{11}\\
&\quad + Q^{12}R^{13} + 19J,\\
P^7Q &= 4Q^{36} - 8Q^{33}R^2 - 4Q^{30}R^4 + Q^{27}R^6 - 3Q^{24}R^8 - 6Q^{21}R^{10}\\
&\quad + 6Q^{18}R^{12} - 8Q^{15}R^{14} + 19J,\\
P^9 &= Q^{45} - 8Q^{42}R^2 + Q^{39}R^4 - 7Q^{36}R^6 + 3Q^{33}R^8 - 9Q^{30}R^{10} - Q^{27}R^{12}\\
&\quad + 4Q^{24}R^{14} - 3Q^{21}R^{16} + Q^{18}R^{18} + 19J.
\end{aligned}$$

We see from (5.16.34) that

$$1 = 8Q^3R - 7R^3 + 19J.$$

We thus find that

$$\begin{aligned}
1 &= 8Q^3R - 7R^3 + 19J,\\
1^2 &= 7Q^6R^2 + 2Q^3R^4 - 8R^6 + 19J,\\
1^3 &= -Q^9R^3 + 5Q^6R^5 - 2Q^3R^7 - R^9 + 19J,\\
1^4 &= -8Q^{12}R^4 + 9Q^9R^6 + 6Q^6R^8 + 6Q^3R^{10} + 7R^{12} + 19J,\\
1^5 &= -7Q^{15}R^5 - 5Q^{12}R^7 + 4Q^9R^9 + 6Q^6R^{11} - 5Q^3R^{13} + 8R^{15} + 19J,\\
1^6 &= Q^{18}R^6 + 9Q^{15}R^8 - 9Q^{12}R^{10} + Q^9R^{12} - 6Q^6R^{14} + 4Q^3R^{16}\\
&\quad + R^{18} + 19J,\\
1^7 &= 8Q^{21}R^7 + 8Q^{18}R^9 - 2Q^{15}R^{11} - 5Q^{12}R^{13} + 2Q^9R^{15} - 2Q^6R^{17}\\
&\quad - Q^3R^{19} - 7R^{21} + 19J,\\
1^8 &= 7Q^{24}R^8 + 8Q^{21}R^{10} + 4Q^{18}R^{12} - 7Q^{15}R^{14} - 6Q^{12}R^{16} + 8Q^9R^{18}\\
&\quad + 6Q^6R^{20} + 8Q^3R^{22} - 8R^{24} + 19J,\\
1^9 &= -Q^{27}R^9 - 4Q^{24}R^{11} - 5Q^{21}R^{13} - 8Q^{18}R^{15} + Q^{15}R^{17} - 8Q^{12}R^{19}\\
&\quad - 8Q^9R^{21} + 3Q^6R^{23} - 6Q^3R^{25} - R^{27} + 19J.
\end{aligned}$$

Using this batch of identities in the previous batch, we find that

$$\begin{aligned}
R^3 \cdot 1^9 &= -Q^{27}R^{12} - 4Q^{24}R^{14} - 5Q^{21}R^{16} - 8Q^{18}R^{18} + Q^{15}R^{20}\\
&\quad - 8Q^{12}R^{22} - 8Q^9R^{24} + 3Q^6R^{26} - 6Q^3R^{28} - R^{30} + 19J,
\end{aligned}$$

$$
\begin{aligned}
Q^3R\cdot 1^9 &= -Q^{30}R^{10} - 4Q^{27}R^{12} - 5Q^{24}R^{14} - 8Q^{21}R^{16} + Q^{18}R^{18}\\
&\quad - 8Q^{15}R^{20} - 8Q^{12}R^{22} + 3Q^9R^{24} - 6Q^6R^{26} - Q^3R^{28} + 19J,\\
PQR^2\cdot 1^8 &= 6Q^{30}R^{10} - 3Q^{27}R^{12} - 9Q^{24}R^{14} + 0Q^{21}R^{16} + 2Q^{18}R^{18}\\
&\quad + 6Q^{15}R^{20} + 9Q^{12}R^{22} + 5Q^9R^{24} - 3Q^6R^{26} + 7Q^3R^{28} + 19J,\\
PQ^4\cdot 1^8 &= 6Q^{33}R^8 - 3Q^{30}R^{10} - 9Q^{27}R^{12} + 0Q^{24}R^{14} + 2Q^{21}R^{16}\\
&\quad + 6Q^{18}R^{18} + 9Q^{15}R^{20} + 5Q^{12}R^{22} - 3Q^9R^{24} + 7Q^6R^{26} + 19J,\\
P^2Q^2R\cdot 1^7 &= 2Q^{33}R^8 + 9Q^{30}R^{10} - 4Q^{27}R^{12} - 4Q^{24}R^{14} - 6Q^{21}R^{16}\\
&\quad + 9Q^{18}R^{18} - 7Q^{15}R^{20} + 0Q^{12}R^{22} - 6Q^9R^{24} + 8Q^6R^{26} + 19J,\\
P^3R^2\cdot 1^6 &= 7Q^{33}R^8 + 0Q^{30}R^{10} - 4Q^{27}R^{12} - 4Q^{24}R^{14} + 8Q^{21}R^{16}\\
&\quad - 3Q^{18}R^{18} + 9Q^{15}R^{20} + 3Q^{12}R^{22} + 3Q^9R^{24} + Q^6R^{26} + 19J,\\
P^3Q^3\cdot 1^6 &= 7Q^{36}R^6 + 0Q^{33}R^8 - 4Q^{30}R^{10} - 4Q^{27}R^{12} + 8Q^{24}R^{14}\\
&\quad - 3Q^{21}R^{16} + 9Q^{18}R^{18} + 3Q^{15}R^{20} + 3Q^{12}R^{22} + Q^9R^{24} + 19J,\\
P^4QR\cdot 1^5 &= -4Q^{36}R^6 - Q^{33}R^8 - 3Q^{30}R^{10} + 4Q^{27}R^{12} + 7Q^{24}R^{14}\\
&\quad + 9Q^{21}R^{16} + 2Q^{18}R^{18} + 8Q^{15}R^{20} + 5Q^{12}R^{22} - 7Q^9R^{24} + 19J,\\
P^5Q^2\cdot 1^4 &= 5Q^{39}R^4 - 7Q^{36}R^6 + Q^{33}R^8 + 6Q^{30}R^{10} + 2Q^{27}R^{12} - 7Q^{24}R^{14}\\
&\quad - Q^{21}R^{16} - Q^{18}R^{18} - 8Q^{15}R^{20} - 8Q^{12}R^{22} + 19J,\\
P^6R\cdot 1^3 &= 8Q^{39}R^4 + 6Q^{36}R^6 - 8Q^{33}R^8 - 6Q^{30}R^{10} - 9Q^{27}R^{12} + 6Q^{24}R^{14}\\
&\quad + 4Q^{21}R^{16} + Q^{18}R^{18} + 0Q^{15}R^{20} - Q^{12}R^{22} + 19J,\\
P^7Q\cdot 1^2 &= 9Q^{42}R^2 + 9Q^{39}R^4 + 0Q^{36}R^6 + 6Q^{33}R^8 - 6Q^{30}R^{10} + Q^{27}R^{12}\\
&\quad - 3Q^{24}R^{14} + 4Q^{21}R^{16} - 7Q^{18}R^{18} + 7Q^{15}R^{20} + 19J,\\
P^9 &= Q^{45} - 8Q^{42}R^2 + Q^{39}R^4 - 7Q^{36}R^6 + 3Q^{33}R^8 - 9Q^{30}R^{10}\\
&\quad - Q^{27}R^{12} + 4Q^{24}R^{14} - 3Q^{21}R^{16} + Q^{18}R^{18} + 19J.
\end{aligned}
$$

If we multiply these last equations by $a, b, c, \ldots,$ respectively, add, and equate coefficients on the right side with the coefficients on the right side of (5.16.36), we obtain a set of sixteen simultaneous congruence relations. Solving these, we find that we can take

$$
\begin{aligned}
&a = 1, \quad b = -5, \quad c = 6, \quad d = 5, \quad e = 0, \quad f = 1,\\
&g = 0, \quad h = -1, \quad i = 2, \quad j = -1, \quad k = -9, \quad \ell = 1,
\end{aligned}
$$

whence we obtain the relation (5.16.32).

We now write

$$
\begin{aligned}
&P^9 - 5P^7Q + 6P^6R + 5P^5Q^2 + P^3Q^3 - P^2Q^2R\\
&+ 2PQ^4 - PQR^2 - 9Q^3R + R^3 + 19J\\
&= \{P^7 + 9P^5Q + 2P^4R + 8P^3Q^2 - 9P^2QR - 4PQ^3 + PR^2 + 7Q^2R\}\\
&\quad \times (P^2 - Q)
\end{aligned}
$$

$$
\begin{aligned}
&+\{6P^6-3P^4Q+7P^3R-5P^2Q^2-PQR+2Q^3-5R^2\}(PQ-R)\\
&+\{-9P^5+P^3Q+6P^2R+4PQ^2+4QR\}(PR-Q^2)\\
&+4(Q^3R-R^3)+19J\\
=&\{-7P^7-6P^5Q+5P^4R+P^3Q^2+6P^2QR+9PQ^3-7PR^2+8Q^2R\}\vartheta P\\
&+\{-P^6-9P^4Q+2P^3R+4P^2Q^2-3PQR+6Q^3+4R^2\}\vartheta Q\\
&+\{P^5+2P^3Q-7P^2R+8PQ^2+8QR\}\vartheta R+4(Q^3R-R^3)+19J\\
=&-8\vartheta P^8-\vartheta P^6Q+\vartheta P^5R+5\vartheta P^4Q^2+2\vartheta P^3QR-5\vartheta P^2Q^3+6\vartheta P^2R^2\\
&+8\vartheta PQ^2R-8\vartheta Q^4+4\vartheta QR^2+4(Q^3R-R^3)+19J\\
=&\,q\frac{dJ}{dq}+4(Q^3R-R^3)+19J.
\end{aligned}
\tag{5.16.37}
$$

Combining (5.16.37) with (5.16.32), we finally deduce that

$$
(Q^3-R^2)^{15}=q\frac{dJ}{dq}+4(Q^3R-R^3)+19J. \tag{5.16.38}
$$

But by (5.1.4) and the binomial theorem,

$$
(Q^3-R^2)^{15}=1728^{15}q^{15}(q;q)_\infty^{360}=-q^{15}\frac{(q^{361};q^{361})_\infty}{(q;q)_\infty}+19J. \tag{5.16.39}
$$

Also, by (5.1.4),

$$
Q^3R-R^3=1728Rq(q;q)_\infty^{24}=-Rq(q;q)_\infty^{24}+19J=-\sum_{n=1}^{\infty}\tau_3(n)q^n+19J. \tag{5.16.40}
$$

Thus, from (5.16.38)–(5.16.40) we deduce that

$$
q^{15}\frac{(q^{361};q^{361})_\infty}{(q;q)_\infty}=q\frac{dJ}{dq}+4\sum_{n=1}^{\infty}\tau_3(n)q^n+19J. \tag{5.16.41}
$$

However,

$$
\tau_3(19n)-\tau_3(19)\tau_3(n)\equiv 0 \pmod{19}, \tag{5.16.42}
$$

and a direct calculation shows that

$$
\tau_3(19)\equiv 6 \pmod{19}. \tag{5.16.43}
$$

Thus, extracting only those terms with powers of the form q^{19n}, using (5.16.42) and (5.16.43) in (5.16.41), and lastly replacing q^{19} by q, we conclude that

$$
(q^{19};q^{19})_\infty\sum_{n=1}^{\infty}p(19n-15)q^n=5\sum_{n=1}^{\infty}\tau_3(n)q^n+19J. \tag{5.16.44}
$$

Furthermore, equating those powers of q^{19} in (5.16.44), employing (5.16.42) and (5.16.43), and replacing q^{19} by q, we deduce that

$$(q;q)_\infty \sum_{n=1}^{\infty} p(19^2n-15)q^n = 5\sum_{n=1}^{\infty} \tau_3(19n)q^n + 19J = -8\sum_{n=1}^{\infty} \tau_3(19n)q^n + 19J. \tag{5.16.45}$$

The identities (5.16.44) and (5.16.45) are, respectively, (5.16.3) and (5.16.8) in Ramanujan's manuscript, and moreover we have determined that in (5.16.8), $c_3 = -8$.

Our last goal is to prove (5.16.4) and (5.16.9) in Ramanujan's manuscript. Again, we follow Rushforth's [305] adaptation of Ramanujan's ideas. A somewhat different proof is given by Rushforth in his paper [306]. To do this, we need to establish the identity

$$\begin{aligned}(Q^3-R^2)^{22} = P^{11} &- 6P^9Q - 8P^8R + 9P^7Q^2 - 3P^6QR - 4P^5Q^3 \\ &- 2P^5R^2 - 3P^4Q^2R + P^3Q^4 - 10P^3QR^2 + 5P^2Q^3R \\ &- 7P^2R^2 + PQ^5 - 5PQ^2R^2 + Q^4R + 7QR^3 + 23J. \end{aligned}\tag{5.16.46}$$

From Ramanujan's tables [275], [281, p. 141, Table I, nos. 11, 12],

$$77683 - 552\Phi_{0,21}(q) = 57183Q^4R + 20500QR^3 \tag{5.16.47}$$

and

$$236364091 + 131040\Phi_{0,23}(q) = 49679091Q^6 + 176400000Q^3R^2 + 10285000R^4. \tag{5.16.48}$$

Using (5.16.47), (5.16.48), (5.16.18), and the elementary congruence

$$\Phi_{0,1}(q) = \Phi_{0,23}(q) + 23J,$$

we find, respectively, that

$$-11 = 5Q^4R + 7QR^3 + 23J, \tag{5.16.49}$$

$$P = 9Q^6 + 2Q^3R^2 - 10R^4 + 23J. \tag{5.16.50}$$

Expanding $(Q^3-R^2)^{22}$, we find that

$$(Q^3-R^2)^{22} = Q^{66} + Q^{63}R^2 + Q^{60}R^4 + \cdots + Q^3R^{42} + R^{44} + 23J, \tag{5.16.51}$$

where each of the coefficients in (5.16.51) is 1. We now express P^{11}, P^9Q, $P^8R, \ldots$ in terms of Q and R by calculating the powers of P from (5.16.50). We find that

$$\begin{aligned}PQ^2R^2 &= 9Q^8R^2 + 2Q^5R^4 - 10Q^2R^6 + 23J,\\ PQ^5 &= 9Q^{11} + 2Q^8R^2 - 10Q^5R^4 + 23J,\\ P^2R^3 &= -11Q^{12}R^3 - 10Q^9R^5 + 8Q^6R^7 + 6Q^3R^9 + 8R^{11} + 23J,\end{aligned}$$

$$
\begin{aligned}
P^2Q^3R &= -11Q^{15}R - 10Q^{12}R^3 + 8Q^9R^5 + 6Q^6R^7 + 8Q^3R^9 + 23J,\\
P^3QR^2 &= -7Q^{19}R^2 + 3Q^{16}R^4 + Q^{13}R^6 + 9Q^{10}R^8 + 4Q^7R^{10} + 2Q^4R^{12}\\
&\quad - 11QR^{14} + 23J,\\
P^3Q^4 &= -7Q^{12} + 3Q^{19}R^2 + 6Q^{16}R^4 + 9Q^{13}R^6 + 4Q^{10}R^8 + 2Q^7R^{10}\\
&\quad - 11Q^4R^{12} + 23J,\\
P^4Q^2R &= 6Q^{26}R - 10Q^{23}R^3 - 7Q^{20}R^5 + 7Q^{17}R^7 - 2Q^{14}R^9 + 5Q^{11}R^{11}\\
&\quad + 3Q^8R^{13} + 4Q^5R^{15} - 5Q^2R^{17} + 23J,\\
P^5R^2 &= 8Q^{30}R^2 - 9Q^{27}R^4 - 5Q^{24}R^6 + 11Q^{21}R^8 - 3Q^{18}R^{10} - 6Q^{15}R^{12}\\
&\quad + 11Q^{12}R^{14} - 8Q^9R^{16} + 2Q^6R^{18} - 4Q^3R^{20} + 4R^{22} + 23J,\\
P^5Q^3 &= 8Q^{33} - 9Q^{30}R^2 - 5Q^{27}R^4 + 11Q^{24}R^6 - 3Q^{21}R^8 - 6Q^{18}R^{10}\\
&\quad + 11Q^{15}R^{12} - 8Q^{12}R^{14} + 2Q^9R^{16} - 4Q^6R^{18} + 4Q^3R^{20} + 23J,\\
P^6QR &= 3Q^{37}R + 4Q^{34}R^3 - 5Q^{31}R^5 - 5Q^{28}R^7 - Q^{25}R^9 - 9Q^{22}R^{11}\\
&\quad + 2Q^{19}R^{13} + 10Q^{16}R^{15} + 7Q^{13}R^{17} + 2Q^{10}R^{19} + 8Q^7R^{21}\\
&\quad + 2Q^4R^{23} + 6QR^{25} + 23J,\\
P^7Q^2 &= 4Q^{44} - 4Q^{41}R^2 + 2Q^{38}R^4 - 3Q^{35}R^6 + 8Q^{32}R^8 - 10Q^{29}R^{10}\\
&\quad + 10Q^{26}R^{12} + 0Q^{23}R^{14} - 6Q^{20}R^{16} + Q^{17}R^{18} + 6Q^{14}R^{20}\\
&\quad - 9Q^{11}R^{22} + Q^8R^{24} - 8Q^5R^{26} + 9Q^2R^{28} + 23J,\\
P^8R &= -10Q^{48}R - 5Q^{45}R^3 - 7Q^{42}R^5 - 6Q^{39}R^7 + 0Q^{36}R^9 + 2Q^{33}R^{11}\\
&\quad - 10Q^{30}R^{13} + 5Q^{27}R^{15} + 7Q^{24}R^{17} - 3Q^{21}R^{19} + Q^{18}R^{21}\\
&\quad - 10Q^{15}R^{23} + 0Q^{12}R^{25} - 3Q^9R^{27} + 9Q^6R^{29} + 6Q^3R^{31} + 2R^{33}\\
&\quad + 23J,\\
P^9Q &= 2Q^{55} + 4Q^{52}R^2 + 4Q^{49}R^4 + 5Q^{46}R^6 - 11Q^{43}R^8 + 9Q^{40}R^{10}\\
&\quad + 6Q^{37}R^{12} + 5Q^{34}R^{14} - 11Q^{31}R^{16} + 6Q^{28}R^{18} + 2Q^{25}R^{20}\\
&\quad + 11Q^{22}R^{22} - 7Q^{19}R^{24} + 4Q^{16}R^{26} + 6Q^{13}R^{28} + 10Q^{10}R^{30}\\
&\quad + 9Q^7R^{32} - 10Q^4R^{34} + 3QR^{36} + 23J,\\
P^{11} &= Q^{66} + 5Q^{63}R^2 + Q^{60}R^4 - 5Q^{57}R^6 + 5Q^{54}R^8 - 8Q^{51}R^{10} + 2Q^{48}R^{12}\\
&\quad + 0Q^{54}R^{14} - 7Q^{42}R^{16} + 8Q^{39}R^{18} + 0Q^{36}R^{20} - 4Q^{33}R^{22}\\
&\quad + 0Q^{30}R^{24} - 10Q^{27}R^{26} - 11Q^{24}R^{28} + 0Q^{21}R^{30} + Q^{18}R^{32}\\
&\quad + 7Q^{15}R^{34} - 6Q^{12}R^{36} + Q^9R^{38} - 10Q^6R^{40} + 7Q^3R^{42} + R^{44} + 23J.
\end{aligned}
$$

We now use (5.16.49) to calculate powers of 1. To that end,

$$
\begin{aligned}
1 &= 10Q^4R - 9QR^3 + 23J,\\
1^3 &= 11Q^{12}R^3 - 9Q^9R^5 - 8Q^6R^7 + 7Q^3R^9 + 23J,\\
1^4 &= -5Q^{16}R^4 - 5Q^{13}R^6 + Q^{10}R^8 + 4Q^7R^{10} + 6Q^4R^{12} + 23J,
\end{aligned}
$$

$$
\begin{aligned}
1^5 &= -4Q^{20}R^5 - 5Q^{17}R^7 + 9Q^{14}R^9 + 8Q^{11}R^{11} + Q^8R^{13} - 8Q^5R^{15} + 23J,\\
1^6 &= 6Q^{24}R^6 + 9Q^{21}R^8 - 3Q^{18}R^{10} - Q^{15}R^{12} + 7Q^{12}R^{14} + 3Q^9R^{16}\\
&\quad + 3Q^6R^{18} + 23J,\\
1^7 &= -9Q^{28}R^7 - 10Q^{25}R^9 + 4Q^{22}R^{11} - 6Q^{19}R^{13} + 10Q^{16}R^{15} - 10Q^{13}R^{17}\\
&\quad + 3Q^{10}R^{19} - 4Q^7R^{21} + 23J,\\
1^8 &= 2Q^{32}R^8 + 4Q^{29}R^{10} - 8Q^{26}R^{12} - 4Q^{23}R^{14} - 7Q^{20}R^{16} - 6Q^{17}R^{18}\\
&\quad + 5Q^{14}R^{20} + 2Q^{11}R^{22} - 10Q^8R^{24} + 23J,\\
1^9 &= -3Q^{36}R^9 - Q^{33}R^{11} - Q^{30}R^{13} + 9Q^{27}R^{15} - 11Q^{24}R^{17} + 3Q^{21}R^{19}\\
&\quad - 11Q^{18}R^{21} - 2Q^{15}R^{23} - 3Q^{12}R^{25} - 2Q^9R^{27} + 23J,\\
1^{10} &= -7Q^{40}R^{10} - 6Q^{37}R^{12} - Q^{34}R^{14} + 7Q^{31}R^{16} - 7Q^{28}R^{18} - 9Q^{25}R^{20}\\
&\quad + Q^{22}R^{22} + 10Q^{19}R^{24} + 11Q^{16}R^{26} + 7Q^{13}R^{28} - 5Q^{10}R^{30} + 23J,\\
1^{11} &= -Q^{44}R^{11} + 3Q^{41}R^{13} - 2Q^{38}R^{15} + 10Q^{35}R^{17} + 5Q^{32}R^{19} - 4Q^{29}R^{21}\\
&\quad - Q^{26}R^{23} - Q^{23}R^{25} - 3Q^{20}R^{27} - 6Q^{17}R^{29} + 2Q^{14}R^{31} - Q^{11}R^{33} + 23J.
\end{aligned}
$$

Using these equalities in the previous set, we find that

$$
\begin{aligned}
QR^3 \cdot 1^{11} &= -Q^{45}R^{14} + 3Q^{42}R^{16} - 2Q^{39}R^{18} + 10Q^{36}R^{20} + 5Q^{33}R^{22}\\
&\quad - 4Q^{30}R^{24} - Q^{27}R^{26} - Q^{24}R^{28} - 3Q^{21}R^{30} - 6Q^{18}R^{32}\\
&\quad + 2Q^{15}R^{34} - Q^{12}R^{36} + 23J,\\
Q^4R \cdot 1^{11} &= -Q^{48}R^{12} + 3Q^{45}R^{14} - 2Q^{42}R^{16} + 10Q^{39}R^{18} + 5Q^{36}R^{20}\\
&\quad - 4Q^{33}R^{22} - Q^{30}R^{24} - Q^{27}R^{26} - 3Q^{24}R^{28} - 6Q^{21}R^{30}\\
&\quad + 2Q^{18}R^{32} - Q^{15}R^{34} + 23J,\\
PQ^2R^2 \cdot 1^{10} &= 6Q^{48}R^{12} + Q^{45}R^{14} + 3Q^{42}R^{16} + 6Q^{39}R^{18} + 7Q^{36}R^{20}\\
&\quad - 4Q^{33}R^{22} - 8Q^{30}R^{24} - 2Q^{27}R^{26} - 6Q^{24}R^{28} + 8Q^{21}R^{30}\\
&\quad - 3Q^{18}R^{32} - 11Q^{15}R^{34} + 4Q^{12}R^{36} + 23J,\\
PQ^5 \cdot 1^{10} &= 6Q^{51}R^{10} + Q^{48}R^{12} + 3Q^{45}R^{14} + 6Q^{42}R^{16} + 7Q^{39}R^{18}\\
&\quad - 4Q^{36}R^{20} - 8Q^{33}R^{22} - 2Q^{30}R^{24} - 6Q^{27}R^{26} + 8Q^{24}R^{28}\\
&\quad - 3Q^{21}R^{30} - 11Q^{18}R^{32} + 4Q^{15}R^{34} + 23J,\\
P^2R^3 \cdot 1^9 &= 10Q^{48}R^{12} - 5Q^{45}R^{14} - 3Q^{42}R^{16} + 0Q^{39}R^{18} - 7Q^{36}R^{20}\\
&\quad - 3Q^{33}R^{22} + 3Q^{30}R^{24} + Q^{27}R^{26} + 10Q^{24}R^{28} - 6Q^{21}R^{30}\\
&\quad + 11Q^{18}R^{32} - 4Q^{15}R^{34} + 10Q^{12}R^{36} + 7Q^9R^{38} + 23J,\\
P^2Q^3R \cdot 1^9 &= 10Q^{51}R^{10} - 5Q^{48}R^{12} - 3Q^{45}R^{14} + 0Q^{42}R^{16} - 7Q^{39}R^{18}\\
&\quad - 3Q^{36}R^{20} + 3Q^{33}R^{22} + Q^{30}R^{24} + 10Q^{27}R^{26} - 6Q^{24}R^{28}\\
&\quad + 11Q^{21}R^{30} - 4Q^{18}R^{32} + 10Q^{15}R^{34} + 7Q^{12}R^{36} + 23J,\\
P^3QR^2 \cdot 1^8 &= 9Q^{51}R^{10} + Q^{48}R^{12} + Q^{45}R^{14} + 3Q^{42}R^{16} + 4Q^{39}R^{18}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 11Q^{36}R^{20} - 4Q^{33}R^{22} - 6Q^{30}R^{24} + 10Q^{27}R^{26} + 0Q^{24}R^{28} \\
&\quad + Q^{21}R^{30} - 6Q^{18}R^{32} + Q^{15}R^{34} + 4Q^{12}R^{36} - 5Q^{9}R^{38} + 23J, \\
P^3Q^4 \cdot 1^8 &= 9Q^{54}R^8 + Q^{51}R^{10} + Q^{48}R^{12} + 3Q^{45}R^{14} + 4Q^{42}R^{16} \\
&\quad + 11Q^{39}R^{18} - 4Q^{36}R^{20} - 6Q^{33}R^{22} + 10Q^{30}R^{24} + 0Q^{27}R^{26} \\
&\quad + Q^{24}R^{28} - 6Q^{21}R^{30} + Q^{18}R^{32} + 4Q^{15}R^{34} - 5Q^{12}R^{36} + 23J, \\
P^4Q^2R \cdot 1^7 &= -8Q^{54}R^8 + 7Q^{51}R^{10} + 3Q^{48}R^{12} + 0Q^{45}R^{14} - 6Q^{42}R^{16} \\
&\quad + 0Q^{39}R^{18} - 10Q^{36}R^{20} + 6Q^{33}R^{22} + 8Q^{30}R^{24} + 6Q^{27}R^{26} \\
&\quad - 6Q^{24}R^{28} - 6Q^{21}R^{30} - 9Q^{18}R^{32} + 4Q^{15}R^{34} - 8Q^{12}R^{36} \\
&\quad - 3Q^{9}R^{38} + 23J, \\
P^5R^2 \cdot 1^6 &= 2Q^{54}R^8 - 5Q^{51}R^{10} + 3Q^{48}R^{12} - 6Q^{45}R^{14} + 0Q^{42}R^{16} \\
&\quad + 8Q^{39}R^{18} - 5Q^{36}R^{20} - 8Q^{33}R^{22} + 2Q^{30}R^{24} - 11Q^{27}R^{26} \\
&\quad - 6Q^{24}R^{28} + 5Q^{21}R^{30} - 8Q^{18}R^{32} - 4Q^{15}R^{34} - Q^{12}R^{36} \\
&\quad + 0Q^{9}R^{38} - 11Q^{6}R^{40} + 23J, \\
P^5Q^3 \cdot 1^6 &= 2Q^{57}R^6 - 5Q^{54}R^8 + 3Q^{51}R^{10} - 6Q^{48}R^{12} + 0Q^{45}R^{14} \\
&\quad + 8Q^{42}R^{16} - 5Q^{39}R^{18} - 8Q^{36}R^{20} + 2Q^{33}R^{22} - 11Q^{30}R^{24} \\
&\quad - 6Q^{27}R^{26} + 5Q^{24}R^{28} - 8Q^{21}R^{30} - 4Q^{18}R^{32} - Q^{15}R^{34} \\
&\quad + 0Q^{12}R^{36} - 11Q^{9}R^{38} + 23J, \\
P^6QR \cdot 1^5 &= 11Q^{57}R^6 - 8Q^{54}R^8 + 4Q^{51}R^{10} - 10Q^{48}R^{12} - 4Q^{45}R^{14} \\
&\quad + 5Q^{42}R^{16} - 3Q^{39}R^{18} + 11Q^{36}R^{20} - Q^{33}R^{22} - 7Q^{30}R^{24} \\
&\quad - 9Q^{27}R^{26} - 3Q^{24}R^{28} + 4Q^{21}R^{30} - 2Q^{18}R^{32} - 7Q^{15}R^{34} \\
&\quad + 9Q^{12}R^{36} - 10Q^{9}R^{38} - 2Q^{6}R^{40} + 23J, \\
P^7Q^2 \cdot 1^4 &= 3Q^{60}R^4 + 0Q^{57}R^6 - 9Q^{54}R^8 - 6Q^{51}R^{10} + 8Q^{48}R^{12} \\
&\quad - 9Q^{45}R^{14} + 8Q^{42}R^{16} + 0Q^{39}R^{18} + 2Q^{36}R^{20} + 5Q^{33}R^{22} \\
&\quad - 4Q^{30}R^{24} - 8Q^{27}R^{26} - 9Q^{24}R^{28} + 10Q^{21}R^{30} - 4Q^{18}R^{32} \\
&\quad - 11Q^{15}R^{34} + 6Q^{12}R^{36} + 11Q^{9}R^{38} + 8Q^{6}R^{40} + 23J, \\
P^8R \cdot 1^3 &= 5Q^{60}R^4 - 11Q^{57}R^6 + 2Q^{54}R^8 - 10Q^{51}R^{10} + 6Q^{48}R^{12} \\
&\quad - 2Q^{45}R^{14} - 9Q^{42}R^{16} - 9Q^{39}R^{18} + 11Q^{36}R^{20} + Q^{33}R^{22} \\
&\quad - 6Q^{30}R^{24} + 0Q^{27}R^{26} - 8Q^{24}R^{28} + 8Q^{21}R^{30} + 10Q^{18}R^{32} \\
&\quad + 9Q^{15}R^{34} - 10Q^{12}R^{36} - 3Q^{9}R^{38} + 3Q^{6}R^{40} - 9Q^{3}R^{42} + 23J, \\
P^9Q \cdot 1^2 &= -7Q^{63}R^2 - 6Q^{60}R^4 + 3Q^{57}R^6 - 11Q^{54}R^8 + 3Q^{51}R^{10} \\
&\quad - 4Q^{48}R^{12} - 2Q^{45}R^{14} + 11Q^{42}R^{16} + 4Q^{39}R^{18} - 5Q^{36}R^{20} \\
&\quad + 0Q^{33}R^{22} + 7Q^{30}R^{24} - 11Q^{27}R^{26} - 2Q^{24}R^{28} + 3Q^{21}R^{30} \\
&\quad - 9Q^{18}R^{32} + 0Q^{15}R^{34} + 7Q^{12}R^{36} + 0Q^{9}R^{38} + 7Q^{6}R^{40}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 10Q^3R^{42} + 23J,\\
P^{11} &= Q^{66} + 5Q^{63}R^2 + Q^{60}R^4 - 5Q^{57}R^6 + 5Q^{54}R^8 - 8Q^{51}R^{10}\\
&\quad + 2Q^{48}R^{12} + 0Q^{45}R^{14} - 7Q^{42}R^{16} + 8Q^{39}R^{18} + 0Q^{36}R^{20}\\
&\quad - 4Q^{33}R^{22} + 0Q^{30}R^{24} - 10Q^{27}R^{26} - 11Q^{24}R^{28} + 0Q^{21}R^{30}\\
&\quad + Q^{18}R^{32} + 7Q^{15}R^{34} - 6Q^{12}R^{36} + Q^9R^{38} - 10Q^6R^{40}\\
&\quad + 7Q^3R^{42} + R^{44} + 23J.
\end{aligned}
$$

Multiplying this last set of equations by $a, b, c, \ldots, p, q$ (with o omitted to avoid confusion), respectively, adding them, and equating coefficients of $Q^{66}, Q^{63}R^2, \ldots$ with the corresponding terms in (5.16.51), we obtain a set of 23 simultaneous linear congruence relations in $a, b, c, \ldots$. These are consistent, and one solution set is given by

$$
\begin{aligned}
&a = 1, \quad b = -6, \quad c = -8, \quad d = 9, \quad e = -3, \quad f = -4, \quad g = -2, \quad h = -3,\\
&i = 1, \quad j = -10, \quad k = 5, \quad \ell = -7, \quad m = 1, \quad n = -5, \quad p = 1, \quad q = 7,
\end{aligned}
$$

which gives the required result, (5.16.46).

But now,

$$
\begin{aligned}
&P^{11} - 6P^9Q - 8P^8R + 9P^7Q^2 - 3P^6QR - 4P^5Q^3 - 2P^5R^2 - 3P^4Q^2R\\
&\quad + P^3Q^4 - 10P^3QR^2 + 5P^2Q^3R - 7P^2R^3 + PQ^5 - 5PQ^2R^2 + Q^4R\\
&\quad + 7QR^3 + 23J\\
&= \left\{P^9 - 11P^7Q + 6P^6R - P^5Q^2 - 2P^4QR - 3P^3Q^3 + 6P^2R^2 - 2P^2Q^2R\right.\\
&\quad \left. + 7PQ^4 - 7PQR^2 - 10Q^3R + 11R^3\right\}(P^2 - Q)\\
&\quad + \left\{6P^8 - 9Q^6Q + 3P^5R - 9P^4Q^2 + 10P^3QR + 10P^2Q^3 + 9P^2R^2\right.\\
&\quad \left. - 5PQ^2R - 6Q^4 + 8QR^2\right\}(PQ - R)\\
&\quad + \left\{-8P^7 - 7P^5Q - 5P^4R - 4P^3Q^2 + 4P^2QR + 9PQ^3 - 9PR^2 - 11Q^2R\right\}\\
&\qquad \times (PR - Q^2) - 3(Q^4R - QR^3) + 23J\\
&= \left\{-11P^9 + 6P^7Q + 3P^6R + 11P^5Q^2 - P^4QR + 10P^3Q^3 + 3P^3R^2\right.\\
&\quad \left. -P^2Q^2R - 8PQ^4 + 8PQR^2 - 5Q^3R - 6R^3\right\}\vartheta P\\
&\quad + \left\{-5P^8 - 4P^6Q + 9P^5R - 4P^4Q^2 + 7P^3QR + 7P^2Q^3 + 4P^2R^2\right.\\
&\quad \left. + 8PQ^2R + 5Q^4 + QR^2\right\}\vartheta Q\\
&\quad + \left\{7P^7 + 9P^5Q - 10P^4R - 8P^3Q^2 + 8P^2QR - 5PQ^3 + 5PR^2 + Q^2R\right\}\vartheta R\\
&\quad - 3(Q^4R - QR^3) + 23J\\
&= -8\vartheta P^{10} - 5\vartheta P^8Q + 7\vartheta P^7R - 2\vartheta P^6Q^2 + 9\vartheta P^5QR - 9\vartheta P^4Q^3\\
&\quad - 5\vartheta P^4R^2 - 8\vartheta P^3Q^2R - 4\vartheta P^2Q^4 + 4\vartheta P^2QR^2 - 5\vartheta PQ^3R - 6\vartheta PR^3\\
&\quad + \vartheta Q^5 - 11\vartheta Q^2R^2 - 3(Q^4R - QR^3) + 23J\\
&= q\frac{dJ}{dq} - 3(Q^4R - QR^3) + 23J. \qquad (5.16.52)
\end{aligned}
$$

On the other hand, by the binomial theorem,

$$(Q^3 - R^2)^{22} = 1728^{22} q^{22} (q;q)_\infty^{528} = q^{22} \frac{(q^{529};q^{529})_\infty}{(q;q)_\infty} + 23J. \tag{5.16.53}$$

By the definition of $\tau_5(n)$ in (5.16.5),

$$Q^4 R - QR^3 = QR \cdot 1728 q (q;q)_\infty^{24} = 3 \sum_{n=1}^{\infty} \tau_5(n) q^n + 23J. \tag{5.16.54}$$

Combining (5.16.46) and (5.16.52)–(5.16.54), we finally conclude that

$$q^{22} \frac{(q^{529};q^{529})_\infty}{(q;q)_\infty} = q \frac{dJ}{dq} - 9 \sum_{n=1}^{\infty} \tau_5(n) q^n + 23J. \tag{5.16.55}$$

However, we know that

$$\tau_5(23n) - \tau_5(23)\tau_5(n) \equiv 0 \pmod{23}. \tag{5.16.56}$$

Also, using the relation

$$\sum_{n=1}^{\infty} \tau_5(n) q^n = QR \sum_{n=1}^{\infty} \tau(n) q^n,$$

we can determine by a direct calculation that

$$\tau_5(23) \equiv 5 \pmod{23}.$$

Therefore, from (5.16.56),

$$\tau_5(23n) - 5\tau_5(n) \equiv 0 \pmod{23}. \tag{5.16.57}$$

Choosing only those terms corresponding to q^{23n} from (5.16.55), using (5.16.56) and (5.16.57), and replacing q^{23} by q, we conclude that

$$(q^{23};q^{23}) \sum_{n=1}^{\infty} p(23n-22) q^n = \sum_{n=1}^{\infty} \tau_5(n) q^n + 23J, \tag{5.16.58}$$

thus proving Ramanujan's assertion (5.16.4). Moreover, extracting the powers q^{23n} from (5.16.58), employing (5.16.56) and (5.16.57), and replacing q^{23} by q, we further conclude that

$$(q;q) \sum_{n=1}^{\infty} p(23^2 n - 22) q^n = 5 \sum_{n=1}^{\infty} \tau_5(n) q^n + 23J,$$

and thus we have proved (5.16.9) with $c_5 = 5$.

The equalities (5.16.4) and (5.16.9) are proved in Rushforth's paper [306]. As with the key results in Section 5.14, the claims (5.16.1)–(5.16.4) and (5.16.7)–(5.16.11) follow from the work of Ono [258].

In (5.16.6), Ramanujan claims that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\tau_2(n)}{n^s}, \quad \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^s},$$
$$\sum_{n=1}^{\infty} \frac{\tau_4(n)}{n^s}, \quad \sum_{n=1}^{\infty} \frac{\tau_5(n)}{n^s}, \quad \sum_{n=1}^{\infty} \frac{\tau_7(n)}{n^s}$$

have Euler products. This is easily verified, since all corresponding modular forms are eigenforms of Hecke operators, for they each lie in a one-dimensional space of cusp forms [118]. In a letter written in 1939 to H. Heilbronn, Watson indicates that Ramanujan's claims about these Dirichlet series have yet to be proved. About this letter, Rankin [292], [69, p. 136] remarks,

> It is rather astonishing that in March 1939 Watson should have believed that the five Euler products require proof, since after Mordell's work [226] one would have thought he could himself have provided one. Moreover, he cannot have been aware of the fundamental work of Hecke [170] on Euler products, which had appeared two years earlier, in which these results appear as particular cases.

At the end of Section 5.16, Ramanujan claims that the two Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\Omega_2(n)}{n^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\Omega_3(n)}{n^s}$$

are both differences of two series with Euler products. In terms of classical modular forms, $\sum_{n=1}^{\infty} \Omega_2(n)q^n \in S_{28}(\Gamma_0(1))$ and $\sum_{n=1}^{\infty} \Omega_3(n)q^n \in S_{30}(\Gamma_0(1))$. Both of these spaces are two-dimensional [118], and one can easily check that

$$S_{28}(\Gamma_0(1)) = \mathbb{C}Q\Delta^2 \oplus \mathbb{C}Q^4\Delta,$$
$$S_{30}(\Gamma_0(1)) = \mathbb{C}R\Delta^2 \oplus \mathbb{C}R^3\Delta,$$

where $\Delta := \Delta(q) = q(q;q)_\infty^{24}$. It is easy to show that the space $S_{28}(\Gamma_0(1))$ is spanned by the eigenforms

$$f_1(q) := \sum_{n=1}^{\infty} a_1(n)q^n := (-5076 + 108\sqrt{18209})Q\Delta^2 + Q^4\Delta,$$
$$f_2(q) := \sum_{n=1}^{\infty} a_2(n)q^n := (-5076 - 108\sqrt{18209})Q\Delta^2 + Q^4\Delta.$$

(For calculations of this sort, see N. Koblitz's text [195, p. 173, Proposition 51].) Since $f_1(q)$ and $f_2(q)$ are eigenforms, the two Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_1(n)}{n^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s}$$

have Euler products as in (5.16.6) with weight 28. The "difference" to which Ramanujan alludes is the identity

$$Q\Delta^2 = \sum_{n=1}^{\infty} \Omega_2(n)q^n = \frac{1}{216\sqrt{18209}}(f_1(q) - f_2(q)).$$

Similarly, one can easily verify that the space $S_{30}(\Gamma_0(1))$ is spanned by the eigenforms

$$g_1(q) := \sum_{n=1}^{\infty} b_1(n)q^n := (5856 + 96\sqrt{51349})R\Delta^2 + R^3\Delta,$$
$$g_2(q) := \sum_{n=1}^{\infty} b_2(n)q^n := (5856 - 96\sqrt{51349})R\Delta^2 + R^3\Delta.$$

Since $g_1(q)$ and $g_2(q)$ are eigenforms, it easily follows that the two Dirichlet series

$$\sum_{n=1}^{\infty} \frac{b_1(n)}{n^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b_2(n)}{n^s}$$

have Euler products as in (5.16.6) with weight 30. The difference to which Ramanujan alludes is the identity

$$R\Delta^2 = \sum_{n=1}^{\infty} \Omega_3(n)q^n = \frac{1}{192\sqrt{51349}}(g_1(q) - g_2(q)).$$

5.17 Divisibility of $\tau(n)$ by 23

The claim (5.17.2) is equivalent to the assertion that $\eta(z)\eta(23z)$ is an eigenform with complex multiplication in the space $S_1(\Gamma_0(23), \chi_{-23})$ [195, p. 127], [139, p. 472], [158], [293], [118], where χ_{-23} is the usual Kronecker character for the quadratic field $\mathbb{Q}(\sqrt{-23})$. Although the claims regarding the Euler products Π_1, Π_2, and Π_3 follow easily from the theory of complex multiplication, one can more easily obtain them from Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

Here one would use an argument similar to that briefly outlined in the commentary for Section 5.8. In analogy with our comments in Sections 5.2 and 5.6, the claim (5.17.8) is false, but the leading term in the asymptotic expansion is indeed $Cx/\sqrt{\log x}$. More precisely, in the notation (5.2.9), according to Ramanujan, $c_2 = \frac{1}{2}$, but the actual value is $c_2 = 0.6083\ldots$ [228].

5.18 The Congruence $p(121n - 5) \equiv 0 \pmod{121}$

The proof of (5.18.1) is quite difficult, but it is given in Rushforth's paper [306]. Ramanujan omitted many details in his assertion (5.18.2). For the remainder of the proof of Ramanujan's congruence modulo 11^2 to be completed, it is necessary to explicitly determine the constants a_4, b_4, and c_3 in (5.18.2). Rushforth does not prove (5.18.2) but proceeds by a different route to (5.18.7). The third congruence in (5.18.3) is proved in Rushforth's paper [306]. The equation to which Ramanujan refers before (5.18.7) is not given in the manuscript, but would arise from (5.18.2) using (5.18.3)–(5.18.6).

5.19 Divisibility of $\tau(n)$ for Almost All Values of n

This manuscript contains many results on the divisibility of $\tau(n)$. In several sections Ramanujan concludes that $\tau(n)$ is a multiple of a given integer M for almost all n. In other words, for such M,

$$\lim_{X\to\infty} \frac{\#\{1 \le n \le X \ : \ \tau(n) \equiv 0 \pmod{M}\}}{X} = 1.$$

Specifically, Ramanujan finds in (5.19.9) that $\tau(n)$ is a multiple of $2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 691$ for almost all n. Various authors have proved versions of (5.19.9) with varying exponents on the six primes. It was first proved by Chowla [112], [113, pp. 639–643] that in fact, the conclusion still holds if the powers of 2, 3, 5, 7, 23, and 691 are replaced by any set of six positive integral powers.

Ramanujan obtained his results by employing the congruences for $\tau(n)$ with modulus $M \in \{2^5, 3^3, 5^2, 7^2, 23, 691\}$. In each case, he found that a positive density of primes p has the property that $\tau(p) \equiv 0 \pmod{M}$. A Tauberian argument based on the multiplicativity of $\tau(n)$ then leads to his conclusion [314, Section 2].

Results of this type depend upon the divisibility of divisor functions. Improving on Watson's theorem [334], Rankin [288] found an asymptotic formula for the number of positive integers $\le x$ for which $\sigma_\nu(n)$ is not divisible by the prime number k. These results were generalized by E.J. Scourfield [309].

Serre [313], [314], [315] has obtained a substantial generalization of Ramanujan's claims for all modular forms of integral weight with respect to congruence subgroups of the full modular group. In particular, if $\sum_{n=1}^\infty a(n)q^n$

($q := e^{2\pi i z}$) is the Fourier expansion of a modular form of integral weight with integral coefficients, then for every positive integer M, $a(n) \equiv 0 \pmod{M}$ for almost all n. M.R. Murty and V.K. Murty [233] have obtained an interesting improvement on the original formulation of Serre's result.

Serre's theorem is based on the existence of ℓ-adic Galois representations associated to modular forms (see the comments on Section 5.10). In addition to providing an arithmetic and group-theoretic description of congruences for Fourier coefficients $a(n)$ of the types found by Ramanujan for $\tau(n)$, their mere existence implies, by the Chebotarev density theorem, that a positive proportion of primes p have the property that $a(p) \equiv 0 \pmod{M}$.

Bambah and Chowla [36], [113, pp. 617–619] state without proofs several congruences for $\tau(n)$. Lahiri [202] gives an enormous number of congruences involving $\tau(n)$. F. van der Blij's beautiful paper [78], giving congruences and other properties of $\tau(n)$, is particularly recommended. Except for those employing the theory of ℓ-adic Galois representations, almost all the authors giving proofs of congruences for $\tau(n)$ whom we have cited use ideas similar to those employed by Ramanujan in this manuscript.

5.20 The Congruence $p(5n+4) \equiv 0 \pmod{5}$, Revisited

Sections 5.20–5.23 contain Ramanujan's proof of his congruences for $p(n)$ modulo any positive integral power of 5, with (5.22.5)–(5.22.8) being the principal congruences. Observe that (5.22.7) and (5.22.8) include (5.1.14). As mentioned earlier, the ideas here were expanded into a more detailed proof given in 1938 by Watson [336], who does not mention Part II of Ramanujan's unpublished manuscript in his paper, although according to Rushforth [306], Watson received a copy from Hardy in 1928.

The details in Section 5.20 are reasonably ample, but beginning with Section 5.21, the details are sparse. In particular, (5.21.1) is more difficult to prove, and a proof may be found in a paper by Hirschhorn and D.C. Hunt [181]. The proof given by Watson possibly follows along somewhat different lines from those indicated by Ramanujan. Readers can likely follow the details for the remainder of Section 5.21. We have added some details for (5.21.6), which is not used in Watson's work. The heart of Ramanujan's proof lies in (5.22.1)–(5.22.6), for which Ramanujan provides no details. Hirschhorn [180] pointed out to the authors that the upper index $5^{\lambda-1}$ in (5.22.3) and (5.22.4) is incorrect. The number of terms is $(5^{\lambda+1}-1)/24$ if λ is odd, and $(5^{\lambda+1}-5)/24$ if λ is even. Furthermore, the upper index 25 in (5.22.1) should be replaced by 26. The details of Ramanujan's argument are developed in Watson's paper [336].

5.23 Congruences for $p(n)$ Modulo Higher Powers of 5, Continued

Ramanujan's conjecture that $c_\lambda = (-2)^\lambda$ is correct; it can be deduced from a theorem of Hirschhorn and Hunt [181, Theorem 1.4].

5.24 The Congruence $p(7n+5) \equiv 0 \pmod 7$

Clearly, Ramanujan intended to follow the same lines of attack for powers of 7 as he did for powers of 5 in Sections 20–23. If he had completed his argument, he would have undoubtedly seen that his original conjecture modulo powers of 7 needed to be corrected. Most likely, his declining health prevented him from working out the remaining details, which were completed by Watson [336].

To verify the equations (5.24.5)–(5.24.7), it suffices to notice that all three equations are essentially claims about the presentation of modular functions with respect to $\Gamma_0(7)$. In each case, one may multiply both sides of the claimed identity by $(q^7;q^7)^8$. After doing so, one needs to compare, up to a shifted power of q, the Fourier expansions of two cusp forms of weight 4. One can then easily deduce these claims from the results in [118].

Ramanujan's proof of (5.24.8) is quite elementary, but the algebraic manipulations are a bit tedious. Ramanujan's argument had not been given in detail in the literature until Berndt, Yee, and Yi [70] provided such an argument. However, Garvan [143] has also given a proof close in spirit to that of Ramanujan. The identity (5.24.8) also appears on page 189 of the lost notebook [283] along with three further identities in the same spirit. All four identities were proved in [70]. In the chapter following this one, proofs of these four identities are given. A perhaps more efficient proof of (5.24.8), but using different ideas, has been given by O. Kolberg [197].

At the end of Part II are two detached fragments. They actually appear at the end of Section 21 in Watson's copy of the manuscript, but it seems to us that they are better placed at the end of the section pertaining to the moduli 7 and 49, and so we have taken the liberty of moving them to the end of the manuscript. The four numbers in the penultimate line are the coefficients of the first four terms of the generating function [143, p. 333], [353]

$$\begin{aligned}\sum_{n=0}^{\infty} p(49n+47)q^n = 2546 \cdot 7^2 \frac{(q^7;q^7)_\infty^4}{(q;q)_\infty^5} + 48934 \cdot 7^4 q \frac{(q^7;q^7)_\infty^8}{(q;q)_\infty^9} \\ + 1418989 \cdot 7^5 q^2 \frac{(q^7;q^7)_\infty^{12}}{(q;q)_\infty^{13}} + 2488800 \cdot 7^7 q^3 \frac{(q^7;q^7)_\infty^{16}}{(q;q)_\infty^{17}} \\ + 2394438 \cdot 7^9 q^4 \frac{(q^7;q^7)_\infty^{20}}{(q;q)_\infty^{21}} + 1437047 \cdot 7^{11} q^5 \frac{(q^7;q^7)_\infty^{24}}{(q;q)_\infty^{25}} \\ + 4043313 \cdot 7^{12} q^6 \frac{(q^7;q^7)_\infty^{28}}{(q;q)_\infty^{29}} + 161744 \cdot 7^{15} q^7 \frac{(q^7;q^7)_\infty^{32}}{(q;q)_\infty^{33}}\end{aligned}$$

$$+ 32136 \cdot 7^{17} q^8 \frac{(q^7;q^7)_\infty^{36}}{(q;q)_\infty^{37}} + 31734 \cdot 7^{18} q^9 \frac{(q^7;q^7)_\infty^{40}}{(q;q)_\infty^{41}}$$
$$+ 3120 \cdot 7^{20} q^{10} \frac{(q^7;q^7)_\infty^{44}}{(q;q)_\infty^{45}} + 204 \cdot 7^{22} q^{11} \frac{(q^7;q^7)_\infty^{48}}{(q;q)_\infty^{49}}$$
$$+ 8 \cdot 7^{24} q^{12} \frac{(q^7;q^7)_\infty^{52}}{(q;q)_\infty^{53}} + 7^{25} q^{13} \frac{(q^7;q^7)_\infty^{56}}{(q;q)_\infty^{57}},$$

which can be found in the papers of Garvan [143, p. 333] and Zuckerman [353]. Ramanujan miscalculated the fourth coefficient, and we have corrected it. There is also a copying error by Watson in the second coefficient, which is correct in Ramanujan's copy. These numbers are the only evidence that we have that Ramanujan calculated a portion of the generating function given above.

6

Theorems about the Partition Function on Pages 189 and 182

6.1 Introduction

Let $p(n)$ denote, as usual, the number of unrestricted partitions of the positive integer n. This chapter contains accounts of two pages, 189 and 182 in [283], which are devoted to $p(n)$. Both pages, especially page 182, are related to Ramanujan's paper [276]. Perhaps page 182 is from a preliminary version of [276] that was considerably shortened before it reached the publisher's desk. Our account of page 182 is taken from a paper that the second author coauthored with C. Gugg and S. Kim [66], and that for page 189 is excerpted from a paper that the second author wrote with A.J. Yee and J. Yi [70]. At the conclusion of the chapter, we offer a few remarks on work on partitions found on pages 207, 208, 248, 252, 326, 331, and 333.

In [276], [281, p. 213], Ramanujan offers the beautiful identities

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \tag{6.1.1}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}, \tag{6.1.2}$$

where, as usual,

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1.$$

References to several proofs of (6.1.1) and (6.1.2) can be found in the latest edition of [281, pp. 372–373] and in the commentaries on Sections 5.4 and 5.8 of Chapter 5. Most proofs of (6.1.1) rely on (5.4.6). Ramanujan gave a brief proof of (6.1.1) in [276]; see also Section 5.4. He did not prove (6.1.2) in [276], but he did give a brief sketch of his proof of (6.1.2) in his unpublished manuscript on

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*, 181
DOI 10.1007/978-1-4614-3810-6_6,

the partition and τ-functions [283, pp. 242–243]; see Section 5.24. Note that (6.1.1) and (6.1.2) immediately yield the congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, respectively. For a connection of (6.1.1) with the Virasoro algebra, see a paper by A. Milas [224].

The two identities (6.1.1) and (6.1.2) are stated on page 189 of Ramanujan's lost notebook in the pagination of [283]. Also given by Ramanujan are two further identities. Define $q_m(n), n \geq 0$, by

$$(q;q)_\infty^m =: \sum_{n=0}^{\infty} q_m(n) q^n. \tag{6.1.3}$$

Note that $q_m(n)$ denotes the number of m-colored partitions of n into an even number of distinct parts minus the number of m-colored partitions of n into an odd number of distinct parts. Then

$$\sum_{n=0}^{\infty} q_5(5n) q^n = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty} \tag{6.1.4}$$

and

$$\sum_{n=0}^{\infty} q_7(7n) q^n = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty} + 49q(q;q)_\infty^4 (q^7;q^7)_\infty^3. \tag{6.1.5}$$

For completeness, in Section 6.2, we begin with essentially Ramanujan's proof of (6.1.1). We then prove (6.1.4).

In Section 6.3, we amplify Ramanujan's sketch in [283] and give a complete proof of (6.1.2). A detailed proof of (6.1.2) along the lines outlined by Ramanujan was first given by Berndt, Yee, and Yi [70], and it is that proof that we give here; the next three sections arise from [70]. It should be remarked that F.G. Garvan gave a proof of Ramanujan's general congruence (5.1.15) also along lines with which Ramanujan would have been comfortable, but for the particular instance (6.1.2), his proof is different from the proof that we give. We also prove (6.1.5) in Section 6.3. Both proofs depend on some theta function identities that Ramanujan stated without proof. Thus, in Section 6.3 we also give proofs of these required identities.

Immediately above the four principal identities (6.1.1), (6.1.2), (6.1.4), and (6.1.5), Ramanujan asserts an identity involving the coefficients of $(q;q)_\infty^{-24s}$, for each positive integer s, and Bernoulli numbers. Although Ramanujan's claim is true for $s=1$, it is false in general. At the bottom of page 189 in [283], Ramanujan offers an elegant assertion on the divisibility of a certain difference of partition functions. Although his claim is true in some cases, it is unfortunately false in general. In Section 6.4 we briefly discuss these two false claims.

As we remarked earlier in this introductory section, page 182 is possibly the only page remaining of a manuscript that might have been a forerunner of his paper [276]. On this page, Ramanujan briefly examines congruences for $p_r(n)$, where

$$\frac{1}{(q;q)_\infty^r} = \sum_{n=0}^{\infty} p_r(n) q^n, \qquad |q| < 1.$$

These congruences are the highlights of our brief discussion of page 182. K.G. Ramanathan [273] also briefly examined this page.

At the conclusion of this chapter, we briefly discuss some miscellaneous entries on partitions found in [283]. In particular, we discuss all of the entries on page 331 of [283], which are related to Hardy and Ramanujan's paper on the asymptotic formula for $p(n)$ [167]. Both on pages 326 and 331, Ramanujan discusses the asymptotic formula for the coefficients of the reciprocal of $\varphi(-q)$, defined in (5.11.1). Because of the historical importance of this asymptotic formula, we conclude this chapter with a discussion of it.

Throughout this chapter, $J, J_1, J_2, \dots$ represent power series with integral coefficients and integral powers, not necessarily the same with each occurrence. We have adhered to Ramanujan's notation, whereas in contemporary notation we would use congruence signs instead.

We emphasize that the theory of modular forms can be utilized to provide proofs of most of the results in this chapter. However, we think that it is instructive to construct proofs that Ramanujan might possibly have given.

6.2 The Identities for Modulus 5

Entry 6.2.1 (p. 189). *If $p(n)$ denotes the ordinary partition function, then*

$$\sum_{n=0}^{\infty} p(5n+4) q^n = 5 \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{6.2.1}$$

Proof. Recall that the Rogers–Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{1}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1.$$

This continued fraction satisfies two beautiful and famous identities [276], [281, p. 212], [55, p. 267, Equations (11.5), (11.6)]:

$$R(q^5) - q - \frac{q^2}{R(q^5)} = \frac{(q;q)_\infty}{(q^{25};q^{25})_\infty} \tag{6.2.2}$$

and

$$R^5(q^5) - 11q^5 - \frac{q^{10}}{R^5(q^5)} = \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty^6}. \tag{6.2.3}$$

(It should be remarked here that the introduction of the continued fraction $R(q)$ is not strictly necessary. In the representations (6.2.2) and (6.2.3) we only need to know that a function $R(q)$, defined by (6.2.2), can be represented as

a power series in q with integral coefficients.) Using the generating function for $p(n)$, (6.2.2) and (6.2.3), and "long division," we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} &= \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty^6} \frac{(q^{25};q^{25})_\infty}{(q;q)_\infty} \\ &= \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \frac{R^5(q^5) - 11q^5 - q^{10}/R^5(q^5)}{R(q^5) - q - q^2/R(q^5)} \\ &= \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \left\{R^4 + qR^3 + 2q^2R^2 + 3q^3R + 5q^4 \right. \\ &\quad \left. -3q^5R^{-1} + 2q^6R^{-2} - q^7R^{-3} + q^8R^{-4}\right\}, \qquad (6.2.4)\end{aligned}$$

where $R := R(q^5)$. Choosing only those terms on each side of (6.2.4) where the powers of q are of the form $5n+4$, we find that

$$\sum_{\substack{n=0 \\ n\equiv 4 \pmod 5}}^{\infty} p(n)q^n = 5q^4 \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6},$$

or

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n} = 5\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}. \qquad (6.2.5)$$

Replacing q^5 by q in (6.2.5), we complete the proof of (6.2.1). □

C. Gugg [160] has given an elegant new proof of (6.2.3). His results in another paper [159] are also connected with (6.2.2) and (6.2.3) and the proof given above.

We remark here that (6.2.5) and (6.2.4) lead to a simple proof of another congruence for $p(n)$.

Corollary 6.2.1. *For each nonnegative integer n,*

$$p(25n+24) \equiv 0 \pmod{25}. \qquad (6.2.6)$$

Proof. Using (6.2.5) and (6.2.4) with q replaced by q^5, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p(5n+4)q^{5n} &= 5\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^5} \frac{1}{(q^5;q^5)_\infty} \\ &= 5\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^5} \frac{(q^{125};q^{125})_\infty^5}{(q^{25};q^{25})_\infty^6} \left\{R^4(q^{25}) + q^5R^3(q^{25}) + \cdots + 5q^{20} - \cdots\right\}.\end{aligned}$$

Apply the binomial theorem to $(q^5;q^5)_\infty^5$ in the denominator above, replace q^5 by q, and equate those terms on each side of the resulting equation with powers of q of the type q^{5n+4}. We then immediately deduce (6.2.6). □

Entry 6.2.2 (p. 189). *If* $q_5(n), n \geq 0$, *is defined by* (6.1.3), *then*

$$\sum_{n=0}^{\infty} q_5(5n)q^n = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}. \tag{6.2.7}$$

Proof. By (6.2.2) and (6.2.3),

$$\begin{aligned}\sum_{n=0}^{\infty} q_5(n)q^n = (q;q)_\infty^5 &= \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty} \frac{(q;q)_\infty^5}{(q^{25};q^{25})_\infty^5} \frac{(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6} \\ &= \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty} \frac{(R(q^5)-q-q^2/R(q^5))^5}{R^5(q^5)-11q^5-q^{10}/R^5(q^5)}. \end{aligned} \tag{6.2.8}$$

Now, the terms where the exponents of q are multiples of 5 in

$$\left(R(q^5) - q - \frac{q^2}{R(q^5)}\right)^5$$

are given by

$$\begin{aligned} &R^5(q^5) - \frac{q^{10}}{R^5(q^5)} - q^5 - \binom{5}{2,1,2} q^5 + \binom{5}{1,3,1} q^5 \\ &= R^5(q^5) - \frac{q^{10}}{R^5(q^5)} - q^5 - 30q^5 + 20q^5 = R^5(q^5) - \frac{q^{10}}{R^5(q^5)} - 11q^5. \end{aligned} \tag{6.2.9}$$

Thus, choosing only those terms from (6.2.8) where the powers of q are multiples of 5, we find upon using (6.2.9) that

$$\sum_{n=0}^{\infty} q_5(5n)q^{5n} = \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty} \frac{R^5(q^5)-11q^5-q^{10}/R^5(q^5)}{R^5(q^5)-11q^5-q^{10}/R^5(q^5)} = \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty}. \tag{6.2.10}$$

Replacing q^5 by q, we complete the proof of (6.2.7). ☐

Corollary 6.2.2. *We have*

$$q_5(5n) \equiv \begin{cases} (-1)^j \pmod{5}, & \textit{if } n = j(3j-1)/2,\ -\infty < j < \infty, \\ 0 \pmod{5}, & \textit{otherwise.} \end{cases}$$

Proof. By Entry 6.2.2 and the binomial theorem,

$$\sum_{n=0}^{\infty} q_5(5n)q^n = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty} \equiv \frac{(q;q)_\infty^6}{(q;q)_\infty^5} = (q;q)_\infty \pmod{5}.$$

The result now follows from Euler's pentagonal number theorem [55, p. 36, Entry 22(iii)]

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \qquad |q| < 1. \tag{6.2.11}$$

☐

To illustrate Corollary 6.2.2, we find, using *Mathematica*, that

$$\begin{aligned}\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty} &= 1 - 6q + 9q^2 + 10q^3 - 30q^4 + q^5 + 5q^6 + 51q^7 + 10q^8 - 100q^9 \\ &\quad + 20q^{10} - 55q^{11} + 109q^{12} + 110q^{13} - 130q^{14} - q^{15} - 110q^{16} \\ &\quad + 160q^{17} + 10q^{18} - 230q^{19} + 100q^{20} + \cdots .\end{aligned}$$

In fact, S.H. Chan pointed out to us that a more general corollary than Corollary 6.2.2 holds, *and the proof is even more elementary than the proof given for* Corollary 6.2.2!

Corollary 6.2.3. *For any prime p,*

$$q_p(pn) \equiv \begin{cases} (-1)^j \pmod p, & \text{if } n = j(3j-1)/2,\ -\infty < j < \infty, \\ 0 \pmod p, & \text{otherwise.} \end{cases} \tag{6.2.12}$$

Proof. By the definition (6.1.3) and the binomial theorem,

$$\sum_{n=0}^{\infty} q_p(n)q^n = (q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod p,$$

and so, extracting only those terms with powers of q that are multiples of p and then replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} q_p(pn)q^n \equiv (q;q)_\infty \pmod p,$$

from which the desired congruence (6.2.12) holds. □

6.3 The Identities for Modulus 7

Our primary goal in this section is to give a completely elementary proof along the lines outlined by Ramanujan in [283], [67, Section 24], or Section 5.24 in this volume, of his famous theorem below, Entry 6.3.1, as well as a proof of the new related theorem, Entry 6.3.2, or (6.1.5).

Entry 6.3.1 (p. 189). *We have*

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{6.3.1}$$

Proof. Using the pentagonal number theorem (6.2.11) in the numerator and then separating the indices of summation in the numerator into residue classes modulo 7, we readily find that

$$\frac{(q^{1/7};q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3, \tag{6.3.2}$$

where J_1, J_2, and J_3 are power series in q with integral coefficients. Now recall Jacobi's identity [55, p. 39, Entry 24(ii)],

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \qquad |q|<1. \tag{6.3.3}$$

Cubing both sides of (6.3.2) and substituting (6.3.3) into the left side, we find that

$$\begin{aligned}
&\frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/14}}{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{7n(n+1)/2}}\\
&= (J_1^3 + 3J_2^2J_3q - 6J_1J_3q) + q^{1/7}(3J_1^2J_2 - 6J_2J_3q + J_3^2q^2)\\
&\quad + 3q^{2/7}(J_1J_2^2 - J_1^2 + J_3q)\\
&\quad + q^{3/7}(J_2^3 - 6J_1J_2 + 3J_1J_3^2q) + 3q^{4/7}(J_1 - J_2^2 + J_2J_3^2q)\\
&\quad + 3q^{5/7}(J_2 + J_1^2J_3 - J_3^2q) + q^{6/7}(6J_1J_2J_3 - 1).
\end{aligned} \tag{6.3.4}$$

On the other hand, by separating the indices of summation in the numerator on the left side of (6.3.4) into residue classes modulo 7, we easily find that

$$\frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/14}}{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{7n(n+1)/2}} = G_1 + q^{1/7}G_2 + q^{3/7}G_3 - 7q^{6/7}, \tag{6.3.5}$$

where G_1, G_2, and G_3 are power series in q with integral coefficients. Comparing coefficients in (6.3.4) and (6.3.5), we conclude that

$$\begin{cases} J_1J_2^2 - J_1^2 + J_3q = 0,\\ J_1 - J_2^2 + J_2J_3^2q = 0,\\ J_2 + J_1^2J_3 - J_3^2q = 0,\\ \qquad 6J_1J_2J_3 - 1 = -7. \end{cases} \tag{6.3.6}$$

Now write (6.3.2) in the form

$$\frac{(\omega q^{1/7};\omega q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1 + \omega q^{1/7}J_2 - \omega^2q^{2/7} + \omega^5q^{5/7}J_3, \tag{6.3.7}$$

where $\omega^7=1$. Multiplying (6.3.7) over all seventh roots of unity, we find that

$$\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} = \prod_{i=0}^{6}(J_1 + \omega^iq^{1/7}J_2 - \omega^{2i}q^{2/7} + \omega^{5i}q^{5/7}J_3). \tag{6.3.8}$$

Using the generating function for $p(n)$, (6.3.2), and (6.3.8), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} &= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8} \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty^8} \frac{(q^{49};q^{49})_\infty}{(q;q)_\infty} \\ &= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8} \frac{\prod_{i=0}^{6}(J_1 + \omega^i qJ_2 - \omega^{2i}q^2 + \omega^{5i}q^5 J_3)}{J_1 + qJ_2 - q^2 + q^5 J_3} \\ &= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8} \left\{ \prod_{i=1}^{6}(J_1 + \omega^i qJ_2 - \omega^{2i}q^2 + \omega^{5i}q^5 J_3) \right\}.\end{aligned} \tag{6.3.9}$$

We need only compute the terms in $\prod_{i=1}^{6}(J_1+\omega^i qJ_2-\omega^{2i}q^2+\omega^{5i}q^5J_3)$ where the powers of q are of the form $7n+5$ to complete the proof. In order to do this, we need to prove the identities

$$J_1^7 + J_2^7 q + J_3^7 q^5 = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} + 14q\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} + 57q^2, \tag{6.3.10}$$

$$J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q, \tag{6.3.11}$$

$$J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 5q. \tag{6.3.12}$$

Since $J_2^2 = J_1+J_2J_3^2q$, $J_1^2 = J_1J_2^2+J_3q$, $J_3^2q = J_2+J_1^2J_3$, and $J_1J_2J_3 = -1$ by (6.3.6), we find that

$$\begin{aligned}J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 &= J_1^3 J_2 + J_1^2 J_2^2 J_3^2 q + J_1^2 J_2^2 J_3^2 q + J_3^3 J_1 q^2 + J_2^3 J_3 q \\ &\quad + J_1^2 J_2^2 J_3^2 q \\ &= J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 3q, \qquad (6.3.13) \\ J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^3 &= J_1 J_2 (J_1 + J_2 J_3^2 q)^2 + J_3 J_1 (J_1 J_2^2 + J_3 q)^2 \\ &\quad + J_2 J_3 (J_2 + J_1^2 J_3)^2 q \\ &= J_1^3 J_2 + 2J_1^2 J_2^2 J_3^2 q + J_1 J_2^3 J_3^4 q^2 + J_3 J_1^3 J_2^4 \\ &\quad + 2J_1^2 J_2^2 J_3^2 q + J_3^3 J_1 q^2 + J_2^3 J_3 q + 2J_1^2 J_2^2 J_3^2 q \\ &\quad + J_2 J_3^3 J_1^4 q \\ &= J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 \\ &\quad - (J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2) + 6q \\ &= 3q, \qquad (6.3.14)\end{aligned}$$

where (6.3.14) is obtained from (6.3.13). (Observe from (6.3.13) that in order to prove (6.3.10)–(6.3.12) it suffices to prove only (6.3.11) or (6.3.12).) By squaring the left side of (6.3.11) and using (6.3.6), (6.3.14), and (6.3.13), we find that

$$(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2)^2 = J_1^6 J_2^2 + J_2^6 J_3^2 q^2 + J_3^6 J_1^2 q^4$$

$$
\begin{aligned}
&+ 2(J_1^3J_2^4J_3q + J_1J_2^3J_3^4q^3 + J_1^4J_2J_3^3q^2)\\
&= J_1^7 + J_1^6J_2J_3^2q + J_2^7q + J_1^2J_2^6J_3q + J_3^7q^5\\
&\quad + J_1J_2^2J_3^6q^4 - 2(J_1^2J_2^3q + J_3^2J_1^3q^2 + J_2^2J_3^3q^3)\\
&= J_1^7 + J_2^7q + J_3^7q^5 - (J_1J_2^5q + J_3J_1^5q + J_2J_3^5q^4)\\
&\quad - 2(J_1^2J_2^3q + J_2^2J_3^3q^3 + J_1^3J_3^2q^2)\\
&= J_1^7 + J_2^7q + J_3^7q^5 - 2q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2)\\
&\quad - 9q^2.
\end{aligned}
$$

Thus,

$$
(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + q)^2 = (J_1^7 + J_2^7q + J_3^7q^5) - 8q^2. \tag{6.3.15}
$$

Expanding the right side of (6.3.8) and using (6.3.6), (6.3.14), and (6.3.13), we obtain

$$
\begin{aligned}
\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} &= J_1^7 + J_2^7q + J_3^7q^5 + 7(J_1J_2^5q + J_3J_1^5q + J_2J_3^5q^4)\\
&\quad + 7(J_1^4J_2^2J_3q + J_1J_2^4J_3^2q^2 + J_2J_3^4J_1^2q^3)\\
&\quad + 7(J_1^3J_2q + J_2^3J_3q^2 + J_3^3J_1q^3) + 14(J_1^2J_2^3q + J_3^2J_1^3q^2 + J_2^2J_3^3q^3)\\
&\quad + 7J_1^2J_2^2J_3^2q^2 + 14J_1J_2J_3q^2 - q^2\\
&= J_1^7 + J_2^7q + J_3^7q^5 + 21q^2 - 7q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2)\\
&\quad + 7q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2)\\
&\quad + 14q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + 3q) + 7q^2 - 14q^2 - q^2\\
&= J_1^7 + J_2^7q + J_3^7q^5 + 14q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2) + 55q^2.
\end{aligned} \tag{6.3.16}
$$

Combining (6.3.15) and (6.3.16), we find that

$$
\begin{aligned}
\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} &= (J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + q)^2 + 8q^2\\
&\quad + 14(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2)q + 55q^2\\
&= (J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + 8q)^2.
\end{aligned}
$$

By (6.3.2), we see that for q sufficiently small and positive, $J_2 < 0$. Thus, taking the square root of both sides above, we find that

$$
J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q, \tag{6.3.17}
$$

which proves (6.3.11). We now see that (6.3.10) follows from (6.3.16) and (6.3.17), and (6.3.12) follows from (6.3.13) and (6.3.17).

Returning to (6.3.9), we are now ready to compute the terms in $\prod_{i=1}^{6}(J_1+\omega^i qJ_2-\omega^{2i}q^2+\omega^{5i}q^5J_3)$ when the powers of q are of the form $7n+5$. Using the computer algebra system Maple, (6.3.11), (6.3.12), and (6.3.14), we find that the desired terms with powers of the form q^{7n+5} are equal to

$$\begin{aligned}
&-(J_1J_2^5+J_3J_1^5+3J_1^3J_2+4J_1^2J_2^3)q^5-(3J_2^3J_3+4J_3^2J_1^3-8)q^{12}\\
&-(4J_2^2J_3^3+3J_3^3J_1)q^{19}-J_2J_3^5q^{26}\\
&\quad=-3(J_1^3J_2+J_2^3J_3q^7+J_3^3J_1q^{14})q^5-4(J_1^2J_2^3+J_3^2J_1^3q^7+J_2^2J_3^3q^{14})q^5\\
&\qquad-(J_1J_2^5+J_3J_1^5+J_2J_3^5q^{21})q^5+8q^{12}\\
&\quad=7\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4}q^5+49q^{12}.
\end{aligned}\tag{6.3.18}$$

Choosing only those terms on each side of (6.3.9) where the powers of q are of the form $7n+5$ and using the calculation from (6.3.18), we find that

$$\sum_{\substack{n=0\\ n\equiv 5\,(\mathrm{mod}\,7)}}^{\infty} p(n)q^n = q^5\frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\left(7\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4}+49q^7\right),$$

or

$$\sum_{n=0}^{\infty} p(7n+5)q^{7n} = 7\frac{(q^{49};q^{49})_\infty^3}{(q^7;q^7)_\infty^4}+49q^7\frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}.\tag{6.3.19}$$

Replacing q^7 by q in (6.3.19), we complete the proof of (6.3.1). □

We show next that we can derive Ramanujan's congruence for $p(n)$ modulo 49 from (6.3.19).

Corollary 6.3.1. *For each nonnegative integer n,*

$$p(49n+47)\equiv 0\,(\mathrm{mod}\,49).\tag{6.3.20}$$

Proof. Using Jacobi's identity (6.3.3), write (6.3.19) in the form

$$\sum_{n=0}^{\infty} p(7n+5)q^{7n} = 7\frac{(q^{49};q^{49})_\infty^3}{(q^7;q^7)_\infty^7}\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{7m(m+1)/2}+49J.\tag{6.3.21}$$

Consider those terms on the right side of (6.3.21) when the power of q is congruent to 42 modulo 49. Using the binomial theorem in the denominator, we see that these terms arise when $m\equiv 3\,(\mathrm{mod}\,7)$ and that the corresponding coefficients are divisible by 49. On the left side, these powers arise when $n\equiv 6\,(\mathrm{mod}\,7)$. The congruence (6.3.20) follows. □

Comparing (6.3.2) with Entry 17(v) in Chapter 19 of Ramanujan's second notebook [282], [55, p. 303], we see that

$$J_1 = \frac{f(-q^2,-q^5)}{f(-q,-q^6)}, \qquad J_2 = -\frac{f(-q^3,-q^4)}{f(-q^2,-q^5)}, \qquad \text{and} \qquad J_3 = \frac{f(-q,-q^6)}{f(-q^3,-q^4)},$$

where

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

In the notation of Section 18 of Chapter 19 in [282], [55, p. 306],

$$\alpha = u^{1/7} = q^{-2/7} J_1, \quad \beta = -v^{1/7} = q^{-1/7} J_2, \quad \text{and} \quad \gamma = w^{1/7} = q^{3/7} J_3. \tag{6.3.22}$$

Thus, the identity (6.3.10) is equivalent to an identity in Entry 18 in Chapter 19 of Ramanujan's second notebook [282], [55, p. 305, Equation (18.2)]. The proof of (6.3.10) given here is much simpler than that given in [55, pp. 306–312].

Entry 6.3.2 (p. 189). *If $q_7(n), n \geq 0$, is defined by* (6.1.3), *then*

$$\sum_{n=0}^{\infty} q_7(7n) q^n = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty} + 49q(q;q)_\infty^4 (q^7;q^7)_\infty^3. \tag{6.3.23}$$

Proof. By (6.3.2), with q replaced by q^7,

$$\begin{aligned}\sum_{n=0}^{\infty} q_7(n) q^n = (q;q)_\infty^7 &= \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty} \frac{(q;q)_\infty^7}{(q^{49};q^{49})_\infty^7} \frac{(q^{49};q^{49})_\infty^8}{(q^7;q^7)_\infty^8} \\ &= \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty} (J_1 + qJ_2 - q^2 + q^5 J_3)^7 \frac{(q^{49};q^{49})_\infty^8}{(q^7;q^7)_\infty^8}.\end{aligned} \tag{6.3.24}$$

Using (6.3.6) and employing (6.3.10)–(6.3.12) with q replaced by q^7, we find that the terms where the exponents of q are multiples of 7 in

$$(J_1 + qJ_2 - q^2 + q^5 J_3)^7$$

are given by

$$\begin{aligned}&\sum_{\substack{u,v,w \geq 0 \\ 7|(u+2v+5w)}} \binom{7}{u,v,w} (-1)^v J_1^{7-u-v-w} J_2^u J_3^w q^{u+2v+5w} \\ &= J_1^7 + J_2^7 q^7 + J_3^7 q^{35} - q^{14} \\ &\quad + (105 J_1^4 J_2^2 J_3 - 42 J_1 J_2^5 + 210 J_1^2 J_2^3 - 42 J_1^5 J_3 - 140 J_1^3 J_2) q^7 \\ &\quad + (105 J_1 J_2^4 J_3^2 - 630 J_1^2 J_2^2 J_3^2 + 210 J_1 J_2 J_3 - 140 J_2^3 J_3 + 210 J_1^3 J_3^2) q^{14} \\ &\quad + (-140 J_1 J_3^3 + 105 J_1^2 J_2 J_3^4 + 210 J_2^2 J_3^3) q^{21} - 42 J_2 J_3^5 q^{28} \\ &= J_1^7 + J_2^7 q^7 + J_3^7 q^{35} - 245(J_1^3 J_2 + J_2^3 J_3 q^7 + J_3^3 J_1 q^{14}) q^7\end{aligned}$$

$$
\begin{aligned}
&- 42(J_1J_2^5 + J_3J_1^5 + J_2J_3^5q^{21})q^7 + 210(J_1^2J_2^3 + J_3^2J_1^3q^7 + J_2^2J_3^3q^{14})q^7 \\
&- 841q^{14} \\
&= J_1^7 + J_2^7q^7 + J_3^7q^{35} - 35(J_1^3J_2 + J_2^3J_3q^7 + J_3^3J_1q^{14})q^7 \\
&+ 630q^{14} - 126q^{14} - 841q^{14} \\
&= \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty^8} + 14q^7\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4} + 57q^{14} - 35q^7\left(- \frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4} - 8q^7\right) \\
&- 337q^{14} \\
&= \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty^8} + 49q^7\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4}. \qquad (6.3.25)
\end{aligned}
$$

Thus, choosing only those terms from (6.3.24) where the powers of q are multiples of 7, we find upon using (6.3.25) that

$$
\begin{aligned}
\sum_{n=0}^{\infty} q_7(7n)q^{7n} &= \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty}\left(\frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty^8} + 49q^7\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4}\right)\frac{(q^{49};q^{49})_\infty^8}{(q^7;q^7)_\infty^8} \\
&= \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty} + 49q^7(q^7;q^7)_\infty^4(q^{49};q^{49})_\infty^3. \qquad (6.3.26)
\end{aligned}
$$

Replacing q^7 by q, we complete the proof of (6.3.23). □

Corollary 6.3.2. *We have*

$$
q_7(7n) \equiv \begin{cases} (-1)^j \pmod 7, & \text{if } n = j(3j-1)/2,\ -\infty < j < \infty, \\ 0 \pmod 7, & \text{otherwise.} \end{cases}
$$

Corollary 6.3.2 can be proved using the same argument as given in the proof of Corollary 6.2.2, or alternatively observe that Corollary 6.3.2 is a special case of Corollary 6.2.3.

To illustrate Corollary 6.3.2, we find, using *Mathematica*, that

$$
\begin{aligned}
\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty} = 1 &- 8q + 20q^2 - 70q^4 + 64q^5 + 56q^6 + q^7 - 133q^8 - 140q^9 \\
&+ 308q^{10} - 70q^{11} + 174q^{12} + 56q^{13} - 518q^{14} - 141q^{15} - 63q^{16} \\
&+ 868q^{17} - 140q^{18} + 238q^{19} + 294q^{20} + \cdots .
\end{aligned}
$$

Apparently, proofs of Entries 6.2.2 and 6.3.2 were first given by M. Newman [238] using modular forms, although he credits D.H. Lehmer with first discovering the identities. Of course, they were unaware that these identities are in the lost notebook. A more complicated proof of Entry 6.2.2 was given by K.G. Ramanathan [273]. Ramanujan's ideas for establishing the congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$ that we have described above have been slightly generalized by J. Malenfant [222].

6.4 Two Beautiful, False, but Correctable Claims of Ramanujan

At the bottom of page 189 in his lost notebook [283], Ramanujan wrote the following:

Entry 6.4.1 (p. 189).

> *"n is the least positive integer such that* $24n-1$ *is divisible by a positive integer* k*. Then*
> $$p(n+vk)-p(n)q(v) \tag{6.4.1}$$
> *is divisible by* k *for all positive integral values of* v*, where*
> $$(x;x)_\infty^{(24n-1)/k} = \sum_{\lambda=0}^{\infty} q(\lambda)x^\lambda.\text{''}$$

Of course, $q(v)$ depends on n (and k). Ramanujan then gives the examples

$$\begin{aligned} p(4), p(9), p(14), \ldots &\equiv 0 \pmod 5, \\ p(5), p(12), p(19), \ldots &\equiv 0 \pmod 7, \\ p(6), p(17), p(28), \ldots &\equiv 0 \pmod{11}, \\ p(24)+1, p(47)+1, p(70), p(93),& \\ p(116)-1, p(139), p(162)-1, p(185), \ldots &\equiv 0 \pmod{23}. \end{aligned}$$

All four sets of congruences would follow from Ramanujan's claim, if it were true. Although it is well known that the first three examples are indeed true, the fourth is false. For example, $p(24)+1 = 1576$ is not divisible by 23. Ramanujan modified his assertion in his unpublished manuscript on the partition and τ-functions [283, pp. 157–162], [67, Sections 15, 16], which is in Chapter 5 of this volume. His reformulation is correct for the examples he calculated, but, as we saw, it must be modified still further. J.-P. Serre [312] and S. Ahlgren and M. Boylan [5] have established corrected versions; see the Commentary of Chapter 5 for more details.

We conclude this chapter with the second false assertion on page 189. For each positive integer s, define the coefficients $u_n = u_n(s)$ by

$$\frac{1}{(q;q)_\infty^{24s}} =: \sum_{n=0}^{\infty} u_n q^n = \sum_{n=0}^{\infty} u_n(s) q^n. \tag{6.4.2}$$

Ramanujan then makes the following claim.

Entry 6.4.2 (p. 189). *Let* $B_j, 0 \le j < \infty$*, denote the* j*th Bernoulli number, and let* $\sigma_k(n) = \sum_{d|n} d^k$*. If* s *is any positive integer, then*

$$\frac{B_{12s+2}}{24s+4} u_s(s) = \sum_{k=0}^{s-1} \sigma_{12s+1}(k+1) u_k(s). \tag{6.4.3}$$

For example, if $s=1$, then (6.4.3) reduces to

$$\frac{B_{14}}{28}u_1(1)=\sigma_{13}(1)u_0(1)=1. \tag{6.4.4}$$

From the definition (6.4.2), it is easy to see that $u_1(1)=24$. Since $B_{14}=7/6$, we see indeed that (6.4.4) is true.

However, unfortunately, (6.4.3) is false in general. We checked (6.4.3) for $s=2,3$ and found that (6.4.3) fails to hold. Fortunately, however, (6.4.3) can be corrected.

Recall that the Eisenstein series $E_{2k}(\tau)$ is defined by

$$E_{2k}(\tau):=1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n, \qquad q=e^{2\pi i\tau}, \qquad \operatorname{Im}\tau>0. \tag{6.4.5}$$

Ramanujan's claim therefore seems to be connected with the Eisenstein series $E_{12s+2}(\tau)$. H.H. Chan and S.H. Chan discovered a corrected version of Entry 6.4.2, which is clearly the version that Ramanujan had in mind, i.e., (6.4.3) was simply misrecorded by Ramanujan.

Entry 6.4.3 (p. 189; Corrected Version). *For any positive integer s,*

$$\frac{B_{12s+2}}{24s+4}u_s(s)=\sum_{k=0}^{s-1}\sigma_{12s+1}(s-k)u_k(s). \tag{6.4.6}$$

If $[q^n]\sum_{k=0}^{\infty}a_kq^k$ denotes a_n and $\Delta(\tau)=q(q;q)_\infty^{24}$, where $q=e^{2\pi i\tau}$, then (6.4.6) can be reformulated as

$$[q^s]\frac{E_{12s+2}(\tau)}{(q;q)_\infty^{24s}}=[q^0]\frac{E_{12s+2}(\tau)}{\Delta^s(\tau)}=0. \tag{6.4.7}$$

We are grateful to Ahlgren for pointing out to us a theorem in R.A. Rankin's book [290, p. 123] from which the assertions above, as well as a generalization, follow. The proof below is due to Ahlgren.

Theorem 6.4.1. *Let $f\in M(\Gamma,2,1)$ (the space of all modular forms of weight 2 and multiplier system identically equal to 1 on Γ, a subgroup of finite index in the full modular group $\Gamma(1)$), and let $\mathbb{F}$ be a proper fundamental region for $\hat{\Gamma}$ (the set of all linear fractional transformations on Γ). Then*

$$\sum_{\zeta\in\mathbb{F}}\operatorname{res}(f,\zeta,\Gamma)=0,$$

where $\operatorname{res}(f,\zeta,\Gamma)$ denotes the residue of f at ζ.

Proof of (6.4.6). We apply Theorem 6.4.1 to the function

$$F(\tau) := \frac{E_{12s+2}(\tau)}{\Delta^s(\tau)},$$

which is a modular form on $\Gamma(1)$ of weight 2 and multiplier system identically equal to 1 (since $E_{12s+2}(\tau)$ has weight $12s+2$ and $\Delta(\tau)$ has weight 12). Since $E_{12s+2}(\tau)$ is analytic on $\mathbb{H} = \{\tau : \operatorname{Im} \tau > 0\}$ and $\Delta(\tau)$ does not vanish on $\mathbb{H}$, the only pole of $F(\tau)$ is a pole of order s at ∞. The residue of $F(\tau)$ at ∞ is the constant term in the Fourier expansion of F at ∞. Hence invoking Theorem 6.4.1, and using (6.4.2) and (6.4.5) to calculate the constant term, we find that

$$-\frac{B_{12s+2}}{24s+4}u_s(s) + \sum_{k=0}^{s-1} \sigma_{12s+1}(s-k)u_k(s) = 0,$$

and so (6.4.6) is immediate. □

In fact, Entry 6.4.3 can be generalized by replacing the Eisenstein series $E_{12s+2}(\tau)$ by any modular form of weight $12s+2$ and multiplier system identically equal to 1 that is analytic on $\mathbb{H}$.

It is doubtful that Ramanujan would have known Theorem 6.4.1. So, we wonder how Ramanujan would have both discovered and proved Entry 6.4.3.

6.5 Page 182

The number (5) is written in the upper right-hand corner of page 182 in [283], likely indicating that this is the fifth page of a handwritten manuscript. The first and second lines on this page are identical to the second and third lines of Equation (11) in [276], [281, p. 211], as Ramanujan begins to relate his elementary proof of $p(5n+4) \equiv 0 \pmod 5$. The tagged equation numbers on the page are (2.2)–(2.5), which clearly indicate that this page is in Section 2 of this manuscript. However, page 182 is not identical to any page or pages in [276]. Ramanujan's proof of $p(5n+4) \equiv 0 \pmod 5$ on page 182 is considerably briefer than it is in [276]. Moreover, central to Ramanujan's thoughts is the more general partition function $p_r(n)$ defined by

$$\frac{1}{(q;q)_\infty^r} = \sum_{n=0}^{\infty} p_r(n)q^n, \qquad |q| < 1.$$

This definition is not provided on page 182, but it is clear that it must have been given somewhere in the missing pages 1–4 of the manuscript. Of course, $p_1(n) = p(n)$. In a letter to Hardy written from Fitzroy House late in 1918 [68, pp. 192–193], Ramanujan writes, "I have considered more or less exhaustively about the congruency of $p(n)$ and in general that of $p_r(n)$ where

$$\sum p_r(n)x^n = \frac{1}{(x;x)_\infty^r},$$

by four different methods." This declaration appears to imply that he had established several results about $p_r(n)$, which quite likely were discussed in the manuscript for which we now unfortunately have only page 5.

After Ramanujan, the function $p_r(n)$, sometimes with the alternative notation $p_{-r}(n)$, was studied by, among others, M. Newman [239], K.G. Ramanathan [272], A.O.L. Atkin [25], J.M. Gandhi [140], B. Gordon [155], I. Kiming and J. Olsson [193], B.K. Sarmah [308], and Garvan [142]. Some, but not all, of these authors confined themselves to congruences satisfied by a small set of primes and powers thereof, in contrast to Ramanujan's theorems satisfying an infinite set of primes.

Ramanujan's elementary methods have been utilized and generalized by several authors. The most extensive applications of this method have been by Andrews and R. Roy [21]; their paper contains several additional references. Our proofs of Entry 6.5.1 and Theorem 6.5.1 are taken from a paper by the second author with C. Gugg and S. Kim [66].

Ramanujan deduces the congruence $p(5n-1) \equiv 0 \pmod 5$ from the congruence $p_{-4}(5n-1) \equiv 0 \pmod 5$, just as he does in [276], but without using this notation. Ramanujan then remarks that, "Precisely in the same way we can show that

$$p_{-4}\left(n\varpi - \frac{\varpi+1}{6}\right) \equiv 0 \pmod{\varpi} \tag{6.5.1}$$

where ϖ is a prime of the form $6\lambda - 1$ ..." He then states a more general theorem.

Entry 6.5.1 (p. 182). *Let δ denote any integer, and let n denote a nonnegative integer. Suppose that ϖ is a prime of the form $6\lambda - 1$. Then*

$$p_{\delta\varpi-4}\left(n\varpi - \frac{\varpi+1}{6}\right) \equiv 0 \pmod{\varpi}. \tag{6.5.2}$$

Proof. Consider

$$\begin{aligned} \sum_{n=0}^{\infty} p_{\delta\varpi-4}(n) q^{n+\lambda} &= (q;q)_\infty^{-\delta\varpi}(q;q)_\infty^3 (q;q)_\infty q^\lambda \\ &\equiv (q^\varpi;q^\varpi)_\infty^{-\delta} \sum_{\mu=0}^{\infty}\sum_{\nu=-\infty}^{\infty} (-1)^{\mu+\nu}(2\mu+1) q^{\frac12\mu(\mu+1)+\frac12\nu(3\nu+1)+\lambda} \pmod{\varpi}, \end{aligned} \tag{6.5.3}$$

upon the use of Euler's pentagonal number theorem (6.2.11) and Jacobi's identity (6.3.3). We want to examine those terms for which

$$\frac12\mu(\mu+1) + \frac12\nu(3\nu+1) + \frac{\varpi+1}{6} \equiv 0 \pmod{\varpi}. \tag{6.5.4}$$

Our goal is to prove that

$$\varpi \mid (2\mu+1). \tag{6.5.5}$$

Multiply (6.5.4) by 24 to obtain the equivalent congruence

$$12\mu(\mu+1)+12\nu(3\nu+1)+4\varpi+4\equiv 0 \pmod{\varpi},$$

or

$$3(2\mu+1)^2+(6\nu+1)^2\equiv 0 \pmod{\varpi}. \tag{6.5.6}$$

Using the fact that, for each prime p, the Legendre symbol $\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}$, and the law of quadratic reciprocity, we find that

$$\left(\frac{-3}{\varpi}\right)=\left(\frac{\varpi}{3}\right)=\left(\frac{-1}{3}\right)=-1.$$

Thus, the only way that (6.5.6) can hold is for (6.5.5) to happen. But then, from the right-hand side of (6.5.3), we can also conclude that

$$p_{\delta\varpi-4}\left(n\varpi-\frac{\varpi+1}{6}\right)\equiv 0 \pmod{\varpi}.$$

Thus, the proof is complete. □

Entry 6.5.2 (p. 182). *For each positive integer n,*

$$p_6(5n-1)\equiv 0 \pmod{5},$$
$$p_7(11n-2)\equiv 0 \pmod{11}.$$

Proof. The first congruence arises from the case $\varpi=5$ and $\delta=2$, while the second arises from the case $\varpi=11$ and $\delta=1$ in Entry 6.5.1. □

Next, Ramanujan gives an elementary proof of the congruence $p(7n-2)\equiv 0 \pmod{7}$. He begins with the same first three lines of [276, Equation (13)], [281, p. 212], and then argues in a somewhat more abbreviated fashion than he does in [276] to deduce the congruence

$$p_{-6}(7n-2)\equiv 0 \pmod{49}, \tag{6.5.7}$$

from which it follows that

$$p(7n-2)\equiv 0 \pmod{7}. \tag{6.5.8}$$

It should be remarked that the stronger congruence (6.5.7) is not mentioned by Ramanujan in [276], although it is implicit in his argument.

Unfortunately, the manuscript ends with (6.5.8). It would seem that Ramanujan would have next offered a theorem analogous to Entry 6.5.1, and so, in analogy to completing Schubert's *Unfinished Symphony*, we shall state and prove such a theorem here, but of course, Ramanujan probably would have had lots more to say to us if his manuscript had survived.

Theorem 6.5.1. *For a prime ϖ with $4 \mid (\varpi+1)$, any integer δ, and any positive integer n,*

$$P_{\delta\varpi-6}\Big(n\varpi - \frac{\varpi+1}{4}\Big) \equiv 0 \pmod{\varpi}. \tag{6.5.9}$$

For $\delta = 3$ and $\varpi = 3$, N.D. Baruah and K.K. Ojah [47] have shown that (6.5.9) holds with $\pmod 3$ replaced by $\pmod{3^2}$.

In the case $\delta = 0$ above, we can strengthen (6.5.9).

Entry 6.5.3 (p. 182). *Under the hypotheses of Theorem 6.5.1,*

$$p_{-6}\Big(n\varpi - \frac{\varpi+1}{4}\Big) \equiv 0 \pmod{\varpi^2}. \tag{6.5.10}$$

Observe that (6.5.7) is the special case $\varpi = 7$ of (6.5.10), and so, with slight exaggeration, we affixed "p. 182" to the entry above.

Proof of theorem 6.5.1. Consider, for $\lambda = (\varpi+1)/4$,

$$\sum_{n=0}^{\infty} p_{\delta\varpi-6}(n)q^{n+\lambda} = (q;q)_\infty^{-\delta\varpi}(q;q)_\infty^6 q^\lambda \tag{6.5.11}$$
$$\equiv (q^\varpi;q^\varpi)_\infty^{-\delta}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}(-1)^{\mu+\nu}(2\mu+1)(2\nu+1)q^{\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)+\lambda} \pmod{\varpi},$$

upon the use of Jacobi's identity (6.3.3). We need to show that if

$$\frac{1}{2}\mu(\mu+1) + \frac{1}{2}\nu(\nu+1) + \frac{\varpi+1}{4} \equiv 0 \pmod{\varpi}, \tag{6.5.12}$$

then

$$\varpi^2 \mid (2\mu+1)(2\nu+1). \tag{6.5.13}$$

The congruence (6.5.10) will then follow from (6.5.13) and (6.5.11). Multiply (6.5.12) by 8 to obtain

$$4\mu(\mu+1) + 4\nu(\nu+1) + 2\varpi + 2 \equiv 0 \pmod{\varpi},$$

or

$$(2\mu+1)^2 + (2\nu+1)^2 \equiv 0 \pmod{\varpi}.$$

Since

$$\Big(\frac{-1}{\varpi}\Big) = -1,$$

we conclude that

$$\varpi \mid (2\mu+1) \quad \text{and} \quad \varpi \mid (2\nu+1),$$

which completes the proof of (6.5.13). ∎

Observe that if $\delta = 0$, then the congruence in (6.5.11) can be replaced by an equality. Hence, in (6.5.9), the congruence modulo ϖ can be replaced by a congruence modulo ϖ^2 in view of (6.5.13). It follows that Entry 6.5.3 is thus valid.

6.6 Further Remarks

Page 248 in the lost notebook [283] contains scratch work and calculations related to Ramanujan's work in Section 6.2 of this chapter.

We find on page 252 of [283] a preliminary version of a table of values of $\omega_{p,q}$ from Hardy and Ramanujan's famous paper [167], [281, p. 306]. On page 208, Ramanujan gives a table of values of A_n, which is again a preliminary version of the table of values of A_n from the same paper [167], [281, p. 307]. It is unfortunate that pages 208 and 252 have not been placed together in [283]. Hardy and Ramanujan's paper [167], [281, p. 308] also contains a table of values of $p(n)$, $1 \leq n \leq 200$. A short table of values of $p(n)$, $1 \leq n \leq 19$, can be found on page 207 in [283]. It is also interesting to note that these three pages belong to the original lost notebook found by the first author. This is further evidence that Ramanujan's mind was occupied with partitions in the last days of his life.

On page 326 in [283], Ramanujan returns to assertions made in his first two letters to Hardy and reiterates the following, which we quote exactly.

Entry 6.6.1 (p. 326). *The coefficient of* x^n *in* $(1-2x+2x^4-2x^9+\cdots)^{-1}$ *is nearly*

$$\frac{1}{4n}\left\{\cosh(\pi\sqrt{n})-\frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}. \tag{6.6.1}$$

He then writes

$$\begin{aligned}(1-2x+2x^4-\cdots)^{-1} &= 1+2x+4x^2+8x^3+14x^4+24x^5+40x^6+64x^7\\ &+100x^8+154x^9+232x^{10}+344x^{11}+504x^{12}+728x^{13}+1040x^{14}+1472x^{15}\\ &+2062x^{16}+2864x^{17}+3948x^{18}+5400x^{19}+7336x^{20}+9904x^{21}+13288x^{22}\\ &+17728x^{23}+23528x^{24}+31066x^{25}+40824x^{26}+53408x^{27}+69568x^{28}\\ &+90248x^{29}+116624x^{30}+150144x^{31}+192612x^{32}+246256x^{33}+313808x^{34}\\ &+398640x^{35}+504886x^{36}+\cdots.\end{aligned}$$

Denote the coefficients described in Entry 6.6.1 by $\overline{p}(n)$. For $n \leq 36$, the following table gives the exact value of $\overline{p}(n)$, as given in the expansion above; the approximate value, as calculated from (6.6.1); and the error made after rounding off the calculation. Observe that Ramanujan's approximation gives the correct value of $\overline{p}(n)$ up to $n = 20$. The error is at most 1 up until $n = 36$, when the error made by (6.6.1) is equal to 2.

Page 331 of [283] is devoted to five further results in the theory of partitions. First, Ramanujan records three asymptotic formulas for partitions, found also in his paper with Hardy [167], [281, p. 304], wherein they prove their asymptotic series for $p(n)$. Observe that Entry 6.6.2 below is another version of Entry 6.6.1 above. Because of the historical importance of these two entries, except for a brief note about page 333, we shall conclude this chapter with a thorough discussion of these two entries.

n	coefficient of x^n	approximation from (6.6.1)	Error
1	2	1.978968842	-
2	4	4.118621867	-
3	8	7.848360679	-
4	14	14.07086490	-
5	24	24.10391358	-
6	40	39.84090926	-
7	64	63.96498067	-
8	100	100.2324016	-
9	154	153.8451840	-
10	232	231.9378086	-
11	344	344.2093550	-
12	504	503.7400063	-
13	728	728.0404078	-
14	1040	1040.393928	-
15	1472	1471.565961	-
16	2062	2061.971642	-
17	2864	2864.414083	-
18	3948	3947.530498	-
19	5400	5400.114184	-
20	7336	7336.516090	$+1$
21	9904	9903.375762	-1
22	13288	13287.98157	-
23	17728	17728.62603	$+1$
24	23528	23527.39749	-1
25	31066	31065.93534	-
26	40824	40824.79469	$+1$
27	53408	53407.17922	-1
28	69568	69567.96717	-
29	90248	90249.12703	$+1$
30	116624	116622.8420	-1
31	150144	150143.8979	-
32	192612	192613.2129	$+1$
33	246256	246254.7187	-1
34	313808	313808.2185	-
35	398640	398641.3567	$+1$
36	504886	504884.3713	-2

Table 6.1. Table of Coefficients: Entry 6.6.1

Entry 6.6.2 (p. 331). *If c_n, $n \geq 0$, is defined by*

$$\sum_{n=0}^{\infty} c_n q^n = \frac{1}{\varphi(-q)},$$

then

$$c_n = \frac{d}{dn} \frac{\cosh(\pi\sqrt{n}) - 1}{2\pi\sqrt{n}} + \sqrt{3} \cdot 2\cos\left(\frac{2n\pi}{3} - \frac{\pi}{6}\right) \frac{d}{dn} \frac{\cosh(\frac{1}{3}\pi\sqrt{n}) - 1}{2\pi\sqrt{n}} + \cdots .$$

Entry 6.6.3 (p. 331). *If c_n, $n \geq 0$, is defined by*

$$\sum_{n=0}^{\infty} c_n q^n = \frac{1}{(q;q^2)_\infty} = (-q;q)_\infty,$$

then

$$\sqrt{2}c_n = \frac{d}{dn} J_0\left(i\pi\sqrt{\tfrac{1}{3}(n + \tfrac{1}{24})}\right) + 2\cos\left(\frac{2n\pi}{3} - \frac{\pi}{9}\right) \frac{d}{dn} J_0\left(\tfrac{1}{3} i\pi\sqrt{\tfrac{1}{3}(n + \tfrac{1}{24})}\right) + \cdots ,$$

where $J_0(x)$ denotes the ordinary Bessel function of order 0.

Entry 6.6.4 (p. 331). *If c_n, $n \geq 0$, is defined by*

$$\sum_{n=0}^{\infty} c_n q^n = (-q;q^2)_\infty,$$

then

$$c_n = \frac{d}{dn} J_0\left(i\pi\sqrt{\tfrac{1}{6}(n - \tfrac{1}{24})}\right) + 2\cos\left(\frac{2n\pi}{3} - \frac{2\pi}{9}\right) \frac{d}{dn} J_0\left(\tfrac{1}{3} i\pi\sqrt{\tfrac{1}{6}(n - \tfrac{1}{24})}\right) + \cdots .$$

Entry 6.6.5 (p. 331). *If c_n, $n \geq 0$, is defined by*

$$\sum_{n=0}^{\infty} c_n q^n = \frac{1}{(aq;q)_\infty},$$

then

$$c_n \sim \sqrt{1-a}\frac{(kn)^{1/4}}{2n\sqrt{\pi}} e^{2\sqrt{kn}}, \tag{6.6.2}$$

where

$$k = \sum_{m=1}^{\infty} \frac{a^m}{m^2} =: \mathrm{Li}_2(a),$$

where $\mathrm{Li}_2(a)$ is the familiar dilogarithm.

Let $p(n, m)$ denote the number of partitions of n into exactly m parts. Then, in Entry 6.6.5,

$$c_n(a) := c_n = \sum_{m=0}^{\infty} p(n, m)a^m.$$

Of course, $c_n(1) = p(n)$. Thus, if we let $a \to 1$, the right side of the asymptotic formula (6.6.2) should tend to the asymptotic formula of Hardy and Ramanujan for $p(n)$ [167], i.e.,

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}, \tag{6.6.3}$$

but clearly this is not the case. Since $\mathrm{Li}_2(1) = \pi^2/6$, we see that the exponential functions in (6.6.2) and (6.6.3) agree, but that the functions multiplying the exponentials do not agree. In particular, the powers of n are not the same. So, the intent of Entry 6.6.5 is unclear to us. Was Ramanujan contemplating finding an asymptotic formula for $p(n, m)$ under certain restrictions on m, say with m small but still tending to infinity with n? There exists a large literature on asymptotic formulas for $p(n, m)$ and similar partition functions. Some of the more important papers are by P. Erdős and J. Lehner [134], F.C. Auluck, S. Chowla, and H. Gupta [29], G. Szekeres [330], C.B. Haselgrove and H.N.V. Temperley [169], and L.B. Richmond [294].

In a postscript to a letter [291], [68, pp. 286, 287], [69, pp. 120, 121] that Hardy wrote to G.N. Watson in April 1930 discussing Ramanujan's miscellaneous papers, he remarks, "I have a pupil working (inter alia) at

$$\sum c_n x^n = \frac{1}{(1-ax)(1-ax^2)\cdots}, \qquad c_n \sim \sqrt{1-a}\frac{(kn)^{1/4}}{2n\sqrt{\pi}} e^{2\sqrt{kn}} \tag{6.6.4}$$

– so don't queer his pitch!" Thus, Hardy had copied (6.6.3) from Ramanujan's fragment and did not notice that the formula was clearly wrong. The "pupil" is not identified, and we are unaware of any paper written on (6.6.4) at that time, or later.

The last entry on page 331 is a briefer version of the transformation formula for

$$\prod_{n=1}^{\infty} \left(1 - e^{2\pi x n^s}\right),$$

where s is an odd positive integer, which is also given on page 330 of [283] and which is discussed in [16, pp. 234–235].

Entries 6.6.1 and 6.6.2 are extremely important results historically, for they relate the first asymptotic formula in the theory of partitions that was found by Ramanujan. We therefore provide a more extensive overview of this work.

In his first letter to Hardy [281, p. xxvii], [68, p. 28], Ramanujan asserted that, "The coefficient of x^n in

$$\begin{aligned}&\frac{1}{1-2x+2x^4-2x^9+2x^{16}-\cdots}\\&= \text{ the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n})-\frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}.\text{''}\end{aligned}$$

In his second letter to Hardy [281, p. 352], [68, pp. 56–57], Ramanujan amended his previous claim and instead asserted that "The coefficient of x^n in $(1-2x+2x^4-\cdots)^{-1}$

$$\begin{aligned}&= \text{ the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n})-\frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}\\&\quad + F(\cos\pi\sqrt{n})+f(\sin\pi\sqrt{n}).\end{aligned}$$

I have not written here the forms of F and f as they are very irregular and complicated, and their values are very small, in most cases a very small proper fraction. In a few cases they assume some small finite values. Hence the coefficient of x^n is an integer very near to

$$\frac{1}{4n}\left\{\cosh(\pi\sqrt{n})-\frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\},$$

and not always the nearest integer to it as I hastily wrote to you in my previous letter. Yet in many cases you will find the coefficient to be the nearest integer though not always. At present we may be contented with the result viz. The coefficient of x^n in the above function divided by

$$\frac{1}{4n}\left\{\cosh(\pi\sqrt{n})-\frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}$$

is very very nearly equal to unity for all values of n, from 0 to ∞ and very rapidly approaches 1 when n becomes infinity."

In their paper [167], [281, p. 304], after having obtained an asymptotic series for the partition function $p(n)$, Hardy and Ramanujan state that the aforementioned coefficient of x^n in the preceding paragraph, namely $\overline{p}(n)$, is equal to

$$\overline{p}(n)=\frac{1}{4\pi}\frac{d}{dn}\left(\frac{e^{\pi\sqrt{n}}}{\sqrt{n}}\right)+\frac{\sqrt{3}}{2\pi}\cos\left(\frac{2}{3}n\pi-\frac{1}{6}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{3}\pi\sqrt{n}}}{\sqrt{n}}\right)+\cdots. \tag{6.6.5}$$

In 1939, H.S. Zuckerman [354] proved the exact formula

$$\overline{p}(n)=\frac{1}{2\pi}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\sqrt{k}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{\omega^2(h,k)}{\omega(2h,k)}e^{-2\pi inh/k}\frac{d}{dn}\left(\frac{\sinh(\pi\sqrt{n}/k)}{\sqrt{n}}\right),$$

where

$$\omega(h,k) := \exp\left(\pi i \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)\right).$$

An excellent account of the formula for $\overline{p}(n)$, both mathematically and historically, as well as of similar formulas for other partition functions, has been given by A.V. Sills [316].

In their brief description of (6.6.5), Hardy and Ramanujan [281, p. 304] remark that $\overline{p}(n)$ “has no very simple arithmetical interpretation.” In fact, $\overline{p}(n)$ does have a simple arithmetical interpretation, but it was brought to fruition and studied only in recent years. See [316] for several references. From the product formula for $\varphi(-q)$ [55, p. 34, eq. (22.7)],

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{1}{\varphi(-q)} = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{6.6.6}$$

we see that the *overpartition function* $\overline{p}(n)$ denotes the number of ways a positive integer n can be represented by a sum of positive integers in nonincreasing order, in which the first appearance of an integer may be overlined. For example, we see that $\overline{p}(3) = 8$, because 3 has the eight representations

$$\overline{3},\ 3,\ \overline{2}+1,\ 2+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ \overline{1}+1+1,\ 1+1+1.$$

Alternatively, $\overline{p}(n)$ denotes the number of partitions of n into (unrestricted) parts of one color, and distinct parts of another.

On page 333 of [283], Ramanujan states Euler’s famous identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$$

and the elementary theta function identity

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

where $\varphi(q)$ is defined in (6.6.6) [55, p. 40, Entry 25(vi)].

7

Congruences for Generalized Tau Functions on Page 178

7.1 Introduction

On page 178 in his lost notebook [283], Ramanujan states nine congruences involving $\tau_2(n)$, $\tau_3(n)$, and $\tau_4(n)$, which are respectively defined by

$$\sum_{n=1}^{\infty} \tau_2(n)q^n = Q(q)\Delta(\tau), \tag{7.1.1}$$

$$\sum_{n=1}^{\infty} \tau_3(n)q^n = R(q)\Delta(\tau), \tag{7.1.2}$$

and

$$\sum_{n=1}^{\infty} \tau_4(n)q^n = Q^2(q)\Delta(\tau). \tag{7.1.3}$$

Here, as usual, $q = e^{2\pi i\tau}$, $Q(q)$ and $R(q)$ are Ramanujan's Eisenstein series, which are defined in (5.1.2) and (5.1.3), respectively, and which in contemporary notation are denoted by $E_4(\tau)$ and $E_6(\tau)$, respectively, and $\Delta(\tau)$ is the discriminant function defined by

$$\Delta(\tau) := q(q;q)_\infty^{24}.$$

Recall that $Q(q)$, $R(q)$, and $\Delta(\tau)$ satisfy the fundamental identity

$$1728\Delta(\tau) = Q^3(q) - R^2(q), \tag{7.1.4}$$

which we shall utilize several times in the sequel. All of the congruences are given in terms of divisor sums

$$\sigma_k(n) = \sum_{d|n} d^k.$$

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The goal of the present chapter is to examine each of these nine congruences for generalized tau functions. The content of this chapter is chiefly based on work of D. Eichhorn [130]. Some of the nine congruences were also established by J.M. Rushforth [305], [306]. J.R. Wilton [340] established a disguised form of (7.1.5).

We now list the nine congruences asserted by Ramanujan.

Entry 7.1.1 (p. 178).

$$\tau_2(n) - \sigma_{15}(n) \equiv 0 \pmod{3617}. \tag{7.1.5}$$

Entry 7.1.2 (p. 178).

$$\tau_2(n) - n\sigma_{13}(n) \equiv 0 \pmod{16170}. \tag{7.1.6}$$

Entry 7.1.3 (p. 178).

$$\tau_2(n) - 2n\sigma_9(n) + n^2\sigma_3(n) \equiv 0 \pmod{600}. \tag{7.1.7}$$

Entry 7.1.4 (p. 178).

$$\tau_3(n) - \sigma_{17}(n) \equiv 0 \pmod{43867}. \tag{7.1.8}$$

Entry 7.1.5 (p. 178).

$$\tau_3(n) - n\sigma_{15}(n) \equiv 0 \pmod{6006}. \tag{7.1.9}$$

Entry 7.1.6 (p. 178).

$$\tau_3(n) - n^2\sigma_1(n) \equiv 0 \pmod{540}. \tag{7.1.10}$$

Entry 7.1.7 (p. 178).

$$\tau_3(n) - 6n^2\sigma_9(n) + 5n\sigma_3(n) \equiv 0 \pmod{150}. \tag{7.1.11}$$

Entry 7.1.8 (p. 178).

$$\tau_3(n) + n\sigma_9(n) + n\sigma_3(n) - 3\tau(n) \equiv 0 \pmod{588}. \tag{7.1.12}$$

Entry 7.1.9 (p. 178).

$$\tau_4(n) - \sigma_{19}(n) \equiv 0 \pmod{174611}. \tag{7.1.13}$$

In fact, the congruences (7.1.10) and (7.1.11) are incorrect. For example, if we let $n = 4$, we find that $\tau_3(4) = 147712 \equiv 292 \pmod{540}$ and $4^2\sigma_1(4) = 112$, vitiating (7.1.10). We prove the following two corrected versions.

Theorem 7.1.1.

$$\tau_3(n) - n^2\sigma_1(n) \equiv 0 \pmod{180}. \tag{7.1.14}$$

Theorem 7.1.2.

$$\tau_3(n) - 6n^2\sigma_9(n) + 5n\sigma_3(n) \equiv 0 \pmod{30}. \tag{7.1.15}$$

7.2 Proofs

As stated above, proofs of the results found on page 178 of Ramanujan's lost notebook have been independently proved by Rushforth [305] and Eichhorn [130]. The proofs below utilize the work of these two mathematicians.

Define $\tau_5(n)$ and $\tau_7(n)$ by

$$\sum_{n=1}^{\infty} \tau_5(n)q^n = Q(q)R(q)\Delta(\tau)$$

and

$$\sum_{n=1}^{\infty} \tau_7(n)q^n = Q^2(q)R(q)\Delta(\tau),$$

respectively. H.P.F. Swinnerton-Dyer [329] observed that the proofs of (7.1.5), (7.1.8), (7.1.13),

$$\tau_5(n) - \sigma_{21}(n) \equiv 0 \pmod{77683}, \tag{7.2.1}$$

and

$$\tau_7(n) - \sigma_{25}(n) \equiv 0 \pmod{657931} \tag{7.2.2}$$

follow along the same lines as the proof of

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{691}, \tag{7.2.3}$$

which is (5.12.7).

In fact, (7.2.1) and (7.2.2) also follow immediately from identities in [275, Table I], [281, p. 141]. Since all five proofs are short, we shall give all of them, even though they are similar. We emphasize that the congruences we have listed above are by no means exhaustive. For example, using the congruences given by Ramanujan in his extensive tables of [275], one can establish many further congruences of the sort proved in this chapter.

For the remainder of the chapter, we delete the arguments q and τ from the functions appearing in our proofs.

Proof of (7.1.5), (7.1.8), (7.1.13), (7.2.1), *and* (7.2.2). We begin with an identity from Ramanujan's paper [275, Table I, no. 8],

$$\begin{aligned} 3617 + 16320\sum_{n=1}^{\infty} \sigma_{15}(n)q^n &= 1617Q^4 + 2000QR^2 \\ &= 3617Q^4 + 2000QR^2 - 2000Q^4 \\ &= 3617Q^4 - 3456000Q\Delta, \end{aligned}$$

where we have used (7.1.4). Now (7.1.5) follows, since $16320 \equiv -3456000 \pmod{3617}$.

Using the identity no. 9 from the same table and (7.1.4), we find that

$$\begin{aligned}43867 - 28728 \sum_{n=1}^{\infty} \sigma_{17}(n) q^n &= 38367 Q^3 R + 5500 R^3 \\ &= 43867 Q^3 R + 5500 R^3 - 5500 Q^3 R \\ &= 43867 Q^3 R - 9504000 R \Delta,\end{aligned}$$

and (7.1.8) follows, since $-28728 \equiv -9504000 \pmod{43867}$.

Employing identity no. 10 from [275, Table I] and (7.1.4) once again, we find that

$$\begin{aligned}174611 + 13200 \sum_{n=1}^{\infty} \sigma_{19}(n) q^n &= 53361 Q^5 + 121250 Q^2 R^2 \\ &= 174611 Q^5 + 121250 Q^2 R^2 - 121250 Q^5 \\ &= 174611 Q^5 - 209520000 Q^2 \Delta,\end{aligned}$$

and (7.1.13) follows, since $13200 \equiv -209520000 \pmod{174611}$.

Next we turn to identity no. 11 from [275, Table I] and with the use of (7.1.4) deduce that

$$\begin{aligned}77683 - 552 \sum_{n=1}^{\infty} \sigma_{21}(n) q^n &= 57183 Q^4 R + 20500 Q R^3 \\ &= 77683 Q^4 R + 20500 Q R^3 - 20500 Q^4 R \\ &= 77683 Q^4 R - 35424000 Q R \Delta.\end{aligned}$$

Since $-552 \equiv -35424000 \pmod{77683}$, we complete the proof of (7.2.1).

Lastly, using identity no. 13 from [275, Table I] as well as (7.1.4), we find that

$$\begin{aligned}657931 - 24 \sum_{n=1}^{\infty} \sigma_{25}(n) q^n &= 392931 Q^5 R + 265000 Q^2 R^3 \\ &= 657931 Q^5 R + 265000 Q^2 R^3 - 265000 Q^5 R \\ &= 657931 Q^5 R - 457920000 Q^2 R \Delta.\end{aligned}$$

Since $-24 \equiv -457920000 \pmod{657931}$, (7.2.2) follows. □

Proof of (7.1.6). Our proof uses identities in [275, Table II], (7.1.4), and congruences from Chapter 5.

Using formula no. 7 from [275, Table II] and (7.1.4), we find that

$$\begin{aligned}144 \sum_{n=1}^{\infty} n \sigma_{13}(n) q^n &= Q(3Q^3 + 4R^2 - 7PQR) \\ &= Q(7Q^3 + 4R^2 - 4Q^3 - 7PQR) \\ &= Q(7Q^3 - 6912\Delta - 7PQR) \\ &= -6912 Q\Delta + 7Q^2(Q^2 - PR). \qquad (7.2.4)\end{aligned}$$

From (5.5.2), we recall that

$$Q^2 \equiv P \pmod 7 \qquad \text{and} \qquad R \equiv 1 \pmod 7.$$

Thus, reducing (7.2.4) modulo 49, we find that

$$\begin{aligned} 3Q\Delta - 3\sum_{n=1}^{\infty} n\sigma_{13}(n)q^n &\equiv 7Q^2(Q^2 - PR) \\ &\equiv 7P(P - PR) \\ &= 7P^2 - 7P^2R \\ &\equiv 0 \pmod{49}. \end{aligned} \tag{7.2.5}$$

We return to the same identity from [275, Table II] and write

$$\begin{aligned} 144\sum_{n=1}^{\infty} n\sigma_{13}(n)q^n &= Q(3Q^3 + 4R^2 - 7PQR) \\ &= Q(Q^3 - R^2 + 2Q^3 + 5R^2 - 7PQR) \\ &= 1728Q\Delta + Q(2Q^3 + 5R^2 - 7PQR). \end{aligned} \tag{7.2.6}$$

Recalling (5.9.2), we record the congruences

$$QR \equiv 1 \pmod{11} \qquad \text{and} \qquad Q^3 - 3R^2 \equiv -2P \pmod{11}.$$

Thus, reducing (7.2.6) modulo 11, we deduce that

$$\begin{aligned} \sum_{n=1}^{\infty} n\sigma_{13}(n)q^n - Q\Delta &\equiv Q(2Q^3 + 5R^2 - 7PQR) \\ &= Q(2Q^3 - 6R^2 + 11R^2 - 7PQR) \\ &\equiv Q(2(-2P) - 7P) \\ &\equiv 0 \pmod{11}. \end{aligned} \tag{7.2.7}$$

Since $Q \equiv 1 \pmod{30}$, it follows trivially that $Q\Delta \equiv \Delta \pmod{30}$. Thus, from (5.12.1), (5.12.4), and (5.2.1), we can conclude, respectively, that

$$Q\Delta \equiv \sum_{n=1}^{\infty} n\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} n\sigma_{13}(n)q^n \pmod 2, \tag{7.2.8}$$

$$Q\Delta \equiv \sum_{n=1}^{\infty} n\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} n\sigma_{13}(n)q^n \pmod 3, \tag{7.2.9}$$

and

$$Q\Delta \equiv \sum_{n=1}^{\infty} n\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} n\sigma_{13}(n)q^n \pmod 5. \tag{7.2.10}$$

Hence, combining together (7.2.5) and (7.2.7)–(7.2.10), we deduce (7.1.6) to complete the proof. ∎

Proof of (7.1.7). Our proof will depend on entries from [275, Tables II, III], (7.2.3), and congruences from Chapter 5.

Since $Q \equiv 1 \pmod{24}$, it follows that $Q\Delta \equiv \Delta \pmod{24}$. Thus, by (5.12.1), we find that

$$Q\Delta \equiv \sum_{n=1}^{\infty} n^3\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} \left(2n\sigma_9(n) - n^2\sigma_3(n)\right) q^n \pmod 8, \tag{7.2.11}$$

where to justify the last congruence, we need to show that

$$n^3\sigma_1(n) \equiv 2n\sigma_9(n) - n^2\sigma_3(n) \pmod 8. \tag{7.2.12}$$

If n is even, the left side of (7.2.12) is a multiple of 8, and so we need to show that

$$2n\sigma_9(n) \equiv n^2\sigma_3(n) \pmod 8. \tag{7.2.13}$$

Considering the cases $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$ separately, we see that (7.2.13) holds, since $\sigma_9(n) \equiv \sigma_3(n) \pmod 2$.

If n is odd, we must show that

$$n\sigma_1(n) \equiv 2n\sigma_9(n) - \sigma_3(n) \pmod 8, \tag{7.2.14}$$

since $n^2 \equiv 1 \pmod 8$. Since $n^k \equiv n^{k-2} \pmod 8$ for $k \geq 3$, it follows that $\sigma_1(n) \equiv \sigma_9(n) \equiv \sigma_3(n) \pmod 8$. Thus, to show (7.2.14), we need to show that $n\sigma_1(n) \equiv 2n\sigma_1(n) - \sigma_1(n) \pmod 8$, i.e.,

$$\sigma_1(n) \equiv n\sigma_1(n) \pmod 8. \tag{7.2.15}$$

For $n \equiv 1 \pmod 8$, (7.2.15) is trivial. When $n \equiv 5 \pmod 8$, n cannot be a square, and so $\sigma_1(n)$ is even. Thus, (7.2.15) holds. For $n \equiv 3 \pmod 4$, there must be at least one prime $p \equiv 3 \pmod 4$ such that $p^{2k+1} \| n$. Thus, $(p+1) \mid \sigma_1(n)$, i.e., $\sigma_1(n) \equiv 0 \pmod 4$, and hence (7.2.15) holds. Thus, the justification of (7.2.12) is complete.

Next, from (5.12.3),

$$Q\Delta \equiv \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} \left(2n\sigma_9(n) - n^2\sigma_3(n)\right) q^n \pmod 3, \tag{7.2.16}$$

provided we can show that

$$n^2\sigma_1(n) \equiv 2n\sigma_9(n) - n^2\sigma_3(n) \pmod 3. \tag{7.2.17}$$

If $n \equiv 0 \pmod 3$, (7.2.17) is trivial. For $n \not\equiv 0 \pmod 3$, we must show that

$$\sigma_1(n) \equiv 2n\sigma_9(n) - \sigma_3(n) \pmod 3, \tag{7.2.18}$$

since $n^2 \equiv 1 \pmod 3$. Since for $k \geq 3$, $n^k \equiv n^{k-2} \pmod 3$, it follows that $\sigma_1(n) \equiv \sigma_9(n) \equiv \sigma_3(n) \pmod 3$. Thus, we need to prove that $\sigma_1(n) \equiv 2n\sigma_1(n) - \sigma_1(n) \pmod 3$, which simplifies to

$$\sigma_1(n) \equiv n\sigma_1(n) \pmod 3. \tag{7.2.19}$$

For $n \equiv 1 \pmod 3$, (7.2.19) is trivial. For $n \equiv 2 \pmod 3$, there exists at least one prime $p \equiv 2 \pmod 3$ such that $p^{2k+1} \| n$. Thus, $(p+1) \mid \sigma_1(n)$, i.e., $\sigma_1(n) \equiv 0 \pmod 3$, and (7.2.19) holds. In summary, (7.2.17) indeed is valid.

We now turn to identities from [275, Table II, no. 5; Table III, no. 2], namely,

$$1584 \sum_{n=1}^{\infty} n\sigma_9(n)q^n = 3Q^3 + 2R^2 - 5PQR \tag{7.2.20}$$

and

$$1728 \sum_{n=1}^{\infty} n^2\sigma_3(n)q^n = P^2Q - 2PR + Q^2, \tag{7.2.21}$$

respectively. Multiply (7.2.20) by 2 and multiply (7.2.21) by 3 and then reduce each of the new equations modulo 25. Taking the difference of the two congruences, we find that

$$\begin{aligned} 9\sum_{n=1}^{\infty} \left(2n\sigma_9(n) - n^2\sigma_3(n)\right) q^n &\equiv 6Q^3 + 4R^2 - 10PQR - 3P^2Q + 6PR - 3Q^2 \\ &\equiv 3Q(2Q^2 - 2PR - Q + P^2) + (R - PQ)(4R + 6P) \pmod{25}. \end{aligned} \tag{7.2.22}$$

From (5.3.1) we know that

$$Q^3 - R^2 \equiv 2Q^2 - 2PR - Q + P^2 \pmod{25}. \tag{7.2.23}$$

Using (7.2.23) in (7.2.22), noting that $Q \equiv 1 \pmod 5$ and $P \equiv R \pmod 5$, and using (7.1.4), we conclude that

$$\begin{aligned} 9\sum_{n=1}^{\infty} \left(2n\sigma_9(n) - n^2\sigma_3(n)\right) q^n &\equiv 3Q(Q^3 - R^2) \\ &\equiv 3Q \cdot 1728\Delta \\ &\equiv 9Q\Delta \pmod{25}. \end{aligned} \tag{7.2.24}$$

Hence, from (7.2.11), (7.2.16), and (7.2.24), we deduce (7.1.7). □

Proof of (7.1.9). Observing that $R \equiv 1 \pmod{42}$, we see that it trivially follows that $R\Delta \equiv \Delta \pmod{42}$. Thus, by (5.12.1), we find that

$$R\Delta \equiv \sum_{n=1}^{\infty} n^3\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} n\sigma_{15}(n)q^n \pmod 2. \tag{7.2.25}$$

Next, from (5.12.3),

$$R\Delta \equiv \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} n\sigma_{15}(n)q^n \pmod 3, \tag{7.2.26}$$

provided we can show that

$$n^2\sigma_1(n) \equiv n\sigma_{15}(n) \pmod 3. \tag{7.2.27}$$

For $n \equiv 0 \pmod 3$, (7.2.27) is trivial. For $n \not\equiv 0 \pmod 3$, we must show that

$$\sigma_1(n) \equiv n\sigma_{15}(n) \pmod 3, \tag{7.2.28}$$

since $n^2 \equiv 1 \pmod 3$. Since $n^k \equiv n^{k-2} \pmod 3$, for $k \geq 3$, it follows that $\sigma_1(n) \equiv \sigma_{15}(n) \pmod 3$. Thus, to prove (7.2.28), we just need to show that $\sigma_1(n) \equiv n\sigma_1(n) \pmod 3$, but this is precisely (7.2.19). Therefore, (7.2.28) and hence also (7.2.27) have been demonstrated.

Next, from (5.6.2), we deduce that

$$R\Delta \equiv \sum_{n=1}^{\infty} n\sigma_3(n)q^n \equiv \sum_{n=1}^{\infty} n\sigma_{15}(n)q^n \pmod 7. \tag{7.2.29}$$

We now turn to an identity from [275, Table I, no. 8],

$$3617 + 16320\sum_{n=1}^{\infty} \sigma_{15}(n)q^n = 1617Q^4 + 2000QR^2. \tag{7.2.30}$$

Applying the operator $q\frac{d}{dq}$ to (7.2.30) and employing Ramanujan's differential equations (5.9.8),

$$q\frac{dQ}{dq} = \frac{PQ - R}{3} \qquad \text{and} \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2},$$

we find that

$$16320\sum_{n=1}^{\infty} n\sigma_{15}(n)q^n = 2156PQ^4 - 4156Q^3R + \frac{8000}{3}PQR^2 - \frac{2000}{3}R^3. \tag{7.2.31}$$

By (5.9.2), we know that

$$QR \equiv 1 \pmod{11} \qquad \text{and} \qquad Q^3 - 3R^2 \equiv -2P \pmod{11}.$$

Thus, reducing (7.2.31) modulo 11, we deduce that

$$\begin{aligned} 7\sum_{n=1}^{\infty} n\sigma_{15}(n)q^n &\equiv 2Q^3R + PQR^2 + 8R^3 \\ &\equiv 7R(Q^3 - R^2) - 5Q^3R + PQR^2 + 15R^3 \\ &\equiv 7R\Delta - 5R(Q^3 - 3R^2) + PR \\ &\equiv 7R\Delta + 10PR + PR \\ &\equiv 7R\Delta \pmod{11}. \end{aligned} \tag{7.2.32}$$

From (5.13.2), we recall that

$$Q^2R \equiv P \pmod{13} \qquad \text{and} \qquad Q^3 - 3R^2 \equiv -2 \pmod{13}.$$

Thus, reducing (7.2.31) modulo 13, we find that

$$\begin{aligned}
5\sum_{n=1}^{\infty} n\sigma_{15}(n)q^n &\equiv -2PQ^4 + 4Q^3R + 6PQR^2 + 5R^3 \\
&= -5R(Q^3 - R^2) - 2PQ^4 + 9Q^3R + 6PQR^2 \\
&\equiv -5R(Q^3 - R^2) + 9PQ - 2PQ(Q^3 - 3R^2) \\
&\equiv 5R\Delta + 9PQ + 4PQ \\
&\equiv 5R\Delta \pmod{13}.
\end{aligned} \tag{7.2.33}$$

Hence, collecting together (7.2.25), (7.2.26), (7.2.29), (7.2.32), and (7.2.33), we deduce (7.1.9) to complete the proof. □

Proof of Theorem 7.1.1. Since $R \equiv 1 \pmod{36}$, it readily follows that $R\Delta \equiv \Delta \pmod{36}$. Thus, by (5.12.1), we find that

$$R\Delta \equiv \sum_{n=1}^{\infty} n^3\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n \pmod{4}. \tag{7.2.34}$$

By (5.12.3), we realize that

$$R\Delta \equiv \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n \pmod{9}. \tag{7.2.35}$$

We again return to Ramanujan's tables [275, Table III, no. 1] to record that

$$1728\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n = 3PQ - 2R - P^3. \tag{7.2.36}$$

So, using the facts $Q \equiv 1 \pmod{5}$ and $P \equiv R \pmod{5}$, we reduce (7.2.36) modulo 5 to find that

$$\begin{aligned}
3\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n &\equiv 3PQ - 2R - P^3 \\
&\equiv R(3Q - 2 - R^2) \\
&\equiv R(Q^3 - R^2) \\
&\equiv 3R\Delta \pmod{5}.
\end{aligned} \tag{7.2.37}$$

Hence, taking (7.2.34), (7.2.35), and (7.2.37) together, we deduce (7.1.14) to complete the proof. □

Proof of Theorem 7.1.2. Since $R \equiv 1 \pmod 6$, it follows that $R\Delta \equiv \Delta \pmod 6$. Thus, by (5.12.1), we have

$$R\Delta \equiv \sum_{n=1}^{\infty} n^3\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} \left(6n^2\sigma_9(n) - 5n\sigma_3(n)\right) q^n \pmod 2. \tag{7.2.38}$$

By (5.12.3), we find that

$$R\Delta \equiv \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} \left(6n^2\sigma_9(n) - 5n\sigma_3(n)\right) q^n \pmod 3. \tag{7.2.39}$$

We take an identity from [275, Table III, no. 5], namely,

$$1728\sum_{n=1}^{\infty} n^2\sigma_9(n)q^n = 6PQ^3 - 5P^2QR + 4PR^2 - 5Q^2R. \tag{7.2.40}$$

Using the congruence $P \equiv R \pmod 5$ and using (7.1.4), we reduce (7.2.40) modulo 5 to find that

$$\begin{aligned} 3\sum_{n=1}^{\infty} n^2\sigma_9(n)q^n &\equiv PQ^3 + 4PR^2 \\ &\equiv RQ^3 - R^3 \\ &= R(Q^3 - R^2) \\ &\equiv 3R\Delta \pmod 5. \end{aligned} \tag{7.2.41}$$

Hence, (7.2.38), (7.2.39), and (7.2.41) together imply (7.1.15) to complete the proof. □

Proof of (7.1.12). Since $R \equiv 1 \pmod{12}$, we see that $R\Delta \equiv \Delta \pmod{12}$. Thus, from (5.12.1),

$$R\Delta \equiv \sum_{n=1}^{\infty} n^3\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} n\sigma_9(n) \equiv \sum_{n=1}^{\infty} n\sigma_3(n) \pmod 4. \tag{7.2.42}$$

By (5.12.1) and the fact that $3 \mid \sigma(3n+2)$, we deduce that

$$R\Delta \equiv \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} n\sigma_9(n) \equiv \sum_{n=1}^{\infty} n\sigma_3(n) \pmod 3. \tag{7.2.43}$$

We utilize two further identities from [275, Table II, nos. 5, 2, resp.], namely,

$$1584\sum_{n=1}^{\infty} n\sigma_9(n)q^n = 3Q^3 + 2R^2 - 5PQR \tag{7.2.44}$$

and

$$720\sum_{n=1}^{\infty} n\sigma_3(n)q^n = PQ - R. \tag{7.2.45}$$

Now multiply (7.2.44) by 10 and multiply (7.2.45) by 22 and then add the resulting two equalities together. Using the facts $Q^2 \equiv P \pmod 7$ and $R \equiv 1 \pmod 7$ and reducing the foregoing sum modulo 49, we find that

$$\begin{aligned}
13\sum_{n=1}^{\infty} \left(n\sigma_9(n) + n\sigma_3(n)\right) q^n &\equiv 30Q^3 + 20R^2 - PQR + 22PQ - 22R \\
&= 3(Q^3 - R^2) - R(Q^3 - R^2) + 27Q^3 + 23R^2 \\
&\quad + RQ^3 - R^3 - PQR + 22PQ - 22R \\
&\equiv 13(3\Delta - R\Delta) + 7Q(4Q^2 + 3P) + 21R(R-1) \\
&\quad + Q(R-1)(Q^2 - P) - R(R^2 - 2R + 1) \\
&\equiv 13(3\Delta - R\Delta) \pmod{49}.
\end{aligned} \tag{7.2.46}$$

Hence, taking (7.2.42), (7.2.43), and (7.2.46) together, we deduce (7.1.12) to finish the proof. □

8

Ramanujan's Forty Identities for the Rogers–Ramanujan Functions

8.1 Introduction

The Rogers–Ramanujan functions in the title of this chapter are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}, \tag{8.1.1}$$

where here and in the sequel we use the customary notation $(a;q)_0 := 1$,

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1,$$
$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q| < 1.$$

These functions satisfy the famous Rogers–Ramanujan identities [303], [277], [281, pp. 214–215]

$$G(q) = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \qquad \text{and} \qquad H(q) = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{8.1.2}$$

At the end of his brief communication [278], [281, p. 231] announcing his proofs of the Rogers–Ramanujan identities (8.1.2), Ramanujan remarks, "I have now found an algebraic relation between $G(q)$ and $H(q)$, viz.:

$$H(q)\left\{G(q)\right\}^{11} - q^2G(q)\left\{H(q)\right\}^{11} = 1 + 11q\left\{G(q)H(q)\right\}^6. \tag{8.1.3}$$

Another noteworthy formula is

$$H(q)G(q^{11}) - q^2G(q)H(q^{11}) = 1. \tag{8.1.4}$$

Each of these formulae is the simplest of a large class." Ramanujan did not indicate how he had proved these two identities, which, as we shall see below,

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are two from a list of forty identities involving $G(q)$ and $H(q)$ that Ramanujan had compiled.

In his paper [304] establishing ten of the identities, L.J. Rogers remarks, "these [identities] were communicated privately to me in February 1919 ..." Rogers did not indicate whether further identities were included in Ramanujan's communication to him. We remark that Ramanujan returned to India on March 13, 1919, and that the short paper [278] was recorded in the minutes of the London Mathematical Society on March 13, 1919. Thus, both the paper to the London Mathematical Society and the letter to Rogers were evidently sent only days prior to Ramanujan's departure.

In 1933, G.N. Watson [333] proved eight of the identities, but with two of them from the group that Rogers had proved. Watson confides, "Among the formulae contained in the manuscripts left by Ramanujan is a set of about forty which involve functions of the types $G(q)$ and $H(q)$; the beauty of these formulae seems to me to be comparable with that of the Rogers–Ramanujan identities. So far as I know, nobody else has discovered any formulae which approach them even remotely; if my belief is well-founded, the undivided credit for the discovery of these formulae is due to Ramanujan." This last statement appears to be so obvious, especially since the manuscript was evidently in Watson's possession, that one wonders why he wrote it.

Ramanujan's forty identities for $G(q)$ and $H(q)$ (which do not include (8.1.2)) were first brought before the mathematical public in their entirety by B.J. Birch [75], who in 1975 found Watson's handwritten copy of Ramanujan's list of forty identities in the Oxford University Library. Ramanujan's original manuscript was in Watson's possession for many years and now resides in the library at Trinity College, Cambridge. Watson's handwritten list was later published along with Ramanujan's lost notebook [283, pp. 236–237] in 1988. Certain pairs of the identities are linked, and so it is natural to place them, in fact, in 35 (not 40) separate entries.

D. Bressoud [81], in his Ph.D. thesis, proved fifteen from the list of forty. His published paper [82] contains proofs of some, but not all, of the general identities from [81], which he developed in order to prove Ramanujan's identities. All the proofs of Rogers, Watson, and Bressoud employ classical means, although it would seem that in many cases the proofs are not like those found by Ramanujan.

After the work of Rogers, Watson, and Bressoud, nine remained to be proved. A.J.F. Biagioli [74] used modular forms to prove eight of them. To those familiar with the theory of modular forms, this theory can be invoked to prove all of the forty identities. About such proofs, Birch [75] opines, "A dull proof would have little value – in fact, all the functions involved in the identities are essentially theta functions, so modular forms of known level with poles of bounded order at known places, so the identities may presumably be verified by just checking that the first hundred or so powers of x are correct." It should be remarked that Biagioli's [74] proofs are more elegant than one might discern from Birch's remarks, for Biagioli used Fricke involutions and

other properties of modular forms to drastically reduce the number of terms that needed to be checked in the scheme envisioned by Birch. In fact, in most cases, Biagioli required only a few terms.

In this chapter, we offer proofs of all forty identities. Some of the proofs that we present were found by either Rogers, Watson, or Bressoud. However, most of the proofs presented in this chapter are from a *Memoir* written by Berndt, G. Choi, Y.-S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi [65], in which 35 of the identities were proved. In [347], Yesilyurt developed a significant generalization of the ideas of Rogers and Bressoud, and consequently proved two further identities. Finally, in a remarkable *tour de force*, Yesilyurt [348] employed his primary theorems from [347] to effect proofs of the remaining three identities, including the one that had never been proved by any means. These last three identities are arguably the most difficult identities of the forty, and Yesilyurt's achievement in proving them is nothing short of stupendous.

Frequently, we provide two or three proofs of an identity. Our goal has been to find proofs for all forty identities that Ramanujan might have given himself. Indeed, in several of our proofs, we utilize modular equations found by Ramanujan and recorded in his notebooks [282]. Although all the proofs offered here are in the spirit of Ramanujan's mathematics, it is to be admitted that for some proofs, knowing the identity beforehand was a distinct advantage to us in finding a proof. In [65], for each of the five identities that at that time did not have proofs that Ramanujan could have given, we provided heuristic arguments showing that both sides of each of the five identities have the same asymptotic expansions as $q \to 1^-$. It is possible that Ramanujan discovered some of his identities for $G(q)$ and $H(q)$ by examining asymptotic expansions. Ramanujan was an expert on asymptotic expansions, and in his last letter to G.H. Hardy, written on January 12, 1920, Ramanujan discussed the asymptotic expansions of his new mock theta functions and compared them to the asymptotic expansion of $G(q)$, with which he opened his letter [68, p. 220]. In the last section of this chapter, Section 8.6, we discuss further identities involving the Rogers–Ramanujan functions that have been discovered by other authors.

In concluding our introduction, we think that modular equations were central in many of Ramanujan's proofs. Although some of our proofs may be those found by Ramanujan, it is clear that all of us, including the aforementioned authors and the present authors, have not unveiled some of Ramanujan's principal ideas, which remain hidden by an impenetrable fog.

8.2 Definitions and Preliminary Results

We first recall Ramanujan's definitions for a general theta function and some of its important special cases. Set

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{8.2.1}$$

Basic properties satisfied by $f(a,b)$ include [55, p. 34, Entry 18]

$$f(a,b) = f(b,a), \tag{8.2.2}$$
$$f(1,a) = 2f(a,a^3), \tag{8.2.3}$$
$$f(-1,a) = 0, \tag{8.2.4}$$

and if n is an integer,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{8.2.5}$$

The basic property (8.2.2) will be used many times in the sequel without comment. The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [55, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{8.2.6}$$

The three most important special cases of (8.2.1) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{8.2.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{8.2.8}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty =: q^{-1/24}\eta(\tau), \tag{8.2.9}$$

where $q = \exp(2\pi i\tau)$, $\operatorname{Im} \tau > 0$, and η denotes the Dedekind eta function. The product representations in (8.2.7)–(8.2.9) are special cases of (8.2.6). Also, after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty. \tag{8.2.10}$$

Using (8.2.6) and (8.2.9), we can rewrite the Rogers–Ramanujan identities (8.1.2) in the forms

$$G(q) = \frac{f(-q^2,-q^3)}{f(-q)} \qquad \text{and} \qquad H(q) = \frac{f(-q,-q^4)}{f(-q)}. \tag{8.2.11}$$

We shall use (8.2.11) many times in the remainder of the chapter. A useful consequence of (8.2.11) in conjunction with the Jacobi triple product identity (8.2.6) is

$$G(q)H(q) = \frac{f(-q^5)}{f(-q)}. \tag{8.2.12}$$

Basic properties of the functions (8.2.7)–(8.2.10) include [55, pp. 39–40, Entries 24, 25(iii)]

$$\frac{f(q)}{f(-q)} = \frac{\psi(q)}{\psi(-q)} = \frac{\chi(q)}{\chi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}}, \tag{8.2.13}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}, \tag{8.2.14}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q), \qquad \chi(q)\chi(-q) = \chi(-q^2), \tag{8.2.15}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2). \tag{8.2.16}$$

It is easy to deduce from (8.2.14) or (8.2.6) that

$$\psi(-q) = \chi(-q)f(-q^4) = \frac{f(-q)}{\chi(-q^2)}, \quad \chi(q)f(-q) = \varphi(-q^2). \tag{8.2.17}$$

We shall use the famous quintuple product identity, which, in Ramanujan's notation (8.2.1), takes the form [55, p. 80, Entry 28(iv)]

$$\frac{f(-a^2,-a^{-2}q)}{f(-a,-a^{-1}q)} = \frac{1}{f(-q)}\left\{f(-a^3q,-a^{-3}q^2) + af(-a^{-3}q,-a^3q^2)\right\}, \tag{8.2.18}$$

where a is any complex number.

The function $f(a,b)$ also satisfies a useful addition formula. For each positive integer n, let

$$U_n := a^{n(n+1)/2}b^{n(n-1)/2} \qquad \text{and} \qquad V_n := a^{n(n-1)/2}b^{n(n+1)/2}.$$

Then [55, p. 48, Entry 31]

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{8.2.19}$$

The Rogers–Ramanujan functions are intimately associated with the Rogers–Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots, \qquad |q|<1, \tag{8.2.20}$$

which first appeared in a paper by Rogers [303] in 1894. Using the Rogers–Ramanujan identities (8.1.2), Rogers proved that

$$R(q) = q^{1/5}\frac{H(q)}{G(q)} = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{8.2.21}$$

This was independently discovered by Ramanujan and can be found in his notebooks [282], [55, p. 79, Chapter 16, Entry 38(iii)].

8.3 The Forty Identities

In our discussion of the forty identities, we attempt to relate the sources for most proofs. However, since many of our proofs are taken from the *Memoir* [65], we usually do not specifically mention that a certain proof is from [65].

Entry 8.3.1.

$$G^{11}(q)H(q) - q^2G(q)H^{11}(q) = 1 + 11qG^6(q)H^6(q). \tag{8.3.1}$$

Entry 8.3.1 is one of two identities stated by Ramanujan without proof in [278], [281, p. 231]. As related in the introduction, Ramanujan [278] claims that, "Each of these formulae is the simplest of a large class." Ramanujan's remark is interesting, because Entry 8.3.1 is the only identity among the forty in which powers of $G(q)$ and $H(q)$ appear. It would seem from Ramanujan's remark that he had further identities involving powers of $G(q)$ or $H(q)$, but no further identities of this sort are known in Ramanujan's work. The first published proof of (8.3.1) is by H.B.C. Darling [120] in 1921. A second proof by Rogers [304] appeared in the same year. One year later, L.J. Mordell [227] found another proof.

By (8.2.21), the identity (8.3.1) is equivalent to a famous identity for the Rogers–Ramanujan continued fraction (8.2.20), namely,

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{8.3.2}$$

This equality was found by Watson in Ramanujan's notebooks [282] and proved by him [332] in order to establish claims about the Rogers–Ramanujan continued fraction communicated by Ramanujan in his first two letters to Hardy [332]. A different proof of (8.3.2) can be found in Berndt's book [55, pp. 265–267]. The identity (8.3.2) can also be found in an unpublished manuscript of Ramanujan, which first appeared in handwritten form with his lost notebook [283, pp. 135–177, 238–243], and which is examined in Chapter 5 of this book. In particular, see (5.20.10).

Entry 8.3.2.

$$G(q)G(q^4) + qH(q)H(q^4) = \chi^2(q) = \frac{\varphi(q)}{f(-q^2)}. \tag{8.3.3}$$

Entry 8.3.2 was first proved in print by Rogers [304]; Watson [333] also found a proof. More recent proofs have been given by C. Gugg [160] and S.H. Son [321]. In fact, Andrews [13, p. 27] has shown that (8.3.3) follows from a very general identity in three variables found in Ramanujan's lost notebook; see our book [16, p. 150].

Entry 8.3.3.

$$G(q)G(q^4) - qH(q)H(q^4) = \frac{\varphi(q^5)}{f(-q^2)}. \tag{8.3.4}$$

Watson [333] gave a proof of (8.3.4), and Gugg [160] and Son [321] later gave further proofs.

Entry 8.3.4.

$$G(q^{11})H(q) - q^2G(q)H(q^{11}) = 1. \tag{8.3.5}$$

Entry 8.3.4 is the second identity offered by Ramanujan without proof in [278], [281, p. 231]. The first published proof was given by Rogers [304]. Watson [333] also gave a proof. R. Blecksmith, J. Brillhart, and I. Gerst [76] have shown that (8.3.5) follows from a very general theta function identity established by them.

Proofs of the next seven entries were first given by Rogers [304]. W. Chu [114, Example 25] has found a new proof of (8.3.6). N.D. Baruah, J. Bora, and N. Saikia [46] and Baruah and Bora [43], [45] have also found proofs of Entry 8.3.6. Z. Cao [96] has developed a very general method for writing certain products of theta functions as a sum of products of theta functions, and using his ideas, he has given proofs of Entries 8.3.6 and 8.3.9–8.3.12.

Entry 8.3.5.

$$G(q^{16})H(q) - q^3G(q)H(q^{16}) = \chi(q^2). \tag{8.3.6}$$

Entry 8.3.6.

$$G(q)G(q^9) + q^2H(q)H(q^9) = \frac{f^2(-q^3)}{f(-q)f(-q^9)}. \tag{8.3.7}$$

Entry 8.3.7.

$$G(q^2)G(q^3) + qH(q^2)H(q^3) = \frac{\chi(-q^3)}{\chi(-q)}. \tag{8.3.8}$$

Entry 8.3.8.

$$G(q^6)H(q) - qG(q)H(q^6) = \frac{\chi(-q)}{\chi(-q^3)}. \tag{8.3.9}$$

Entry 8.3.9.

$$G(q^7)H(q^2) - qG(q^2)H(q^7) = \frac{\chi(-q)}{\chi(-q^7)}. \tag{8.3.10}$$

Entry 8.3.10.

$$G(q)G(q^{14}) + q^3H(q)H(q^{14}) = \frac{\chi(-q^7)}{\chi(-q)}. \tag{8.3.11}$$

Entry 8.3.11.

$$G(q^8)H(q^3) - qG(q^3)H(q^8) = \frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}. \tag{8.3.12}$$

Generalizing slightly the approach of Rogers [304], L.-C. Zhang [351] proved four general theorems from which Entries 8.3.4, 8.3.5, 8.3.6, 8.3.9, 8.3.10, 8.3.11, and 8.3.12 follow as special cases. Unfortunately, he was not able to find any new examples to illustrate any of his general theorems. At the end of this section, we briefly discuss his general theorems.

Entry 8.3.12.

$$G(q)G(q^{24}) + q^5H(q)H(q^{24}) = \frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}. \tag{8.3.13}$$

Entry 8.3.13.

$$G(q^9)H(q^4) - qG(q^4)H(q^9) = \frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}. \tag{8.3.14}$$

Entry 8.3.14.

$$G(q^{36})H(q) - q^7G(q)H(q^{36}) = \frac{\chi(-q^6)\chi(-q^9)}{\chi(-q^2)\chi(-q^3)}. \tag{8.3.15}$$

Entries 8.3.12–8.3.14 were first proved by Bressoud in his doctoral dissertation [81].

Entry 8.3.15.

$$G(q^3)G(q^7) + q^2H(q^3)H(q^7) = G(q^{21})H(q) - q^4G(q)H(q^{21}) \tag{8.3.16}$$

$$= \frac{1}{2\sqrt{q}}\chi(q^{1/2})\chi(-q^{3/2})\chi(q^{7/2})\chi(-q^{21/2}) - \frac{1}{2\sqrt{q}}\chi(-q^{1/2})\chi(q^{3/2})\chi(-q^{7/2})\chi(q^{21/2}). \tag{8.3.17}$$

The only previously known proofs of (8.3.16) and (8.3.17) are by Biagioli [74], who used the theory of modular forms.

Entry 8.3.16.

$$G(q^2)G(q^{13}) + q^3H(q^2)H(q^{13}) = G(q^{26})H(q) - q^5G(q)H(q^{26}) \tag{8.3.18}$$

$$= \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}. \tag{8.3.19}$$

Up to the appearance of [65], the only known proof of (8.3.18) was by Bressoud [81], while Biagioli, using the theory of modular forms, had established the only known proof of (8.3.19). Biagioli's [74] formulation of (8.3.19) contains two misprints; the formula is also misnumbered as #17 instead of #18.

Proofs of the next four identities, (8.3.20)–(8.3.23), have been given by Bressoud [81].

Entry 8.3.17.

$$G(q)G(q^{19}) + q^4 H(q)H(q^{19}) = \frac{1}{4\sqrt{q}}\chi^2(q^{1/2})\chi^2(q^{19/2}) - \frac{1}{4\sqrt{q}}\chi^2(-q^{1/2})\chi^2(-q^{19/2}) - \frac{q^2}{\chi^2(-q)\chi^2(-q^{19})}. \quad (8.3.20)$$

Entry 8.3.18.

$$G(q^{31})H(q) - q^6 G(q)H(q^{31}) = \frac{1}{2q}\chi(q)\chi(q^{31}) - \frac{1}{2q}\chi(-q)\chi(-q^{31}) + \frac{q^3}{\chi(-q^2)\chi(-q^{62})}. \quad (8.3.21)$$

Entry 8.3.19.

$$\begin{aligned}&\left\{G(q)G(q^{39}) + q^8 H(q)H(q^{39})\right\} f(-q)f(-q^{39}) \\ &= \left\{G(q^{13})H(q^3) - q^2 G(q^3)H(q^{13})\right\} f(-q^3)f(-q^{13}) \quad (8.3.22) \\ &= \frac{1}{2q}\left(\varphi(-q^3)\varphi(-q^{13}) - \varphi(-q)\varphi(-q^{39})\right). \quad (8.3.23)\end{aligned}$$

Entry 8.3.20.

$$G(q)H(-q) + G(-q)H(q) = \frac{2}{\chi^2(-q^2)} = \frac{2\psi(q^2)}{f(-q^2)}. \quad (8.3.24)$$

Entry 8.3.21.

$$G(q)H(-q) - G(-q)H(q) = \frac{2q\psi(q^{10})}{f(-q^2)}. \quad (8.3.25)$$

Watson [333] constructed proofs of both (8.3.24) and (8.3.25). For more recent proofs of these identities, see papers by W. Chu [114, Example 22], Gugg [159], Son [321], and M.D. Hirschhorn [179].

Entry 8.3.22.

$$G(-q)G(-q^4) + qH(-q)H(-q^4) = \chi(q^2). \quad (8.3.26)$$

Entry 8.3.23.

$$G(-q^2)G(-q^3) + qH(-q^2)H(-q^3) = \frac{\chi(q)\chi(q^6)}{\chi(q^2)\chi(q^3)}. \quad (8.3.27)$$

Entry 8.3.24.

$$G(-q^6)H(-q) - qH(-q^6)G(-q) = \frac{\chi(q^2)\chi(q^3)}{\chi(q)\chi(q^6)}. \quad (8.3.28)$$

Bressoud [81] established the three previous entries.

Entry 8.3.25.

$$G(-q)G(q^9) - q^2H(-q)H(q^9) = \frac{\chi(-q)\chi(q^9)}{\chi(-q^6)}. \tag{8.3.29}$$

Equality (8.3.29) is a corrected version of that given by Watson [283] and was first proved by Bressoud [81].

Entry 8.3.26.

$$\begin{aligned} &G(q^{11})H(-q) + q^2G(-q)H(q^{11}) \\ &\qquad = \frac{\chi(q^2)\chi(q^{22})}{\chi(-q^2)\chi(-q^{22})} - \frac{2q^3}{\chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44})}. \end{aligned} \tag{8.3.30}$$

Watson [333] established (8.3.30). The minus sign in front of the second expression on the right side of (8.3.30) is missing in Watson's list [283].

Our formulations of Entries 8.3.27 and 8.3.28 are slightly different from those of Ramanujan, who had reversed the hypotheses in each entry. In other words, he intended that the formulas for U and V be the conclusions in each case, with the pairs of equations (8.3.33), (8.3.34) and (8.3.36), (8.3.37) being the conditions under which the formulas for U and V should hold. Watson proved Entry 8.3.27 under the same interpretation as we have given.

Entry 8.3.27. *Define*

$$U := U(q) := G(q)G(q^{44}) + q^9H(q)H(q^{44}) \tag{8.3.31}$$

and

$$V := V(q) := G(q^4)G(q^{11}) + q^3H(q^4)H(q^{11}). \tag{8.3.32}$$

Then

$$U^2 + qV^2 = \chi^3(q)\chi^3(q^{11}) \tag{8.3.33}$$

and

$$UV + q = \chi^2(q)\chi^2(q^{11}). \tag{8.3.34}$$

Entry 8.3.28. *Define*

$$U := G(q^{17})H(q^2) - q^3G(q^2)H(q^{17}) \quad \textit{and} \quad V := G(q)G(q^{34}) + q^7H(q)H(q^{34}). \tag{8.3.35}$$

Then

$$\frac{U}{V} = \frac{\chi(-q)}{\chi(-q^{17})} \tag{8.3.36}$$

and

$$U^4V^4 - qU^2V^2 = \frac{\chi^3(-q^{17})}{\chi^3(-q)}\left(1 + q^2\frac{\chi^3(-q)}{\chi^3(-q^{17})}\right)^2. \tag{8.3.37}$$

Bressoud proved (8.3.36) in his thesis [81]. Biagioli [74] intended to prove (8.3.37), but his proof was omitted. The first proof of (8.3.37) by any means was given by Yesilyurt [348].

Entry 8.3.29.

$$\begin{aligned}&\left\{G(q^2)G(q^{23})+q^5H(q^2)H(q^{23})\right\}\left\{G(q^{46})H(q)-q^9G(q)H(q^{46})\right\}\\&=\chi(-q)\chi(-q^{23})+q+\frac{2q^2}{\chi(-q)\chi(-q^{23})}.\end{aligned}\tag{8.3.38}$$

Entry 8.3.30.

$$\frac{G(q^{19})H(q^4)-q^3G(q^4)H(q^{19})}{G(q^{76})H(-q)+q^{15}G(-q)H(q^{76})}=\frac{\chi(-q^2)}{\chi(-q^{38})}.\tag{8.3.39}$$

Entry 8.3.31.

$$\frac{G(q^2)G(q^{33})+q^7H(q^2)H(q^{33})}{G(q^{66})H(q)-q^{13}H(q^{66})G(q)}=\frac{\chi(-q^3)}{\chi(-q^{11})}.\tag{8.3.40}$$

Entry 8.3.32.

$$\frac{G(q^3)G(q^{22})+q^5H(q^3)H(q^{22})}{G(q^{11})H(q^6)-qG(q^6)H(q^{11})}=\frac{\chi(-q^{33})}{\chi(-q)}.\tag{8.3.41}$$

Using the theory of modular forms, Biagioli [74] constructed proofs of Entries 8.3.29–8.3.32. A difficult but beautiful proof of Entry 8.3.29 is given by Yesilyurt in [348], while proofs of the latter three entries were effected earlier by Yesilyurt in [347]. The equivalence of Entries 8.3.31 and 8.3.32 was earlier established in the *Memoir* [65].

Entry 8.3.33.

$$\frac{G(q)G(q^{54})+q^{11}H(q)H(q^{54})}{G(q^{27})H(q^2)-q^5G(q^2)H(q^{27})}=\frac{\chi(-q^3)\chi(-q^{27})}{\chi(-q)\chi(-q^9)}.\tag{8.3.42}$$

Entry 8.3.34.

$$\begin{aligned}&\left\{G(q)G(-q^{19})-q^4H(q)H(-q^{19})\right\}\left\{G(-q)G(q^{19})-q^4H(-q)H(q^{19})\right\}\\&=G(q^2)G(q^{38})+q^8H(q^2)H(q^{38}).\end{aligned}\tag{8.3.43}$$

Proofs of (8.3.42) and (8.3.43) have been found by Bressoud [81], who corrected a misprint in Watson's [283] formulation of (8.3.43).

Entry 8.3.35.

$$\begin{aligned}&\left\{G(q)G(q^{94})+q^{19}H(q)H(q^{94})\right\}\left\{G(q^{47})H(q^2)-q^9G(q^2)H(q^{47})\right\}\\&=\chi(-q)\chi(-q^{47})+2q^2+\frac{2q^4}{\chi(-q)\chi(-q^{47})}\\&\quad+q\sqrt{4\chi(-q)\chi(-q^{47})+9q^2+\frac{8q^4}{\chi(-q)\chi(-q^{47})}}.\end{aligned}\tag{8.3.44}$$

The only previously known proof [74] of Entry 8.3.35 employs the theory of modular forms. Yesilyurt's [348] beautiful proof is given here.

Observe that in most of the forty identities, $G(q)$ and $H(q)$ occur in the combinations

$$G(q^r)G(q^s) + q^{(r+s)/5}H(q^r)H(q^s), \qquad \text{when } r+s \equiv 0 \pmod 5, \tag{8.3.45}$$

$$G(q^r)H(q^s) - q^{(r-s)/5}H(q^r)G(q^s), \qquad \text{when } r-s \equiv 0 \pmod 5, \tag{8.3.46}$$

or when one or both of q^r and q^s are replaced by $-q^r$ and $-q^s$, respectively, in either (8.3.45) or (8.3.46) above.

We pointed out after Entry 8.3.11 that Zhang [351] has established four general identities for the expressions in (8.3.45) and (8.3.46). For illustration, we record two of them and refer readers to his paper for the remaining two.

Theorem 8.3.1. *Let (s,t) be a pair of positive integers such that $5 \mid (s+t)$ and such that there exists another pair of positive integers (α,β) such that*

$$st = \alpha\beta, \qquad \frac{s+t}{5} = \frac{\alpha+\beta}{3} := \lambda, \qquad \textit{and} \qquad \frac{s\pm\alpha}{\lambda} \qquad \textit{is an integer.}$$

Then

$$G(q^s)G(q^t) + q^{(s+t)/5}H(q^s)H(q^t) = \frac{f(-q^\alpha)f(-q^\beta)}{f(-q^s)f(-q^t)}. \tag{8.3.47}$$

Theorem 8.3.2. *Let (s,t) be a pair of positive integers such that $5 \mid (s-t)$ and such that there exists another pair of positive integers (α,β) such that*

$$st = \alpha\beta, \qquad \frac{9s+t}{5} = \frac{\alpha+\beta}{3} := \lambda, \qquad \textit{and} \qquad \frac{3s\pm\alpha}{\lambda} \qquad \textit{is an integer.}$$

Then

$$G(q^t)H(q^s) - q^{(t-s)/5}H(q^t)G(q^s) = \frac{f(-q^\alpha)f(-q^\beta)}{f(-q^s)f(-q^t)}. \tag{8.3.48}$$

Note that Entries 8.3.6 and 8.3.10 are instances of (8.3.47), while Entries 8.3.4 and 8.3.9 arise from (8.3.48).

Ramanujan's identities are remarkable for several reasons. The Rogers–Ramanujan functions are associated with modular equations of degree 5 and q-products with base q^5. However, the "5" is missing on all of the right sides of the identities, except for Entries 8.3.3 and 8.3.21. One would expect to see in such identities theta functions with arguments q^{5n}, for certain positive integers n, but such functions do not generally appear! At the end of Section 4.3 in [65], we provided some heuristic thoughts about this phenomenon.

Next, observe that the right sides in almost all of the identities are expressed entirely in terms of the modular function χ with no other theta function appearing. We have no explanation for this phenomenon. It seems likely

that the function χ played a more important role in Ramanujan's thinking than we are able to discern.

As we shall see in the proofs throughout this chapter, some of the identities are amenable to general techniques established either by Watson, Rogers, Bressoud, or the authors of [65]. However, for those identities that are more difficult to prove (and there are many), these ideas do not appear to be useful. Yesilyurt's [347], [348] generalization of Rogers's method is the most general and powerful tool that we have, and it could be that Ramanujan used arguments of this sort. However, Ramanujan appeared to have had at least one key idea that all researchers to date have missed. Moreover, each of the forty identities, in principle, can be associated with modular equations of a certain degree. It happens that for each such degree, Ramanujan recorded at least one modular equation of that degree in his notebooks [282], [55]. We are certain that modular equations were at the heart of Ramanujan's methods.

Before embarking on the proofs, we summarize here those proofs that we have borrowed from others. The proofs of Entries 8.3.18 and 8.3.28 that we give are due to Bressoud [81]. Our proof of Entry 8.3.34 is a modification of his proof [81]. Our proof of Entry 8.3.19 begins at the same point as that of Bressoud but diverges thereafter. We give two proofs of Entry 8.3.12, one of which is due to Bressoud [81]. Watson's proofs of Entries 8.3.3, 8.3.21, 8.3.26, and 8.3.27 are provided. Rogers's proofs of Entries 8.3.9–8.3.11 are given. The proofs of Entries 8.3.28 (second part), 8.3.29–8.3.36, and 8.3.35 are due to Yesilyurt [347], [348]. Lastly, we emphasize that many of the proofs that follow appeared in the *Memoir* [65] for the first time.

8.4 The Principal Ideas Behind the Proofs

In this section, we describe the main ideas behind the proofs given by Watson [333], Rogers [304], Bressoud [81], the authors of [65], and Yesilyurt [347], [348].

We first discuss an idea of Watson [333]. In these proofs, one expresses the left sides of the identities in terms of theta functions using (8.2.11). In some cases, after clearing fractions, the right side can be expressed as a product of two theta functions, say with summation indices m and n. One then tries to find a change of indices of the form

$$\alpha m + \beta n = 5M + a \qquad \text{and} \qquad \gamma m + \delta n = 5N + b,$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side. We emphasize that this method works only when the right side is a product of two theta functions, and even then, only in some cases does this kind of change of variables produce the desired equality. This method was probably not that used by Ramanujan, because it would seem that the identity to be proved must be explicitly known in advance.

We next present a modest generalization of Rogers's method [304]. We let p and m denote odd positive integers with $p > 1$, and let α, β, and λ be real numbers such that

$$\alpha m^2 + \beta = \lambda p. \tag{8.4.1}$$

The special case that α, β, and λ are integers is given by Rogers [304]. Consider, for any real number v, the product

$$\begin{aligned} &q^{p\alpha m^2 v^2} f(-q^{p\alpha+2p\alpha mv}, -q^{p\alpha-2p\alpha mv}) q^{p\beta v^2} f(-q^{p\beta+2p\beta v}, -q^{p\beta-2p\beta v}) \\ &= \sum_{r=-\infty}^{\infty} (-1)^r q^{p\alpha(r+mv)^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{p\beta(s+v)^2} = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (-1)^{r+s} q^I, \end{aligned} \tag{8.4.2}$$

where

$$I = p\alpha(r+mv)^2 + p\beta(s+v)^2.$$

For fixed s, write $r = ms + t$. Then, by (8.4.1),

$$\begin{aligned} I &= p\alpha\left\{(s+v)m+t\right\}^2 + p\beta(s+v)^2 \\ &= \lambda p^2(s+v)^2 + 2p\alpha mt(s+v) + p\alpha t^2 \\ &= \lambda\left\{p(s+v) + \frac{\alpha mt}{\lambda}\right\}^2 - \frac{\alpha^2 m^2 t^2}{\lambda} + p\alpha t^2 \\ &= \lambda\left\{p(s+v) + \frac{\alpha mt}{\lambda}\right\}^2 + \frac{\alpha\beta}{\lambda} t^2. \end{aligned} \tag{8.4.3}$$

Note also that since m is odd,

$$(-1)^{r+s} = (-1)^t. \tag{8.4.4}$$

Now let

$$S_p := \left\{\frac{1}{2p}, \frac{3}{2p}, \dots, \frac{2p-1}{2p}\right\}. \tag{8.4.5}$$

Thus, using (8.4.2)–(8.4.5), we find that

$$\begin{aligned} &\sum_{v\in S_p} q^{p\alpha m^2 v^2} f(-q^{p\alpha+2p\alpha mv}, -q^{p\alpha-2p\alpha mv}) q^{p\beta v^2} f(-q^{p\beta+2p\beta v}, -q^{p\beta-2p\beta v}) \\ &= \sum_{v\in S_p} \sum_{r=-\infty}^{\infty} (-1)^r q^{p\alpha(r+mv)^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{p\beta(s+v)^2} = \sum_{k=1}^{p} \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^I, \end{aligned} \tag{8.4.6}$$

where

$$I = I(r,s,t) := \lambda\left\{p\left(s + \frac{2k-1}{2p}\right) + \frac{\alpha mt}{\lambda}\right\}^2 + \frac{\alpha\beta}{\lambda} t^2$$

$$= \lambda \left\{ ps + k - 1 + \frac{1}{2} + \frac{\alpha m t}{\lambda} \right\}^2 + \frac{\alpha\beta}{\lambda} t^2$$
$$= \lambda \left\{ u + \frac{1}{2} + \frac{\alpha m t}{\lambda} \right\}^2 + \frac{\alpha\beta}{\lambda} t^2, \tag{8.4.7}$$

upon letting $u := ps + k - 1$. Hence, (8.4.6) can now be expressed as

$$\sum_{v \in S_p} q^{p\alpha m^2 v^2} f(-q^{p\alpha + 2p\alpha m v}, -q^{p\alpha - 2p\alpha m v}) q^{p\beta v^2} f(-q^{p\beta + 2p\beta v}, -q^{p\beta - 2p\beta v})$$
$$= \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^I, \tag{8.4.8}$$

with I as given in (8.4.7).

The strategy of Rogers is to find two sets of parameters $\{\alpha_1, \beta_1, m_1, p_1, \lambda_1\}$ and $\{\alpha_2, \beta_2, m_2, p_2, \lambda_2\}$ both giving rise to the same function on the right-hand side of (8.4.8). This would establish an identity between two sums of products of two theta functions each of the form (8.4.2). For instance, if we choose the two sets of parameters such that

$$\alpha_1\beta_1 = \alpha_2\beta_2, \qquad \lambda_1 = \lambda_2, \qquad \text{and} \qquad \frac{\alpha_1 m_1}{\lambda_1} \pm \frac{\alpha_2 m_2}{\lambda_2} \qquad \text{is an integer,} \tag{8.4.9}$$

then both sets of parameters would satisfy the formula for I in (8.4.7), thus giving rise to the same function on the right-hand side of (8.4.8).

We next show that the contributions of the terms with indices k and $p - k + 1$ are identical. Applying (8.2.5) with $n = -m$, we find that

$$q^{\alpha m^2 (2k-1)^2/(4p)} f(-q^{p\alpha + \alpha m(2k-1)}, -q^{p\alpha - \alpha m(2k-1)})$$
$$= q^{\alpha m^2 (2k-1)^2/(4p) + m^2 p\alpha - m^2 \alpha(2k-1)}$$
$$\times f(-q^{p\alpha + \alpha m(2k-1) - 2p\alpha m}, -q^{p\alpha - \alpha m(2k-1) + 2p\alpha m})$$
$$= q^{\alpha m^2 (2p-2k+1)^2/(4p)} f(-q^{p\alpha + \alpha m(2p-2k+1)}, -q^{p\alpha - \alpha m(2p-2k+1)}), \tag{8.4.10}$$

where we have used the fact that p is odd. The same argument holds for the other theta function in (8.4.2). This establishes our claim.

Next, we show that the contribution of the term with $k = (p+1)/2$, i.e., $v = 1/2$, equals 0. Thus, we examine

$$\sum_{r=-\infty}^{\infty} (-1)^r q^{p\alpha(r+m/2)^2} = q^{p\alpha m^2/4} f(-q^{p\alpha(1-m)}, -q^{p\alpha(1+m)}). \tag{8.4.11}$$

To the theta function in (8.4.11), we apply (8.2.5) with $n = (m-1)/2$. Thus, for some constant c,

$$f(-q^{p\alpha(1-m)}, -q^{p\alpha(1+m)}) = q^c f(-1, -q^{2p\alpha}) = 0, \tag{8.4.12}$$

by (8.2.4). The same argument shows that the other theta function appearing in (8.4.6) also vanishes when $v = 1/2$.

Using (8.4.10) and (8.4.12) in (8.4.6), we deduce that when p is odd,

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} F(\alpha,\beta,m,p,\lambda,k)\\
&:= \sum_{k=1}^{(p-1)/2} \sum_{r=-\infty}^{\infty} (-1)^r q^{p\alpha(r+m(2k-1)/(2p))^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{p\beta(s+(2k-1)/(2p))^2}\\
&= \sum_{k=1}^{(p-1)/2} q^{\alpha m^2(2k-1)^2/(4p)} f(-q^{p\alpha+\alpha m(2k-1)}, -q^{p\alpha-\alpha m(2k-1)})\\
&\quad \times q^{\beta(2k-1)^2/(4p)} f(-q^{p\beta+\beta(2k-1)}, -q^{p\beta-\beta(2k-1)})\\
&= \frac{1}{2} \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^I, \qquad (8.4.13)
\end{aligned}$$

where I is given in (8.4.7).

If p is even and if α is even, then the same argument shows that the terms with indices k and $p-k+1$ are identical. Hence, for p even,

$$\begin{aligned}
&\sum_{k=1}^{p/2} F(\alpha,\beta,m,p,\lambda,k)\\
&= \sum_{k=1}^{p/2} \sum_{r=-\infty}^{\infty} (-1)^r q^{p\alpha(r+m(2k-1)/(2p))^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{p\beta(s+(2k-1)/(2p))^2}\\
&= \sum_{k=1}^{p/2} q^{\alpha m^2(2k-1)^2/(4p)} f(-q^{p\alpha+\alpha m(2k-1)}, -q^{p\alpha-\alpha m(2k-1)})\\
&\quad \times q^{\beta(2k-1)^2/(4p)} f(-q^{p\beta+\beta(2k-1)}, -q^{p\beta-\beta(2k-1)})\\
&= \frac{1}{2} \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^I, \qquad (8.4.14)
\end{aligned}$$

where I is given in (8.4.7).

For later applications, we record some special cases of (8.4.13) and (8.4.14). If $p = 5$ and $m = 1$,

$$\begin{aligned}
\sum_{k=1}^{2} F(\alpha,\beta,1,5,\lambda,k) &= q^{(\alpha+\beta)/20} f(-q^{4\alpha}, -q^{6\alpha}) f(-q^{4\beta}, -q^{6\beta})\\
&\quad + q^{9(\alpha+\beta)/20} f(-q^{2\alpha}, -q^{8\alpha}) f(-q^{2\beta}, -q^{8\beta}). \qquad (8.4.15)
\end{aligned}$$

If $p = 5$ and $m = 3$,

$$\sum_{k=1}^{2} F(\alpha,\beta,3,5,\lambda,k) = q^{(9\alpha+\beta)/20} f(-q^{2\alpha},-q^{8\alpha}) f(-q^{4\beta},-q^{6\beta}) \\ - q^{(\alpha+9\beta)/20} f(-q^{4\alpha},-q^{6\alpha}) f(-q^{2\beta},-q^{8\beta}), \quad (8.4.16)$$

where we applied (8.2.5) with $n = 1$. If $p = 3$ and $m = 1$,

$$\sum_{k=1}^{1} F(\alpha,\beta,1,3,\lambda,k) = q^{(\alpha+\beta)/12} f(-q^{2\alpha}) f(-q^{2\beta}). \quad (8.4.17)$$

If $p = 2$ and $m = 1$, by (8.2.8),

$$\sum_{k=1}^{1} F(\alpha,\beta,1,2,\lambda,k) = q^{(\alpha+\beta)/8} f(-q^{\alpha},-q^{3\alpha}) f(-q^{\beta},-q^{3\beta}) \\ = q^{(\alpha+\beta)/8} \psi(-q^{\alpha}) \psi(-q^{\beta}). \quad (8.4.18)$$

Rogers's ideas were extended by Bressoud [81], but we have not employed Bressoud's more general theorems in this chapter. We have used Rogers's method, however, in proving further identities in Ramanujan's list.

A third approach is a method of *elimination*. Here one sets $T(q)$, say, equal to the left side of the identity to be proved. By changes of variable, if necessary, one records two further (previously proved) identities involving $G(q)$ and $H(q)$, each involving a pair of the same Rogers–Ramanujan functions appearing in the identity to be proved. Thus, we have three equations involving the same three Rogers–Ramanujan functions, which we proceed to eliminate from the three equations. There remains then an identity involving $T(q)$ and (usually) theta functions to be proved. It must be emphasized that this method can be applied only if one can find two identities related to the one to be proved. In particular, the method cannot be utilized in those cases in which Ramanujan offered only one or two identities of a given degree. The theta function identity to be verified is usually difficult, and generally one should convert it to a modular equation. Hopefully, the modular equation is a known one, in particular, one of the couple hundred that Ramanujan found, but of course, it may not be. For completeness, we next define a modular equation.

We give the definition of a modular equation, as understood by Ramanujan. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, ℓ, and $\ell' := \sqrt{1-\ell^2}$, respectively, where $0 < k$, $\ell < 1$. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \quad (8.4.19)$$

for some positive rational integer n. A relation between k and ℓ induced by (8.4.19) is called a *modular equation of degree n*. Following Ramanujan, set

$$\alpha = k^2 \qquad \text{and} \qquad \beta = \ell^2.$$

We often say that β has degree n over α. If

$$q = \exp(-\pi K'/K), \tag{8.4.20}$$

one of the most fundamental relations in the theory of elliptic functions is given by the formula [55, pp. 101–102]

$$\varphi^2(q) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} =: \frac{2}{\pi}K(k). \tag{8.4.21}$$

The first equality in (8.4.21) and elementary theta function identities make it possible to write each modular equation as a theta function identity. (The second equality in (8.4.21) arises from expanding the integrand in a binomial series and integrating termwise.) Lastly, the multiplier m of degree n is defined by

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{8.4.22}$$

Ramanujan derived an extensive "catalogue" of formulas [55, pp. 122–124] giving the "evaluations" of $f(-q)$, $\varphi(q)$, $\psi(q)$, and $\chi(q)$ at various powers of the arguments in terms of

$$z := z_1 := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha), \qquad \alpha, \qquad \text{and} \qquad q.$$

If q is replaced by q^n, then the evaluations are given in terms of

$$z_n := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta), \qquad \beta, \qquad \text{and} \qquad q^n,$$

where β has degree n over α.

In this chapter, we utilize a new, fourth, approach in which $G(q)$ and $H(q)$ are expressed as linear combinations of G and H with arguments q^n for certain positive integers n. Watson [333] discovered the first pair of formulas of this sort, but used them to prove only one of the forty identities. We develop further formulas of this kind and employ them in proving over a dozen of the forty identities.

We provide here statements and proofs of the lemmas from [73] that we use in the sequel to establish several of Ramanujan's forty identities. Some of our proofs below *actually use some of Ramanujan's forty identities.* Indeed, some of our arguments are circular. However, in all such instances, we exhibit at least one further proof of each particular entry, which is independent of the other entries. Moreover, our arguments then show that certain pairs of entries are equivalent; for example, Entries 8.3.7 and 8.3.12 are equivalent.

We begin with Watson's lemma [333], Lemma 8.4.1. Watson's proof of (8.4.23) [333, p. 60] is based on Entries 8.3.2 and 8.3.3. Here, we provide a direct proof.

Lemma 8.4.1. *With $f(-q)$ defined by (8.2.9),*

$$G(q) = \frac{f(-q^8)}{f(-q^2)} \left(G(q^{16}) + qH(-q^4) \right), \tag{8.4.23}$$

$$H(q) = \frac{f(-q^8)}{f(-q^2)} \left(q^3 H(q^{16}) + G(-q^4) \right). \tag{8.4.24}$$

Proof. Employing (8.2.18) with q replaced by q^{10} and a replaced by q, we find that

$$\frac{f(-q^2,-q^8)f(-q^{10})}{f(-q,-q^9)} = f(-q^{13},-q^{17}) + qf(-q^7,-q^{23}). \tag{8.4.25}$$

The left-hand side of (8.4.25), by (8.2.6) and (8.2.11), is easily seen to be equal to $f(-q^2)G(q)$, and so we conclude that

$$f(-q^2)G(q) = f(-q^{13},-q^{17}) + qf(-q^7,-q^{23}). \tag{8.4.26}$$

Similarly, replacing q by q^{10}, q^5, q^5, and a by $q^7, -q, -q^2$, respectively, in (8.2.18) and using (8.2.6) and (8.2.11), we find that

$$f(-q^2)H(q) = f(-q^{11},-q^{19}) + q^3 f(-q,-q^{29}), \tag{8.4.27}$$

$$f(-q)G(q^2) = f(q^7,q^8) - qf(q^2,q^{13}), \tag{8.4.28}$$

and

$$f(-q)H(q^2) = f(q^4,q^{11}) - qf(q,q^{14}). \tag{8.4.29}$$

Using (8.2.19) twice with $n = 2$, and with $U_r = (-1)^r q^{15r^2-2r}$, $V_r = (-1)^r q^{15r^2+2r}$ and $U_r = (-1)^r q^{15r^2-8r}$, $V_r = (-1)^r q^{15r^2+8r}$, respectively, we separate each term on the right side of (8.4.26) into its even and odd parts and so find that

$$\begin{aligned} f(-q^2)G(q) &= f(q^{56},q^{64}) - q^{13} f(q^4,q^{116}) + q\left(f(q^{44},q^{76}) - q^7 f(q^{16},q^{104})\right) \\ &= f(q^{56},q^{64}) - q^8 f(q^{16},q^{104}) + q\left(f(q^{44},q^{76}) - q^{12} f(q^4,q^{116})\right) \\ &= f(-q^8)G(q^{16}) + qf(-q^8)H(-q^4), \end{aligned}$$

where in the last step we used (8.4.28) and (8.4.27) with q replaced by q^8 and $-q^4$, respectively. This proves (8.4.23). The related identity (8.4.24) is proved in a similar way, and so we omit the details. ∎

Lemma 8.4.2. *With χ defined by (8.2.10),*

$$\chi(-q)\chi(q^3)G(q) = \frac{\chi(q^6)}{\chi(-q^4)} G(-q^6) - q^5 \frac{\chi(q^2)}{\chi(-q^{12})} H(q^{24}), \tag{8.4.30}$$

$$\chi(-q)\chi(q^3)H(q) = -q\frac{\chi(q^6)}{\chi(-q^4)} H(-q^6) + \frac{\chi(q^2)}{\chi(-q^{12})} G(q^{24}). \tag{8.4.31}$$

Proof. By two applications of Entry 8.3.7, the second with q replaced by $-q$, and by Entry 8.3.20 with q replaced by q^3,

$$\frac{\chi(-q^3)}{\chi(-q)}G(-q^3) - \frac{\chi(q^3)}{\chi(q)}G(q^3) \tag{8.4.32}$$
$$= \left(G(q^2)G(q^3) + qH(q^2)H(q^3)\right)G(-q^3) \tag{8.4.33}$$
$$- \left(G(q^2)G(-q^3) - qH(q^2)H(-q^3)\right)G(q^3)$$
$$= qH(q^2)\left\{H(q^3)G(-q^3) + H(-q^3)G(q^3)\right\} = 2q\frac{H(q^2)}{\chi^2(-q^6)},$$

which, by (8.2.15), simplifies to

$$\chi(q)\chi(-q^3)G(-q^3) - \chi(-q)\chi(q^3)G(q^3) = 2q\frac{\chi(-q^2)}{\chi^2(-q^6)}H(q^2). \tag{8.4.34}$$

Employing (8.4.23) with q replaced by $-q^3$ and q^3, respectively, in (8.4.34), we find that

$$L(q) := \chi(q)\chi(-q^3)\left\{G(q^{48}) - q^3H(-q^{12})\right\} \tag{8.4.35}$$
$$- \chi(-q)\chi(q^3)\left\{G(q^{48}) + q^3H(-q^{12})\right\} \tag{8.4.36}$$
$$= 2q\frac{f(-q^6)\chi(-q^2)}{f(-q^{24})\chi^2(-q^6)}H(q^2) = 2q\chi(-q^2)\chi(q^6)H(q^2),$$

by (8.2.14) and (8.2.15). Collecting terms on the left side of (8.4.35) and using (8.5.7.10) below, we find that

$$L(q) = \left\{\chi(q)\chi(-q^3) - \chi(-q)\chi(q^3)\right\}G(q^{48})$$
$$- q^3\left\{\chi(q)\chi(-q^3) + \chi(-q)\chi(q^3)\right\}H(-q^{12})$$
$$= 2q\frac{\chi(q^4)}{\chi(-q^{24})}G(q^{48}) - 2q^3\frac{\chi(q^{12})}{\chi(-q^8)}H(-q^{12}). \tag{8.4.37}$$

Hence, by (8.4.35) and (8.4.37),

$$2q\frac{\chi(q^4)}{\chi(-q^{24})}G(q^{48}) - 2q^3\frac{\chi(q^{12})}{\chi(-q^8)}H(-q^{12}) = 2q\chi(-q^2)\chi(q^6)H(q^2).$$

Dividing both sides by $2q$ and then replacing q^2 by q, we deduce (8.4.31). The companion equality (8.4.30) is proved in a similar way, and so we omit the details. $\square$

Lemma 8.4.3. *We have*

$$\chi(q)\chi(-q^3)G(q^9) - \chi(-q)\chi(q^3)G(-q^9) = 2q\frac{G(q^4)}{\chi(-q^{18})} \tag{8.4.38}$$

and

$$\chi(q)\chi(-q^3)H(q^9) + \chi(-q)\chi(q^3)H(-q^9) = 2\frac{H(q^4)}{\chi(-q^{18})}. \tag{8.4.39}$$

Proof. The proofs of (8.4.38) and (8.4.39) are very similar to the proofs of (8.4.30) and (8.4.31), except that Entry 8.3.13 is used instead of Entry 8.3.20. We prove only (8.4.39), since the proof of (8.4.38) follows along the same lines.

By two applications of Entry 8.3.13 and one application of Entry 8.3.20 with q replaced by q^9,

$$\begin{aligned}
&\frac{\chi(q)\chi(-q^6)}{\chi(q^3)\chi(-q^{18})}H(q^9)+\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}H(-q^9)\\
&=\left\{G(-q^9)H(q^4)+qG(q^4)H(-q^9)\right\}H(q^9)\\
&\quad+\left\{G(q^9)H(q^4)-qG(q^4)H(q^9)\right\}H(-q^9)\\
&=H(q^4)\left\{G(-q^9)H(q^9)+G(q^9)H(-q^9)\right\}=2\frac{H(q^4)}{\chi^2(-q^{18})}.
\end{aligned}$$

Using (8.2.15) above, we complete the proof of (8.4.39). □

Lemma 8.4.4. *If*

$$a(q)=\frac{\chi^2(q)\chi(-q^2)}{\chi(-q^6)} \qquad \textit{and} \qquad b(q)=\frac{\chi(-q)\chi(-q^2)}{\chi(-q^3)\chi(-q^6)}, \tag{8.4.40}$$

then

$$G(q)=a(q)G(q^6)-qb(q)H(q^4), \tag{8.4.41}$$

$$H(q)=qa(q)H(q^6)+b(q)G(q^4). \tag{8.4.42}$$

First Proof of Lemma 8.4.4. The equality (8.4.41) can be rewritten in the form

$$\frac{\chi(-q^6)}{\chi(-q^2)}G(q)=\chi^2(q)G(q^6)-q\frac{\chi(-q)}{\chi(-q^3)}H(q^4). \tag{8.4.43}$$

When the identities for $\dfrac{\chi(-q^6)}{\chi(-q^2)}$, $\dfrac{\chi(-q)}{\chi(-q^3)}$, and $\chi^2(q)$ are substituted from (8.3.8), (8.3.9), and (8.3.3), respectively, it is easy to see that (8.4.43) is trivially satisfied. The proof of (8.4.42) follows along the same lines. □

Second Proof of Lemma 8.4.4. Define

$$B(q):=G(q)+qH(q^4) \qquad \text{and} \qquad qA(q):=-H(q)+G(q^4). \tag{8.4.44}$$

Let us also define

$$s(q):=\frac{\chi(-q^3)}{\chi(-q)}. \tag{8.4.45}$$

From the definition (8.4.44) and (8.3.3), we see that

$$-q^2A(q)H(q^4)+B(q)G(q^4)=G(q)G(q^4)+qH(q)H(q^4)=\chi^2(q). \tag{8.4.46}$$

Similarly, by (8.4.44), (8.3.9), (8.3.8), and (8.4.45), we find that

$$
\begin{aligned}
&qA(q)G(q^6) + qB(q)H(q^6) \\
&= -H(q)G(q^6) + qG(q)H(q^6) + \left\{G(q^4)G(q^6) + q^2H(q^4)H(q^6)\right\} \\
&= -\frac{1}{s(q)} + s(q^2). \qquad (8.4.47)
\end{aligned}
$$

Using (8.3.8) and (8.4.45), we solve for $B(q)$ and $qA(q)$ in (8.4.46) and (8.4.47) and find that

$$
\begin{aligned}
B(q) &= \frac{\chi^2(q)}{s(q^2)}G(q^6) - q\frac{1}{s(q)s(q^2)}H(q^4) + qH(q^4), \\
qA(q) &= -\frac{1}{s(q)s(q^2)}G(q^4) - q\frac{\chi^2(q)}{s(q^2)}H(q^6) + G(q^4),
\end{aligned}
$$

which, by (8.4.44), immediately yield (8.4.41) and (8.4.42). □

Our fifth approach uses a formula of Blecksmith, Brillhart, and Gerst [77] to provide a representation for a product of two theta functions as a sum of m products of pairs of theta functions, under certain conditions. This formula generalizes formulas of H. Schröter [55, pp. 65–72], which have been enormously useful in establishing many of Ramanujan's modular equations [55].

Define, for $\epsilon \in \{0,1\}$ and $|ab| < 1$,

$$
f_\epsilon(a,b) = \sum_{n=-\infty}^{\infty} (-1)^{\epsilon n}(ab)^{n^2/2}(a/b)^{n/2}, \qquad (8.4.48)
$$

or equivalently,

$$
f_k(a,b) = \begin{cases} f(a,b), & \text{if } k \equiv 0 \pmod 2, \\ f(-a,-b), & \text{if } k \equiv 1 \pmod 2. \end{cases} \qquad (8.4.49)
$$

Theorem 8.4.1. *Let a, b, c, and d denote positive numbers with $|ab|, |cd| < 1$. Suppose that there exist positive integers α, β, and m such that*

$$
(ab)^\beta = (cd)^{\alpha(m-\alpha\beta)}. \qquad (8.4.50)
$$

Let $\epsilon_1, \epsilon_2 \in \{0,1\}$, and define $\delta_1, \delta_2 \in \{0,1\}$ by

$$
\delta_1 \equiv \epsilon_1 - \alpha\epsilon_2 \pmod 2 \qquad \textit{and} \qquad \delta_2 \equiv \beta\epsilon_1 + p\epsilon_2 \pmod 2, \qquad (8.4.51)
$$

respectively, where $p = m - \alpha\beta$. Then, if R denotes any complete residue system modulo m,

$$
\begin{aligned}
f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) &= \sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2} \qquad (8.4.52) \\
&\times f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right) \\
&\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p}, \frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right).
\end{aligned}
$$

Proof. Setting $s = k - \alpha n$, we find that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{n,s=-\infty}^{\infty} (-1)^{\epsilon_1 n+\epsilon_2 s}(ab)^{n^2/2}(a/b)^{n/2}(cd)^{s^2/2}(c/d)^{s/2}$$
$$= \sum_{n,k=-\infty}^{\infty} (-1)^{\epsilon_1 n+\epsilon_2 (k-\alpha n)}(ab)^{n^2/2}(a/b)^{n/2}(cd)^{(k-\alpha n)^2/2}(c/d)^{(k-\alpha n)/2}.$$

Expand into residue classes modulo m and set $k = tm + r$, $-\infty < t < \infty$, $r \in R$, to deduce that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}\sum_{n,t=-\infty}^{\infty} (-1)^{\epsilon_1 n+\epsilon_2 (tm+r-\alpha n)}$$
$$\times (ab)^{n^2/2}(a/b)^{n/2}(cd)^{(tm+r-\alpha n)^2/2}(c/d)^{(tm+r-\alpha n)/2}.$$

Next, setting $n = \ell + \beta t$, $-\infty < \ell < \infty$, we find that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}\sum_{\ell,t=-\infty}^{\infty} (-1)^{\epsilon_1(\ell+\beta t)+\epsilon_2(tm+r-\alpha(\ell+\beta t))}$$
$$\times (ab)^{(\ell+\beta t)^2/2}(a/b)^{(\ell+\beta t)/2}(cd)^{(tm+r-\alpha(\ell+\beta t))^2/2}(c/d)^{(tm+r-\alpha(\ell+\beta t))/2}.$$

Recalling that $p = m - \alpha\beta$ and noting that $tm + r - \alpha(\ell + \beta t) = tp + r - \alpha\ell$, we find that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}\sum_{\ell,t=-\infty}^{\infty} (-1)^{\epsilon_1(\ell+\beta t)}(-1)^{\epsilon_2(tp+r-\alpha\ell)}$$
$$\times (ab)^{(\ell+\beta t)^2/2}(a/b)^{(\ell+\beta t)/2}(cd)^{(tp+r-\alpha\ell)^2/2}(c/d)^{(tp+r-\alpha\ell)/2}.$$

Now, by (8.4.50) and the definition $p = m - \alpha\beta$, we find that

$$\begin{aligned}(ab)^{\beta(\ell t+\beta t^2/2)}(cd)^{t^2p^2/2-tp\alpha\ell} &= (cd)^{\alpha p(\ell t+\beta t^2/2)}(cd)^{t^2p^2/2-tp\alpha\ell}\\ &= (cd)^{t^2p(\alpha\beta+p)/2}\\ &= (cd)^{t^2pm/2}.\end{aligned}$$

Hence, recalling the definitions of δ_1 and δ_2 from (8.4.51) and the definition of $f_\epsilon(a,b)$ from (8.4.48), we find that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}\sum_{\ell,t=-\infty}^{\infty} (-1)^{\delta_1\ell}(-1)^{\delta_2 t}(-1)^{\epsilon_2 r}(cd)^{r^2/2}(c/d)^{r/2}$$
$$\times \left(ab(cd)^{\alpha^2}\right)^{\ell^2/2}\left(\frac{a}{b}\left(\frac{c}{d}\right)^{-\alpha}(cd)^{-2r\alpha}\right)^{\ell/2}$$

$$\times \left((cd)^{mp}\right)^{t^2/2}\left(\left(\frac{a}{b}\right)^{\beta}\left(\frac{c}{d}\right)^{p}(cd)^{2pr}\right)^{t/2}$$
$$=\sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2}f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^{\alpha}},\frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^{\alpha}}\right)$$
$$\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^{p}},\frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^{p}}\right),$$

after some elementary algebra and elementary manipulation. □

Lastly, we offer the results of Yesilyurt [347] that are needed to prove five of the identities. As in the work of Rogers [304], the principal idea is to construct two equal sums of theta functions arising from two different sets of parameters. Choosing the parameters adroitly then leads to a proof of the identity in question.

Let m, δ, and ϵ be integers, and let α, β, p, and λ be positive integers such that

$$\alpha m^2+\beta=p\lambda. \tag{8.4.53}$$

Let l and t be real, and let x and y be nonzero complex numbers. Recall that the general theta function f_k is defined by (8.4.49). With these parameters, we set

$$R(\epsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)$$
$$:=\sum_{\substack{k=0\\ n=2k+t}}^{p-1}(-1)^{\epsilon k}y^k q^{\{\lambda n^2+p\alpha l^2+2\alpha nml\}/4}f_{\delta}(xq^{(1+l)p\alpha+\alpha nm},x^{-1}q^{(1-l)p\alpha-\alpha nm})$$
$$\times f_{\epsilon p+m\delta}(x^{-m}y^{p}q^{p\beta+\beta n},x^{m}y^{-p}q^{p\beta-\beta n}). \tag{8.4.54}$$

We first state two theorems providing theta function expansions (8.4.54) with the parameters under certain conditions. We follow these two theorems with two further theorems giving conditions for two sums of theta functions being equal [347, Lemmas 4.1, 4.2; Theorems 4.3, 4.4]. We emphasize that the parameters have different meanings in different theorems.

Theorem 8.4.2. *Let l, t, and z be integers with $z=\pm1$. Define*

$$\delta_1:=\epsilon p+m\delta$$

and assume that

$$\epsilon(p+t)+\delta(l+m)\equiv 1\pmod 2.$$

Then,

$$R_1(z,\epsilon,\delta,l,t,\alpha,\beta,m,p):=R(\epsilon,\delta,l-\tfrac13 zm,t+\tfrac13 zp,\alpha,\beta,m,p,\lambda,1,1)$$
$$=(-1)^{(z+1)(1+\delta_1)/2}q^{\frac14\{p\alpha l^2+p\beta/9\}}f(-q^{2p\beta/3})\{S_1+(-1)^{\epsilon t/2}S_2\}, \tag{8.4.55}$$

where

$$S_1 = \sum_{\substack{n=1 \\ n \equiv t \pmod 2}}^{p-1} (-1)^{\epsilon(n-t)/2} q^{\frac{1}{4}\left\{\lambda n^2 + 2\alpha mnl - 2n\beta/3\right\}} \frac{f(-q^{2\beta n/3}, -q^{2p\beta/3 - 2\beta n/3})}{f_{\delta_1}(q^{\beta n/3}, q^{2p\beta/3 - \beta n/3})} \times f_\delta(q^{(1+l)p\alpha + \alpha mn}, q^{(1-l)p\alpha - \alpha mn}) \tag{8.4.56}$$

and

$$S_2 = \begin{cases} f_\delta(q^{(1+l)p\alpha}, q^{(1-l)p\alpha}), & \text{if } t \equiv \delta_1 + 1 \equiv 0 \pmod 2, \\ 0, & \text{otherwise.} \end{cases} \tag{8.4.57}$$

Theorem 8.4.3. *Let l and t be integers. Define*

$$\delta_1 := \epsilon p + m\delta$$

and assume that

$$\epsilon t + \delta(l+1) \equiv 1 \pmod 2. \tag{8.4.58}$$

Define

$$R_2(\epsilon, \delta, l, t, \alpha, \beta, m, p) := R(\epsilon, \delta, l - \tfrac{1}{3}, t, \alpha, \beta, m, p, \lambda, 1, 1).$$

If $\gcd(m,p) = 1$, *then,*

$$R_2(\epsilon, \delta, l, t, \alpha, \beta, m, p) = q^{p\alpha/36} f(-q^{2p\alpha/3})\{S_3 + S_4\}, \tag{8.4.59}$$

where

$$S_3 = \sum_{\substack{n=1 \\ n \equiv t \pmod 2}}^{p-1} (-1)^{\epsilon(n-t)/2} q^{\frac{1}{4}\left\{\lambda n^2 + 2\alpha mn(l-1/3) + p\alpha l(l-2/3)\right\}} \times \frac{f(-q^{2\alpha(nm+lp)/3}, -q^{2p\alpha/3 - 2\alpha(nm+lp)/3})}{f_\delta(q^{\alpha(nm+lp)/3}, q^{2p\alpha/3 - \alpha(nm+lp)/3})} f_{\delta_1}(q^{p\beta+\beta n}, q^{p\beta - \beta n}) \tag{8.4.60}$$

and

$$S_4 = \begin{cases} (-1)^{(l+t\epsilon)/2} \tau_{\delta_1}(q^{p\beta}), & \text{if } t \equiv 0 \pmod 2, \\ 2(-1)^{(m+l+\epsilon(p-t))/2} q^{p\beta/4} \psi(q^{2p\beta}), & \text{if } p \equiv t \equiv \delta \equiv 1 + m + l \equiv 1 \pmod 2, \\ 0, & \text{otherwise.} \end{cases} \tag{8.4.61}$$

Theorem 8.4.4. *Let α, β, m, p, and λ be as before with*

$$\alpha m^2 + \beta = p\lambda,$$

and let ϵ, δ, l, and t be integers with

$$(1+l)\delta + t\epsilon \equiv 1 \pmod 2.$$

Assume further that $3 \mid \alpha m$ *and* $\gcd(3,\lambda)=1$. *Recall that* R_1 *and* R_2 *are defined by* (8.4.55) *and* (8.4.59), *respectively. Let* α_1, β_1, m_1, *and* p_1 *be another set of parameters such that*

$$\alpha_1 m_1^2 + \beta_1 = p_1\lambda, \qquad \alpha\beta = \alpha_1\beta_1,$$

and set $a := (\alpha m - \alpha_1 m_1)/\lambda$. *Then*

$$R_2(\epsilon,\delta,l,t,\alpha,\beta,m,p) = R_1(z,\delta,\epsilon,l_1,t_1,1,\alpha\beta,\alpha m,\lambda), \tag{8.4.62}$$

where $l_1 := t + \alpha m z/3$, $t_1 := l - 1/3 - z\lambda/3$, *and* $z = \pm 1$ *with* $z \equiv -\lambda \pmod 3$. *Moreover, if* $3 \mid \alpha_1 m_1$, *then*

$$R_2(\epsilon,\delta,l,t,\alpha,\beta,m,p) = R_2(\epsilon,\delta + a\epsilon,l,t_2,\alpha_1,\beta_1,m_1,p_1), \tag{8.4.63}$$

where $t_2 = t + a(l - 1/3)$. *If* $3 \mid \beta_1$ *and* $\gcd(3,\alpha_1 m_1) = 1$, *then*

$$R_2(\epsilon,\delta,l,t,\alpha,\beta,m,p) = R_1(y,\epsilon,\delta + a\epsilon,l_3,t_3,\alpha_1,\beta_1,m_1,p_1), \tag{8.4.64}$$

where $y = \pm 1$ *with* $y \equiv m_1 \pmod 3$, $l_3 = l - 1/3 + y m_1/3$, *and* $t_3 = t + a(l - 1/3) - y p_1/3$.

Theorem 8.4.5. *Let* α, β, m, p, *and* λ *be as before with*

$$\alpha m^2 + \beta = p\lambda,$$

and let ϵ, δ, l, *and* t *be integers with*

$$\epsilon(p+t) + \delta(l+m) \equiv 1 \pmod 2.$$

Assume that $y = \pm 1$ *with* $y \equiv m \pmod 3$. *Assume further that* $3 \mid \beta$ *and* $\gcd(3, m\lambda) = 1$. *Recall that* R_1 *and* R_2 *are defined by* (8.4.59) *and* (8.4.55), *respectively. Let* α_1, β_1, m_1, *and* p_1 *be another set of parameters as in the previous theorem. Set* $a := (\alpha m - \alpha_1 m_1)/\lambda$. *Then*

$$R_1(z,\epsilon,\delta,l,t,\alpha,\beta,m,p) = R_1(y,\delta,\epsilon,l_1,t_1 1,\alpha\beta,\alpha m,\lambda), \tag{8.4.65}$$

where $l_1 = t + (zp + \alpha m y)/3$, $t_1 = l - (zm + y\lambda)/3$, *and* $z = \pm 1$ *with* $z \equiv -\lambda \pmod 3$. *Moreover, if* $3 \mid \beta_1$ *and* $\gcd(3,\alpha_1 m_1) = 1$, *then*

$$R_1(y,\epsilon,\delta,l,t,\alpha,\beta,m,p) = R_1(y_1,\epsilon,\delta + a\epsilon,l_2,t_2,\alpha_1,\beta_1,m_1,p_1),$$

where $l_2 = l - (ym - y_1 m_1)/3$, $t_2 = t + al + (yp - y_1 p_1 - aym)/3$, *and* $y_1 = \pm 1$ *with* $y_1 \equiv m_1 \pmod 3$. *If* $3 \mid \alpha_1 m_1$, *then*

$$R_1(y,\epsilon,\delta,l,t,\alpha,\beta,m,p) = R_2(\epsilon,\delta + a\epsilon,l_3,t_3,\alpha_1,\beta_1,m_1,p_1),$$

where $l_3 = l + (1 - ym)/3$ *and* $t_3 = t + al + y(p - am)/3$.

8.5 Proofs of the 40 Entries

8.5.1 Proof of Entry 8.3.1

We begin by proving the following identity from Chapter 19 of Ramanujan's second notebook [282], [55, p. 80, Entry 38(iv); p. 262, Entry 10(iii)]. Our proof is taken from [55, pp. 81–82].

Lemma 8.5.1. *We have*

$$f^2(-q^2,-q^3) - q^{2/5} f^2(-q,-q^4) = f(-q)\left\{ f(-q^{1/5}) + q^{1/5} f(-q^5) \right\}. \tag{8.5.1.1}$$

Proof. Apply (8.2.19) with $a = -q$, $b = -q^2$, and $n = 5$. Then

$$U_n = (-1)^n q^{n(3n-1)/2} \qquad \text{and} \qquad V_n = (-1)^n q^{n(3n+1)/2}.$$

Thus, by (8.2.9) and (8.2.19),

$$\begin{aligned} f(-q) = f(-q,-q^2) &= f(-q^{35},-q^{40}) - q f(-q^{50},-q^{25}) + q^5 f(-q^{65},-q^{10}) \\ &\quad - q^{12} f(-q^{80},-q^{-5}) + q^{22} f(-q^{95},-q^{-20}) \\ &= -q f(-q^{25}) + \left\{ f(-q^{35},-q^{40}) + q^5 f(-q^{10},-q^{65}) \right\} \\ &\quad - q^2 \left\{ f(-q^{20},-q^{55}) + q^{10} f(-q^{-5},-q^{80}) \right\}, \end{aligned} \tag{8.5.1.2}$$

where we applied (8.2.5). We now invoke the quintuple product identity (8.2.18) twice, with q replaced by q^{25} and $a = q^5, q^{10}$, respectively. We therefore find that (8.5.1.2) can be written as

$$f(-q) + q f(-q^{25}) = f(-q^{25}) \left\{ \frac{f(-q^{15},-q^{10})}{f(-q^{20},-q^5)} - q^2 \frac{f(-q^5,-q^{20})}{f(-q^{15},-q^{10})} \right\}. \tag{8.5.1.3}$$

By (8.2.6),

$$f(-q,-q^4) f(-q^2,-q^3) = f(-q) f(-q^5). \tag{8.5.1.4}$$

Multiplying both sides of (8.5.1.3) by $f(-q)$, but with q replaced by $q^{1/5}$, and using (8.5.1.4), we deduce that

$$\begin{aligned} & f(-q)\left\{ f(-q^{1/5}) + q^{1/5} f(-q^5) \right\} \\ &\quad = f(-q) f(-q^5) \left\{ \frac{f(-q^2,-q^3)}{f(-q,-q^4)} - q^{2/5} \frac{f(-q,-q^4)}{f(-q^2,-q^3)} \right\} \\ &\quad = f^2(-q^2,-q^3) - q^{2/5} f^2(-q,-q^4), \end{aligned}$$

which completes the proof. $\square$

Proof of Entry 8.3.1. Replace $q^{1/5}$ by $q^{1/5}\zeta$ in (8.5.1.1), where ζ is a fifth root of unity, and multiply all five identities together. We then find that

$$\begin{aligned} &f^5(-q)\prod_{\zeta} f(-q^{1/5}\zeta) \\ &=\prod_{\zeta}\left\{f^2(-q^2,-q^3)-q^{2/5}\zeta^2 f^2(-q,-q^4)-q^{1/5}\zeta f(-q)f(-q^5)\right\}. \end{aligned} \tag{8.5.1.5}$$

Multiplying out the products on each side of (8.5.1.5), we find that

$$\begin{aligned} \frac{f^{11}(-q)}{f(-q^5)} &= f^{10}(-q^2,-q^3)-q^2 f^{10}(-q,-q^4)-qf^5(-q)f^5(-q^5) \\ &\quad -5qf^2(-q^2,-q^3)f^2(-q,-q^4)f^3(-q)f^3(-q^5) \\ &\quad -5qf^4(-q^2,-q^3)f^4(-q,-q^4)f(-q)f(-q^5). \end{aligned} \tag{8.5.1.6}$$

By (8.5.1.4), (8.5.1.6) simplifies to

$$\begin{aligned} \frac{f^{11}(-q)}{f(-q^5)} &= f^{10}(-q^2,-q^3)-q^2 f^{10}(-q,-q^4)-qf^5(-q)f^5(-q^5) \\ &\quad -5qf^5(-q)f^5(-q^5)-5qf^5(-q)f^5(-q^5) \\ &= f^{10}(-q^2,-q^3)-q^2 f^{10}(-q,-q^4)-11qf^5(-q)f^5(-q^5). \end{aligned} \tag{8.5.1.7}$$

Multiplying both sides of (8.5.1.7) by $f(-q^5)/f^{11}(-q)$, using (8.5.1.4), and lastly employing (8.2.11), we conclude that

$$\begin{aligned} 1 &= \frac{f^{11}(-q^2,-q^3)f(-q,-q^4)}{f^{12}(-q)} - q^2\frac{f^{11}(-q,-q^4)f(-q^2,-q^3)}{f^{12}(-q)} \\ &\quad -11q\frac{f^6(-q,-q^4)f^6(-q^2,-q^3)}{f^{12}(-q)} \\ &= G^{11}(q)H(q)-q^2H^{11}(q)G(q)-11qG^6(q)H^6(q), \end{aligned}$$

which completes the proof of Entry 8.3.1. □

8.5.2 Proofs of Entry 8.3.2

Proof. Using (8.4.23) and (8.4.24) in (8.3.6), we find that

$$\begin{aligned} \chi(q^2) &= G(q^{16})H(q)-q^3H(q^{16})G(q) \\ &= \frac{f(-q^8)}{f(-q^2)}\left\{G(q^{16})\left(q^3H(q^{16})+G(-q^4)\right)\right. \\ &\quad \left.-q^3H(q^{16})\left(G(q^{16})+qH(-q^4)\right)\right\} \\ &= \frac{f(-q^8)}{f(-q^2)}\left\{G(-q^4)G(q^{16})-q^4H(-q^4)H(q^{16})\right\}. \end{aligned}$$

Therefore, by (8.2.14) and (8.2.15), we deduce that

$$\begin{aligned}G(-q^4)G(q^{16})-q^4H(-q^4)H(q^{16})&=\frac{f(-q^2)}{f(-q^8)}\chi(q^2)\\&=\frac{\chi(-q^2)f(-q^4)\chi(q^2)}{f(-q^4)/\chi(-q^4)}=\chi^2(-q^4),\end{aligned}$$

which is Entry 8.3.2 with q replaced by $-q^4$. □

In his lost notebook [283, p. 27], Ramanujan offers the beautiful identity [16, p. 153, Entry 7.2.2]

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n^2}}{(q^4;q^4)_n}\sum_{n=0}^{\infty}\frac{a^{-2n}q^{4n^2}}{(bq^4;q^4)_n}+\sum_{n=0}^{\infty}\frac{a^nb^nq^{(n+1)^2}}{(q^4;q^4)_n}\sum_{n=0}^{\infty}\frac{a^{-2n-1}q^{4n^2+4n}}{(bq^4;q^4)_n}\\=\frac{f(aq,q/a)}{(bq^4;q^4)_\infty}-(1-b)\sum_{n=0}^{\infty}a^{n+1}q^{(n+1)^2}\sum_{j=0}^{n}\frac{b^j}{(q^4;q^4)_j}.\end{aligned}\tag{8.5.2.1}$$

If we set $a=b=1$ in (8.5.2.1) and multiply both sides by $(-q^2;q^2)_\infty$, we see that (8.5.2.1) reduces to

$$(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}G(q^4)+(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}}{(q^4;q^4)_n}H(q^4)=\frac{\varphi(q)}{f(-q^2)}.\tag{8.5.2.2}$$

However, Rogers [303] proved that

$$(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}=G(q)$$

and

$$(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n}=H(q),$$

and so (8.5.2.2) reduces to (8.3.3).

8.5.3 Proof of Entry 8.3.3

Entry 8.3.3 follows from combining Entry 8.3.2 with the following lemma.

Lemma 8.5.2. *We have*

$$\varphi^2(q)-\varphi^2(q^5)=4qf^2(-q^{10})\frac{\chi(q)}{\chi(q^5)}.\tag{8.5.3.1}$$

Proof. By Entry 10(iv) in Chapter 19 of Ramanujan's second notebook [282], [55, p. 262] and the Jacobi triple product identity (8.2.6),

$$\begin{aligned}\varphi^2(q)-\varphi^2(q^5) &= 4qf(q,q^9)f(q^3,q^7)\\ &= 4q(-q;q^{10})_\infty(-q^9;q^{10})_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^{10};q^{10})_\infty^2\\ &= 4qf^2(-q^{10})\frac{(-q;q^2)_\infty}{(-q^5;q^{10})_\infty}\\ &= 4qf^2(-q^{10})\frac{\chi(q)}{\chi(q^5)},\end{aligned}$$

and the proof is complete. □

The identity (8.5.3.1) is an analogue of

$$\varphi^2(q)-5\varphi^2(q^5)=-4f^2(-q^2)\frac{\chi(q^5)}{\chi(q)},$$

which is found in Ramanujan's lost notebook [283] and was first proved by S.-Y. Kang [190, Theorem 2.2(i)].

Proof of Entry 8.3.3*.* The proof that follows is due to Watson [333]. From Entry 8.3.2, (8.2.11), and (8.5.1.4), we find that

$$\begin{aligned}\left\{G(q)G(q^4)-qH(q)H(q^4)\right\}^2 &= \left\{G(q)G(q^4)+qH(q)H(q^4)\right\}^2\\ &\quad -4qG(q)H(q)G(q^4)H(q^4)\\ &= \frac{\varphi^2(q)}{f^2(-q^2)}-4q\frac{f(-q^5)f(-q^{20})}{f(-q)f(-q^4)}.\end{aligned}\tag{8.5.3.2}$$

A straightforward calculation shows that

$$\chi(q)=\frac{f^2(-q^2)}{f(-q)f(-q^4)}.\tag{8.5.3.3}$$

Using (8.5.3.3) twice, we find that (8.5.3.2) can be written in the form

$$\left\{G(q)G(q^4)-qH(q)H(q^4)\right\}^2=\frac{\varphi^2(q)-4qf^2(-q^{10})\dfrac{\chi(q)}{\chi(q^5)}}{f^2(-q^2)}=\frac{\varphi^2(q^5)}{f^2(-q^2)},$$

where we applied Lemma 8.5.2. Taking the square root of both sides above, we complete the proof. □

8.5.4 Proof of Entry 8.3.4

By employing (8.2.11), we easily find that the proposed identity is equivalent to the identity

$$f(-q,-q^4)f(-q^{22},-q^{33})-q^2f(-q^2,-q^3)f(-q^{11},-q^{44})=f(-q)f(-q^{11}). \tag{8.5.4.1}$$

To prove (8.5.4.1), we apply the ideas of Rogers, in particular, (8.4.13) with the two sets of parameters $\alpha_1=1$, $\beta_1=11$, $m_1=3$, $p_1=5$, $\lambda_1=4$ and $\alpha_2=1$, $\beta_2=11$, $m_2=1$, $p_2=3$, $\lambda_2=4$. The requisite conditions (8.4.9) are readily seen to be satisfied. Using (8.4.16) and (8.4.17), we derive the identity

$$f(-q^2,-q^8)f(-q^{44},-q^{66})-q^4f(-q^4,-q^6)f(-q^{22},-q^{88})=f(-q^2)f(-q^{22}),$$

which is the same as (8.5.4.1), but with q replaced by q^2.

8.5.5 Proof of Entry 8.3.5

The proof of Entry 8.3.5 is very similar to that for Entry 8.3.22 below. In fact, we reduce the desired equality to the same new modular equation (8.5.21.11) of degree 5. Remarkably, Ramanujan derived 27 modular equations of degree 5, although several are "reciprocals" of others [55, pp. 280–282, Entry 13]. In Ramanujan's terminology, let β have degree 5 over α.

Lemma 8.5.3. *If β has degree 5 over α, then*

$$(1-\beta)^{1/4}-(1-\alpha)^{1/4}=2^{2/3}(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/24}. \tag{8.5.5.1}$$

Proof. Let

$$m=\frac{\varphi^2(q)}{\varphi^2(q^5)}$$

denote the *multiplier* of degree 5. As in [55, p. 284, Equation (13.3)], define

$$\rho:=\sqrt{m^3-2m^2+5m}. \tag{8.5.5.2}$$

We need the following parameterizations for certain algebraic functions of α and β, namely [55, pp. 285–286, Equations (13.8), (13.10), (13.11)],

$$\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}=\frac{(m-1)(5-m)}{4m}, \tag{8.5.5.3}$$

$$\left\{(1-\alpha)^3(1-\beta)\right\}^{1/8}=\frac{\rho-3m+5}{4m}, \tag{8.5.5.4}$$

and

$$\left\{(1-\alpha)(1-\beta)^3\right\}^{1/8}=\frac{\rho-m^2+3m}{4m}, \tag{8.5.5.5}$$

where ρ is defined by (8.5.5.2). Using (8.5.5.3)–(8.5.5.5), we find that

$$\frac{(1-\beta)^{1/4}-(1-\alpha)^{1/4}}{2^{2/3}(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/24}}$$

$$
\begin{aligned}
&= \frac{\left\{(1-\alpha)(1-\beta)^3\right\}^{1/8} - \left\{(1-\alpha)^3(1-\beta)\right\}^{1/8}}{2^{2/3}\left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/6}} \\
&= \frac{(\rho - m^2 + 3m) - (\rho - 3m + 5)}{(m-1)(5-m)} \\
&= \frac{m^2 - 6m + 5}{(m-1)(m-5)} = 1,
\end{aligned}
$$

which completes the proof. $\square$

We begin the proof of Entry 8.3.5 with the system of equations

$$
\begin{aligned}
G(q^{16})H(q) - q^3G(q)H(q^{16}) &=: T(q), \\
G(q^{16})G(q^4) + q^4H(q^4)H(q^{16}) &= \frac{\varphi(q^4)}{f(-q^8)}, \\
G(q^{16})G(q^4) - q^4H(q^4)H(q^{16}) &= \frac{\varphi(q^{20})}{f(-q^8)}.
\end{aligned}
$$

From the first equation above, we see that our task is to prove that $T(q) = \chi(q^2)$. The second and third equations are simply (8.3.3) and (8.3.4), respectively, but with q replaced by q^4. Regarding this system in the variables $G(q^{16})$, $q^3H(q^{16})$, and -1, we see that

$$
\begin{vmatrix}
H(q) & -G(q) & T(q) \\
G(q^4) & qH(q^4) & \dfrac{\varphi(q^4)}{f(-q^8)} \\
G(q^4) & -qH(q^4) & \dfrac{\varphi(q^{20})}{f(-q^8)}
\end{vmatrix} = 0. \tag{8.5.5.6}
$$

Expanding the determinant in (8.5.5.6) along the last column, we find that

$$
\begin{aligned}
-2qG(q^4)H(q^4)T(q) - \frac{\varphi(q^4)}{f(-q^8)}\left\{G(q)G(q^4) - qH(q)H(q^4)\right\}& \\
+ \frac{\varphi(q^{20})}{f(-q^8)}\left\{G(q)G(q^4) + qH(q)H(q^4)\right\} = 0.& \qquad (8.5.5.7)
\end{aligned}
$$

Using (8.2.12), (8.3.3), (8.3.4), and

$$
\frac{f(-q^4)}{f(-q^8)} = \chi(-q^4) = \chi(-q^2)\chi(q^2), \tag{8.5.5.8}
$$

which arises from (8.2.15), we can write (8.5.5.7) in the form

$$
-2q\frac{f(-q^{20})}{f(-q^4)}T(q) - \frac{\varphi(q^4)}{f(-q^8)}\frac{\varphi(q^5)}{f(-q^2)} + \frac{\varphi(q^{20})}{f(-q^8)}\frac{\varphi(q)}{f(-q^2)} = 0. \tag{8.5.5.9}
$$

Rearranging (8.5.5.9) while using (8.5.5.8), we find that

$$2qT(q) = \frac{\chi(-q^2)\chi(q^2)}{f(-q^2)f(-q^{20})}\left\{\varphi(q)\varphi(q^{20}) - \varphi(q^5)\varphi(q^4)\right\}. \tag{8.5.5.10}$$

Recall the representations [55, pp. 122–124, Entries 10(i), (v), (ii), 11(v), 12(iii), (iv), (vii)]

$$\varphi(q) = \sqrt{z_1}, \qquad \varphi(q^4) = \tfrac{1}{2}\sqrt{z_1}\left\{1 + (1-\alpha)^{1/4}\right\}, \tag{8.5.5.11}$$

$$\varphi(-q) = \sqrt{z_1}(1-\alpha)^{1/4}, \qquad \psi(q^8) = \frac{1}{4q}\sqrt{z_1}\left\{1 - (1-\alpha)^{1/4}\right\}, \tag{8.5.5.12}$$

$$f(-q^2) = \sqrt{z_1}2^{-1/3}\left(\frac{\alpha(1-\alpha)}{q}\right)^{1/12}, \; f(-q^4) = \sqrt{z_1}2^{-2/3}\left(\frac{(1-\alpha)\alpha^4}{q^4}\right)^{1/24}, \tag{8.5.5.13}$$

and

$$\chi(-q^2) = 2^{1/3}\left(\frac{(1-\alpha)q^2}{\alpha^2}\right)^{1/24}, \tag{8.5.5.14}$$

where

$$z_n := \varphi(q^n).$$

Recall from the theory of modular equations that if n is the degree of the modular equation, then (8.5.5.13) also holds with z_1, q, and α replaced by z_n, q^n, and β, respectively, where β has degree n over α. Hence, from (8.5.5.13) and (8.5.5.14), we find that after simplification,

$$\frac{\chi(-q^2)}{f(-q^2)f(-q^{20})} = \frac{2^{4/3}q}{\sqrt{z_1z_5}(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}}. \tag{8.5.5.15}$$

Employing (8.5.5.15) and (8.5.5.11) in (8.5.5.10), we deduce that

$$\begin{aligned}
2qT(q) &= \frac{\chi(q^2)2^{4/3}q}{\sqrt{z_1z_5}(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}} \\
&\quad \times \sqrt{z_1z_5}\left\{\tfrac{1}{2}\left\{1+(1-\beta)^{1/4}\right\} - \tfrac{1}{2}\left\{1+(1-\alpha)^{1/4}\right\}\right\} \\
&= \frac{\chi(q^2)2^{1/3}q\left\{(1-\beta)^{1/4} - (1-\alpha)^{1/4}\right\}}{(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}} \\
&= 2q\chi(q^2),
\end{aligned} \tag{8.5.5.16}$$

by Lemma 8.5.3. Equation (8.5.5.16) is trivially equivalent to (8.3.6), and so the proof is complete.

8.5.6 Proofs of Entry 8.3.6

First Proof of Entry 8.3.6. Using (8.2.11), we find that in order to prove Entry 8.3.6, it suffices to prove that

$$f(-q^2,-q^3)f(-q^{18},-q^{27})+q^2f(-q,-q^4)f(-q^9,-q^{36})=f^2(-q^3). \quad (8.5.6.1)$$

We apply (8.4.13) with $\alpha_1=1,\beta_1=9,m_1=1,p_1=5,\lambda_1=2$ and with $\alpha_2=3,\beta_2=3,m_2=1,p_2=3,\lambda_2=2$. We easily check that these two sets of parameters satisfy the conditions (8.4.9). From (8.4.13) and (8.4.15), we then deduce the identity

$$f(-q^4,-q^6)f(-q^{36},-q^{54})+q^4f(-q^2,-q^8)f(-q^{18},-q^{72})=f^2(-q^6),$$

which is precisely (8.5.6.1), but with q replaced by q^2. This then completes the proof of Entry 8.3.6. □

Second Proof of Entry 8.3.6. We rewrite (8.5.6.1) in the form

$$\begin{aligned}&\sum_{\substack{m=-\infty\\ m\equiv 0\,(\mathrm{mod}\,3)}}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{m+n}q^{(5m^2-3m+5n^2-n)/2}\\ &\quad+q^2\sum_{\substack{m=-\infty\\ m\equiv 0\,(\mathrm{mod}\,3)}}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{m+n}q^{(5m^2-9m+5n^2-3n)/2}\\ &=\sum_{\substack{m,n=-\infty\\ m\equiv 0,n\equiv 0\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{m+n}q^{(m^2-m+n^2-n)/2}=:F(q).\end{aligned} \quad (8.5.6.2)$$

Now, for $(a,b)\in\{0,\pm1,\pm2\}$, set

$$2m+n=5M+a \qquad \text{and} \qquad -m+2n=5N+b.$$

Hence,

$$m=2M-N+(2a-b)/5 \qquad \text{and} \qquad n=M+2N+(a+2b)/5,$$

where the parameters a and b are given in the first two lines of the table below. The corresponding values of m and n are given in the table's last two lines.

a	0	1	-1	2	-2
b	0	2	-2	-1	1
m	$2M-N$	$2M-N$	$2M-N$	$2M-N+1$	$2M-N-1$
n	$M+2N$	$M+2N+1$	$M+2N-1$	$M+2N$	$M+2N$

Recalling that $m,n\equiv 0\,(\mathrm{mod}\,3)$, for the five cases in the table above, we find that, respectively,

$$M \equiv N \equiv 0 \pmod 3, \quad M \equiv 1, N \equiv -1 \pmod 3, \quad M \equiv -1, N \equiv 1 \pmod 3,$$
$$M \equiv N \equiv -1 \pmod 3, \qquad M \equiv N \equiv 1 \pmod 3.$$

Calculating the corresponding values of m^2+n^2-m-n, we find that

$$\begin{aligned}
F(q) &= \sum_{\substack{M,N=-\infty \\ M\equiv 0, N\equiv 0 \pmod 3}}^{\infty} (-1)^{M+N} q^{(5M^2-3M+5N^2-N)/2} \\
&\quad + \sum_{\substack{M,N=-\infty \\ M\equiv 1, N\equiv -1 \pmod 3}}^{\infty} (-1)^{M+N+1} q^{(5M^2-M+5N^2+3N)/2} \\
&\quad + \sum_{\substack{M,N=-\infty \\ M\equiv -1, N\equiv 1 \pmod 3}}^{\infty} (-1)^{M+N+1} q^{(5M^2-5M+5N^2-5N+2)/2} \\
&\quad + \sum_{\substack{M,N=-\infty \\ M\equiv -1, N\equiv -1 \pmod 3}}^{\infty} (-1)^{M+N+1} q^{(5M^2+M+5N^2-3N)/2} \\
&\quad + \sum_{\substack{M,N=-\infty \\ M\equiv 1, N\equiv 1 \pmod 3}}^{\infty} (-1)^{M+N-1} q^{(5M^2-7M+5N^2+N+2)/2} \\
&=: S_1+S_2+S_3+S_4+S_5.
\end{aligned} \tag{8.5.6.3}$$

First, setting $M=3m-1$, we find that

$$\begin{aligned}
\sum_{\substack{M=-\infty \\ M\equiv -1 \pmod 3}}^{\infty} (-1)^M q^{5(M^2-M)/2} &= -\sum_{m=-\infty}^{\infty} (-1)^m q^{45m(m-1)/2} \\
&= -f(-1,-q^{45}) = 0,
\end{aligned}$$

by (8.2.4). Hence,

$$S_3 = 0. \tag{8.5.6.4}$$

Replacing M by $M+1$, and then changing the signs of M and N, we readily find that

$$S_5 = \sum_{\substack{M,N=-\infty \\ M\equiv 0, N\equiv -1 \pmod 3}}^{\infty} (-1)^{M+N} q^{(5M^2+5N^2-3M-N)/2}.$$

Changing the signs of M and N and then replacing M by $M-1$, we deduce that

$$S_2 = q^2 \sum_{\substack{M,N=-\infty \\ M\equiv 0, N\equiv 1 \pmod 3}}^{\infty} (-1)^{M+N} q^{(5M^2+5N^2-9M-3N)/2}.$$

Replacing M by $M-1$, we easily see that

$$S_4 = q^2 \sum_{\substack{M,N=-\infty \\ M\equiv 0, N\equiv -1 \pmod 3}}^{\infty} (-1)^{M+N} q^{(5M^2+5N^2-9M-3N)/2}.$$

Hence,

$$S_1 + S_5 = \sum_{\substack{M,N=-\infty \\ M\equiv 0, N\equiv 0,-1 \pmod 3}}^{\infty} (-1)^{M+N} q^{(5M^2+5N^2-3M-N)/2} \tag{8.5.6.5}$$

and

$$S_2 + S_4 = q^2 \sum_{\substack{M,N=-\infty \\ M\equiv 0, N\equiv \pm 1 \pmod 3}}^{\infty} (-1)^{M+N} q^{(5M^2+5N^2-9M-3N)/2}. \tag{8.5.6.6}$$

Substituting (8.5.6.4)–(8.5.6.6) in (8.5.6.3) and comparing this with (8.5.6.2), we see that in order prove (8.5.6.2), we need to show that

$$\sum_{\substack{m,n=-\infty \\ m\equiv 0, n\equiv 1 \pmod 3}}^{\infty} (-1)^{m+n} q^{(5m^2+5n^2-3m-n)/2} + q^2 \sum_{\substack{m,n=-\infty \\ m,n\equiv 0 \pmod 3}}^{\infty} (-1)^{m+n} q^{(5m^2+5n^2-9m-3n)/2} = 0. \tag{8.5.6.7}$$

If we set $m = 3k$, $n = 3l+1$ and $m = -3l$, $n = 3k$ in the first and second sums of (8.5.6.7), respectively, we easily deduce (8.5.6.7), and so the proof is complete. □

Entry 8.3.6 is a natural companion to Entry 8.3.13; in Section 8.5.13, a third proof of Entry 8.3.6 will be given concomitantly with a proof of Entry 8.3.13.

8.5.7 Proofs of Entry 8.3.7

First Proof of Entry 8.3.7. Using (8.2.11), we can write (8.3.8) in the alternative form

$$f(-q^4,-q^6)f(-q^6,-q^9)+qf(-q^2,-q^8)f(-q^3,-q^{12}) = f(-q^2)f(-q^3)\frac{\chi(-q^3)}{\chi(-q)}. \tag{8.5.7.1}$$

Using

$$\chi(-q^3) = \frac{\varphi(-q^3)}{f(-q^3)} \qquad \text{and} \qquad \chi(-q) = \frac{f(-q^2)}{\psi(q)} \tag{8.5.7.2}$$

from (8.2.14), we rewrite (8.5.7.1) as

$$f(-q^4,-q^6)f(-q^6,-q^9)+qf(-q^2,-q^8)f(-q^3,-q^{12})=\psi(q)\varphi(-q^3). \tag{8.5.7.3}$$

For a and b in the set $\{0,\pm1,\pm2\}$, let

$$m+3n=5M+a \qquad \text{and} \qquad m-2n=5N+b,$$

from which it follows that

$$n=M-N+(a-b)/5 \qquad \text{and} \qquad m=2M+3N+(2a+3b)/5.$$

It follows easily that $a=b$, and so $m=2M+3N+a$ and $n=M-N$, where $-2\le a\le 2$. Thus, there is a one-to-one correspondence between the set of all pairs of integers (m,n), $-\infty<m,n<\infty$, and triples of integers (M,N,a), $-\infty<M,N<\infty$, $-2\le a\le 2$. From the definitions (8.2.8) and (8.2.7) of $\psi(q)$ and $\varphi(-q^3)$, the indicated changes of indices of summation, and (8.2.4), we deduce that

$$\begin{aligned}
2\psi(q)\varphi(-q^3) &= \sum_{m,n=-\infty}^{\infty}(-1)^n q^{m(m+1)/2+3n^2}\\
&= \sum_{a=-2}^{2} q^{a(a+1)/2}\sum_{M=-\infty}^{\infty}(-1)^M q^{5M^2+(1+2a)M}\\
&\quad\times\sum_{N=-\infty}^{\infty}(-1)^N q^{15N^2/2+3N/2+3aN}\\
&= \sum_{a=-2}^{2} q^{a(a+1)/2} f(-q^{4-2a},-q^{6+2a})f(-q^{6-3a},-q^{9+3a})\\
&= 2f(-q^4,-q^6)f(-q^6,-q^9)+2qf(-q^2,-q^8)f(-q^3,-q^{12})\\
&\quad+q^3f(-1,-q^{10})f(-1,-q^{15})\\
&= 2f(-q^4,-q^6)f(-q^6,-q^9)+2qf(-q^2,-q^8)f(-q^3,-q^{12}),
\end{aligned}$$

which is (8.5.7.3). So we complete our proof. □

Second Proof of Entry 8.3.7. Using (8.4.23) and (8.4.24) in (8.3.13), we arrive at

$$\begin{aligned}
\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)} &= G(q)G(q^{24})+q^5H(q)H(q^{24})\\
&= \frac{f(-q^8)}{f(-q^2)}\left\{G(q^{24})\left(G(q^{16})+qH(-q^4)\right)\right.\\
&\quad\left.+q^5H(q^{24})\left(q^3H(q^{16})+G(-q^4)\right)\right\}\\
&= \frac{f(-q^8)}{f(-q^2)}\Big\{G(q^{16})G(q^{24})+q^8H(q^{16})H(q^{24})\\
&\quad+q\left(H(-q^4)G(q^{24})+q^4G(-q^4)H(q^{24})\right)\Big\}.
\end{aligned} \tag{8.5.7.4}$$

By several applications of (8.2.14), we deduce from (8.5.7.4) that

$$\begin{aligned} &G(q^{16})G(q^{24}) + q^8 H(q^{16})H(q^{24}) + q\left(H(-q^4)G(q^{24}) + q^4 G(-q^4)H(q^{24})\right) \\ &= \frac{\chi(-q^3)\chi(-q^{12})f(-q^2)}{\chi(-q)\chi(-q^4)f(-q^8)} = \chi(q)\chi(-q^3)\chi(-q^{12}). \end{aligned} \tag{8.5.7.5}$$

Therefore, it suffices to find the even and the odd parts of $\chi(q)\chi(-q^3)$. By (8.2.6), (8.2.8), and (8.2.17),

$$\begin{aligned} f(-q,-q^5) &= (q;q^6)_\infty (q^5;q^6)_\infty (q^6;q^6)_\infty \\ &= \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty}(q^6;q^6)_\infty \\ &= \chi(-q)\psi(q^3) = \chi(-q)\chi(q^3)f(-q^{12}). \end{aligned} \tag{8.5.7.6}$$

Employing (8.2.19) with $n=2$, $a=q$, and $b=q^5$, we also find that

$$f(q,q^5) = f(q^8,q^{16}) + qf(q^4,q^{20}). \tag{8.5.7.7}$$

It is also easily verified that (see [55, p. 350, Equation (2.3)])

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{8.5.7.8}$$

Therefore, by (8.5.7.6), (8.5.7.7), and (8.5.7.8), we find that

$$\begin{aligned} \chi(q)\chi(-q^3) &= \frac{f(q,q^5)}{f(-q^{12})} = \frac{f(q^8,q^{16})}{f(-q^{12})} + q\frac{f(q^4,q^{20})}{f(-q^{12})} \\ &= \frac{\varphi(-q^{24})}{\chi(-q^8)f(-q^{12})} + q\frac{\chi(q^4)\chi(-q^{12})f(-q^{48})}{f(-q^{12})}. \end{aligned} \tag{8.5.7.9}$$

Next, by several applications of (8.2.14), we deduce from (8.5.7.9) that

$$\chi(q)\chi(-q^3) = \frac{\chi(q^{12})}{\chi(-q^8)} + q\frac{\chi(q^4)}{\chi(-q^{24})}. \tag{8.5.7.10}$$

Therefore, by (8.5.7.5), (8.5.7.10), and (8.2.16),

$$\begin{aligned} &G(q^{16})G(q^{24}) + q^8 H(q^{16})H(q^{24}) + q\left(H(-q^4)G(q^{24}) + q^4 G(-q^4)H(q^{24})\right) \\ &= \frac{\chi(q^{12})\chi(-q^{12})}{\chi(-q^8)} + q\frac{\chi(q^4)\chi(-q^{12})}{\chi(-q^{24})} = \frac{\chi(-q^{24})}{\chi(-q^8)} + q\frac{\chi(q^4)}{\chi(q^{12})}. \end{aligned} \tag{8.5.7.11}$$

Equating the even parts on both sides of (8.5.7.11), we obtain Entry 8.3.7 with q replaced by q^8. Similarly, equating the odd parts gives Entry 8.3.8 with q replaced by $-q^4$. $\square$

8.5.8 Proof of Entry 8.3.8

We have just given a proof of Entry 8.3.8 along with one of our proofs of Entry 8.3.7. We provide a second proof here.

If we use (8.2.11), we can put Entry 8.3.8 in the form

$$f(-q^{12},-q^{18})f(-q,-q^4)-qf(-q^2,-q^3)f(-q^6,-q^{24})=\frac{\chi(-q)}{\chi(-q^3)}f(-q)f(-q^6). \tag{8.5.8.1}$$

Using (8.2.7)–(8.2.10), we can rewrite (8.5.8.1) as

$$f(-q^{12},-q^{18})f(-q,-q^4)-qf(-q^2,-q^3)f(-q^6,-q^{24})=\psi(q^3)\varphi(-q). \tag{8.5.8.2}$$

In the representation

$$2\psi(q^3)\varphi(-q)=f(1,q^3)f(-q,-q)=\sum_{m,n=-\infty}^{\infty}(-1)^n q^{(3m^2+3m+2n^2)/2}, \tag{8.5.8.3}$$

we make the change of indices

$$3m-2n=5M+a \qquad \text{and} \qquad m+n=5N+b, \tag{8.5.8.4}$$

where a and b have values selected from the integers 0, ±1, ±2. Since

$$m=M+2N+(a+2b)/5 \qquad \text{and} \qquad n=-M+3N+(3b-a)/5, \tag{8.5.8.5}$$

we see that values of a and b are associated as in the following table:

a	0	±1	±2
b	0	±2	∓1

Thus, there is a one-to-one correspondence between all pairs of integers (m,n) and all sets of integers (M,N,a) as given above. We therefore deduce from (8.5.8.3), (8.2.4), and (8.2.5) that

$$\begin{aligned}
2\psi(q^3)\varphi(-q) &= \sum_{m,n=-\infty}^{\infty}(-1)^n q^{(3m^2+3m+2n^2)/2}\\
&= -qf(-q^2,-q^3)f(-q^6,-q^{24})-qf(-q^2,-q^3)f(-q^6,-q^{24})\\
&\quad +f(-q,-q^4)f(-q^{12},-q^{18})-q^4f(-1,-q^5)f(-1,-q^{30})\\
&\quad -qf(-q^6,-q^{-1})f(-q^{12},-q^{18})\\
&= -2qf(-q^2,-q^3)f(-q^6,-q^{24})+f(-q,-q^4)f(-q^{12},-q^{18})\\
&\quad +f(-q,-q^4)f(-q^{12},-q^{18})\\
&= 2f(-q,-q^4)f(-q^{12},-q^{18})-2qf(-q^2,-q^3)f(-q^6,-q^{24}).
\end{aligned}$$

By (8.5.8.2), we see that the proof is complete.

8.5.9 Proof of Entry 8.3.9

Using (8.2.11) and (8.2.14), we find that Entry 8.3.9 is equivalent to the identity

$$f(-q^{14},-q^{21})f(-q^2,-q^8)-qf(-q^4,-q^6)f(-q^7,-q^{28}) \\ = f(-q^2)f(-q^7)\frac{\chi(-q)}{\chi(-q^7)} = f(-q)f(-q^{14}). \quad (8.5.9.1)$$

We invoke (8.4.13) with the two sets of parameters $\alpha_1=2$, $\beta_1=7$, $m_1=3$, $p_1=5$, $\lambda_1=5$ and $\alpha_2=1$, $\beta_2=14$, $m_2=1$, $p_2=3$, $\lambda_2=5$. The conditions in (8.4.9) are easily seen to be met. Using (8.4.14) and (8.4.15), we find that

$$f(-q^{28},-q^{42})f(-q^4,-q^{16})-q^2f(-q^8,-q^{12})f(-q^{14},-q^{56}) = f(-q^2)f(-q^{28}).$$

Replacing q^2 by q in the last equality, we deduce (8.5.9.1) to complete the proof.

8.5.10 Proof of Entry 8.3.10

Using (8.2.11) and (8.2.14), we find that Entry 8.3.10 is equivalent to the identity

$$f(-q^2,-q^3)f(-q^{28},-q^{42})+q^3f(-q,-q^4)f(-q^{14},-q^{56}) \\ = f(-q)f(-q^{14})\frac{\chi(-q^7)}{\chi(-q)} = f(-q^2)f(-q^7). \quad (8.5.10.1)$$

We now apply (8.4.13) with $\alpha_1=1$, $\beta_1=14$, $m_1=1$, $p_1=5$, $\lambda_1=3$ and $\alpha_2=2$, $\beta_2=7$, $m_2=1$, $p_2=3$, $\lambda_2=3$. We easily find that these sets of parameters satisfy the conditions in (8.4.9). Employing (8.4.13) and (8.4.15), we find that

$$f(-q^4,-q^6)f(-q^{56},-q^{84})+q^6f(-q^2,-q^8)f(-q^{28},-q^{112}) = f(-q^4)f(-q^{14}),$$

which is (8.5.10.1), but with q replaced by q^2, and so the proof is complete.

8.5.11 Proofs of Entry 8.3.11

First Proof of Entry 8.3.11*.* Employing (8.2.11), we see that Entry 8.3.11 is equivalent to the identity

$$f(-q^{16},-q^{24})f(-q^3,-q^{12})-qf(-q^6,-q^9)f(-q^8,-q^{32}) \\ = f(-q^3)f(-q^8)\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}. \quad (8.5.11.1)$$

We apply (8.4.13) with the two sets of parameters $\alpha_1 = 3$, $\beta_1 = 8$, $m_1 = 3$, $p_1 = 5$, $\lambda_1 = 7$ and $\alpha_2 = 2$, $\beta_2 = 12$, $m_2 = 1$, $p_2 = 2$, $\lambda_2 = 7$. The conditions (8.4.9) are easily seen to be satisfied. Using (8.4.14) and (8.4.16), we find that

$$f(-q^{32},-q^{48})f(-q^6,-q^{24})-q^2f(-q^{12},-q^{18})f(-q^{16},-q^{64}) = \psi(-q^2)\psi(-q^{12}). \tag{8.5.11.2}$$

After replacing q^2 by q in (8.5.11.2) and comparing the result with (8.5.11.1), we find that it suffices to show that

$$\frac{\psi(-q)\psi(-q^6)}{f(-q^3)f(-q^8)} = \frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}. \tag{8.5.11.3}$$

Using the product representations for $\psi(-q^a)$ and $f(-q^a)$ in (8.2.8) and (8.2.9), respectively, we find that

$$\begin{aligned}
\frac{\psi(-q)\psi(-q^6)}{f(-q^3)f(-q^8)} &= \frac{(q^2;q^2)_\infty(q^{12},q^{12})_\infty}{(-q;q^2)_\infty(-q^6,q^{12})_\infty(q^3;q^3)_\infty(q^8;q^8)_\infty} \\
&= \frac{(q^2;q^2)_\infty(q;q^2)_\infty(q^{12},q^{12})_\infty(q^6;q^{12})_\infty}{(q^2;q^4)_\infty(q^{12},q^{24})_\infty(q^3;q^3)_\infty(q^8;q^8)_\infty} \\
&= \frac{\chi(-q)\chi(-q^4)}{\chi(-q^{12})}\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty} \\
&= \frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}.
\end{aligned}$$

Thus, the proof of (8.5.11.3) is complete, and so that of Entry 8.3.11 is also complete. ☐

Second Proof of Entry 8.3.11. Using (8.4.23) and (8.4.24) in (8.3.9), we find that

$$\begin{aligned}
&\frac{\chi(-q)}{\chi(-q^3)} = G(q^6)H(q) - qG(q)H(q^6) \\
&= \frac{f(-q^8)}{f(-q^2)}\left\{G(q^6)\left(q^3H(q^{16})+G(-q^4)\right) - qH(q^6)\left(G(q^{16})+qH(-q^4)\right)\right\} \\
&= \frac{f(-q^8)}{f(-q^2)}\Big\{G(-q^4)G(q^6) - q^2H(-q^4)H(q^6) \\
&\qquad\qquad - q\left(H(q^6)G(q^{16}) - q^2G(q^6)H(q^{16})\right)\Big\}.
\end{aligned} \tag{8.5.11.4}$$

By (8.2.15) and (8.2.17), and by (8.5.7.10) with q replaced by $-q$, we deduce from (8.5.11.4) that

$$\begin{aligned}
&G(-q^4)G(q^6) - q^2H(-q^4)H(q^6) - q\left(H(q^6)G(q^{16}) - q^2G(q^6)H(q^{16})\right) \\
&\quad = \frac{\chi(-q)f(-q^2)}{\chi(-q^3)f(-q^8)} = \frac{f(-q^2)}{f(-q^8)\chi(-q^6)}\chi(-q)\chi(q^3)
\end{aligned} \tag{8.5.11.5}$$

$$= \frac{f(-q^2)}{f(-q^8)\chi(-q^6)} \left\{ \frac{\chi(q^{12})}{\chi(-q^8)} - q\frac{\chi(q^4)}{\chi(-q^{24})} \right\}$$
$$= \frac{\chi(-q^2)\chi(q^{12})}{\chi(q^4)\chi(-q^6)} - q\frac{\chi(-q^2)\chi(-q^8)}{\chi(-q^6)\chi(-q^{24})}.$$

Equating the even and odd parts on the extremal sides of the equations (8.5.11.5), we obtain Entries 8.3.23 and 8.3.11 with q replaced by $-q^2$ and q^2, respectively. ☐

8.5.12 Proofs of Entry 8.3.12

The first proof that we give is due to Bressoud [81].

First Proof of Entry 8.3.12. Using (8.2.11), we readily find that Entry 8.3.12 is equivalent to the identity

$$f(-q^2,-q^3)f(-q^{48},-q^{72}) + q^5 f(-q,-q^4)f(-q^{24},-q^{96}) = f(-q)f(-q^{24})\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}. \tag{8.5.12.1}$$

We apply (8.4.13) with the two sets of parameters $\alpha_1 = 1$, $\beta_1 = 24$, $m_1 = 1$, $p_1 = 5$, $\lambda_1 = 5$ and $\alpha_2 = 4$, $\beta_2 = 6$, $m_2 = 1$, $p_2 = 2$, $\lambda_2 = 5$. We find that the conditions in (8.4.9) are satisfied. Hence, using (8.4.15) and (8.4.18), we deduce the identity

$$f(-q^4,-q^6)f(-q^{96},-q^{144}) + q^{10} f(-q^2,-q^8)f(-q^{48},-q^{192}) = \psi(-q^4)\psi(-q^6). \tag{8.5.12.2}$$

Replacing q^2 by q in (8.5.12.2) and comparing it with (8.5.12.1), we find that it suffices to prove that

$$\frac{\psi(-q^2)\psi(-q^3)}{f(-q)f(-q^{24})} = \frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}. \tag{8.5.12.3}$$

Using the product representations of $\psi(-q^a)$ and $f(-q^a)$ from (8.2.8) and (8.2.9), respectively, we find that

$$\frac{\psi(-q^2)\psi(-q^3)}{f(-q)f(-q^{24})} = \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{(-q^2;q^4)_\infty(-q^3;q^6)_\infty(q;q)_\infty(q^{24};q^{24})_\infty}$$
$$= \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty(q^2;q^4)_\infty(q^3;q^6)_\infty}{(q^4;q^8)_\infty(q^6;q^{12})_\infty(q;q)_\infty(q^{24};q^{24})_\infty}$$
$$= \frac{(q^2;q^2)_\infty(q^6;q^6)_\infty\chi(-q^3)}{\chi(-q^4)(q;q)_\infty(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}$$
$$= \frac{(q^6;q^6)_\infty\chi(-q^3)}{\chi(-q)\chi(-q^4)(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}$$

$$= \frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)},$$

which establishes (8.5.12.3), and so the proof is complete. □

Second Proof of Entry 8.3.12. Using (8.4.23) and (8.4.24) in (8.3.8) with q replaced by q^3, we arrive at

$$\begin{aligned}
&\frac{\chi(-q^3)}{\chi(-q)} = G(q^2)G(q^3) + qH(q^2)H(q^3) \\
&= \frac{f(-q^{24})}{f(-q^6)} \left\{G(q^2)\left(G(q^{48}) + q^3H(-q^{12})\right) + qH(q^2)\left(q^9H(q^{48}) + G(-q^{12})\right)\right\} \\
&= \frac{f(-q^{24})}{f(-q^6)} \Big\{G(q^2)G(q^{48}) + q^{10}H(q^2)H(q^{48}) \\
&\qquad\qquad + q\left(H(q^2)G(-q^{12}) + q^2G(q^2)H(-q^{12})\right)\Big\}.
\end{aligned} \tag{8.5.12.4}$$

That is,

$$\begin{aligned}
G(q^2)G(q^{48}) + q^{10}H(q^2)H(q^{48}) + q\left(H(q^2)G(-q^{12}) + q^2G(q^2)H(-q^{12})\right) \\
= \frac{f(-q^6)\chi(-q^3)}{f(-q^{24})\chi(-q)}.
\end{aligned} \tag{8.5.12.5}$$

Therefore, by (8.5.7.10), (8.2.15), and (8.2.14),

$$\frac{f(-q^6)\chi(-q^3)}{f(-q^{24})\chi(-q)} = \frac{f(-q^6)\chi(q)\chi(-q^3)}{f(-q^{24})\chi(-q^2)} = \frac{\chi(-q^6)\chi(-q^{24})}{\chi(-q^2)\chi(-q^8)} + q\frac{\chi(q^4)\chi(-q^6)}{\chi(-q^2)\chi(q^{12})}. \tag{8.5.12.6}$$

Returning to (8.5.12.5), we use (8.5.12.6) to equate the odd parts on both sides of the equation, and upon replacing q^2 by $-q$, we find that

$$H(-q)G(-q^6) - qG(-q)H(-q^6) = \frac{\chi(q^2)\chi(q^3)}{\chi(q)\chi(q^6)},$$

which is Entry 8.3.24. Similarly, equating the even parts in (8.5.12.5), employing (8.5.12.6), and replacing q^2 by q, we deduce that

$$G(q)G(q^{24}) + q^5H(q)H(q^{24}) = \frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)},$$

which is Entry 8.3.12. □

8.5.13 Proofs of Entries 8.3.13 and 8.3.14

Throughout this section, we use several times without comment the elementary identity (8.2.5). To prove Entries 8.3.13 and 8.3.14, we also need the following lemma.

Lemma 8.5.4. *For* $|\, q \,|< 1$, $x \neq 0$, *and* $y \neq 0$,

$$\begin{aligned} &f(-x,-x^{-1}q)f(-y,-y^{-1}q) \qquad (8.5.13.1)\\ &= f(xy,(xy)^{-1}q^2)f(x^{-1}yq,xy^{-1}q) - xf(xyq,(xy)^{-1}q)f(x^{-1}y,xy^{-1}q^2). \end{aligned}$$

In this form, Lemma 8.5.4 is given as Theorem 1.1 in [171, p. 649]. However, it is easily seen that Lemma 8.5.4 can be obtained by adding Entries 29(i) and (ii) in Chapter 16 of Ramanujan's second notebook [55, p. 45]. (The theta functions $f(-x^n,-x^{-n}y)$, where n is an integer, are connected with Somos sequences.)

Lemma 8.5.5. *We have*

$$f^2(-q^{27},-q^{45}) + q^9 f^2(-q^9,-q^{63}) = f(-q^3,q^6)\psi(-q^9)\chi(q^3). \qquad (8.5.13.2)$$

Proof. Replacing q, x, and y by q^{36}, $-q^9$, and q^{18}, respectively, in Lemma 8.5.4, we find that

$$\begin{aligned} f^2(-q^{27},-q^{45}) + q^9 f^2(-q^9,-q^{63}) &= f(q^9,q^{27})f(-q^{18},-q^{18})\\ &= \psi(q^9)\varphi(-q^{18}). \qquad (8.5.13.3) \end{aligned}$$

Using (8.2.13)–(8.2.15), or using (8.2.7)–(8.2.9), we can easily conclude that

$$\varphi(-q^2)\psi(q) = \varphi(q)\psi(-q). \qquad (8.5.13.4)$$

Therefore, by (8.5.7.8) with q replaced by $-q^3$, and by (8.5.13.4) with q replaced by q^9, we find that

$$f(-q^3,q^6)\psi(-q^9)\chi(q^3) = \psi(-q^9)\varphi(q^9) = \psi(q^9)\varphi(-q^{18}). \qquad (8.5.13.5)$$

Thus, we have proved Lemma 8.5.5. ∎

Lemma 8.5.6. *We have*

$$\begin{aligned} &f(-q^{21},-q^{51})f(-q^{27},-q^{45}) + q^6 f(-q^3,-q^{69})f(-q^{27},-q^{45})\\ &\quad + q^3 f(-q^{33},-q^{39})f(-q^9,-q^{63}) - q^6 f(-q^{15},-q^{57})f(-q^9,-q^{63})\\ &= \psi^2(-q^9)\chi(q^3). \qquad (8.5.13.6) \end{aligned}$$

Proof. Replacing q, x, and y by q^{36}, q^6, and $-q^{15}$, respectively, in Lemma 8.5.4, we find that

$$\begin{aligned} &f(-q^{21},-q^{51})f(-q^{27},-q^{45}) - q^6 f(-q^{15},-q^{57})f(-q^9,-q^{63})\\ &= f(-q^6,-q^{30})f(q^{15},q^{21}), \qquad (8.5.13.7) \end{aligned}$$

and replacing q, x, and y by q^{36}, $-q^3$, and q^6, respectively, in Lemma 8.5.4, we find that

$$\begin{aligned}&f(-q^9,-q^{63})f(-q^{33},-q^{39})+q^3f(-q^{27},-q^{45})f(-q^3,-q^{69})\\&=f(q^3,q^{33})f(-q^6,-q^{30}).\end{aligned}\tag{8.5.13.8}$$

By (8.2.19) with $a=q^3$, $b=q^6$, and $n=2$, we deduce that

$$f(q^3,q^6)=f(q^{15},q^{21})+q^3f(q^3,q^{33}).\tag{8.5.13.9}$$

Thus, by (8.5.13.7)–(8.5.13.9), we find that

$$\begin{aligned}&f(-q^{21},-q^{51})f(-q^{27},-q^{45})+q^6f(-q^3,-q^{69})f(-q^{27},-q^{45})\\&\quad+q^3f(-q^{33},-q^{39})f(-q^9,-q^{63})-q^6f(-q^{15},-q^{57})f(-q^9,-q^{63})\\&=f(q^3,q^6)f(-q^6,-q^{30}).\end{aligned}\tag{8.5.13.10}$$

One can easily verify that

$$\psi^2(-q)=\psi(q^2)\varphi(-q).\tag{8.5.13.11}$$

By (8.5.7.8), (8.5.7.6), and (8.5.13.11) with q replaced by q^3, q^6, and q^9, respectively, and by (8.2.15), we conclude that

$$\begin{aligned}f(q^3,q^6)f(-q^6,-q^{30})&=\frac{\varphi(-q^9)}{\chi(-q^3)}\chi(-q^6)\psi(q^{18})\\&=\varphi(-q^9)\psi(q^{18})\chi(q^3)=\psi^2(-q^9)\chi(q^3).\end{aligned}\tag{8.5.13.12}$$

Thus, we have proved Lemma 8.5.6. □

Theorem 8.5.1. *For* $|\,q\,|<1$,

$$\begin{aligned}&f(-q,-q^7)f(-q^{27},-q^{45})-q^4f(-q^3,-q^5)f(-q^9,-q^{63})\\&=(q^4;q^4)_\infty(q^9;q^9)_\infty\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}.\end{aligned}\tag{8.5.13.13}$$

Proof. Replacing n, a, and b by 3, $-q$, and $-q^7$, respectively, in (8.2.19), we find that

$$f(-q,-q^7)=f(-q^{27},-q^{45})-qf(-q^{21},-q^{51})-q^7f(-q^3,-q^{69});\tag{8.5.13.14}$$

replacing n, a, and b by 3, $-q^3$, and $-q^5$, respectively, in (8.2.19), we find that

$$f(-q^3,-q^5)=f(-q^{33},-q^{39})-q^3f(-q^{15},-q^{57})-q^5f(-q^9,-q^{63});\tag{8.5.13.15}$$

and replacing n, a, and b by 3, $-q$, and $-q^3$, respectively, in (8.2.19), we find that (see also [55, p. 49, Corollary])

$$\begin{aligned}\psi(-q)=f(-q,-q^3)&=f(-q^{15},-q^{21})-qf(-q^9,-q^{27})-q^3f(-q^3,-q^{33})\\&=f(-q^3,q^6)-q\psi(-q^9),\end{aligned}\tag{8.5.13.16}$$

where in the last step, we used (8.5.13.9) with q replaced by $-q$. Then, by (8.5.13.14) and (8.5.13.15), the left-hand side of (8.5.13.13) equals

$$\begin{aligned}&f^2(-q^{27},-q^{45})-qf(-q^{21},-q^{51})f(-q^{27},-q^{45})\\&-q^7f(-q^3,-q^{69})f(-q^{27},-q^{45})-q^4f(-q^{33},-q^{39})f(-q^9,-q^{63})\\&+q^7f(-q^{15},-q^{57})f(-q^9,-q^{63})+q^9f^2(-q^9,-q^{63}).\end{aligned}\tag{8.5.13.17}$$

Therefore, by (8.5.13.17), Lemma 8.5.5, Lemma 8.5.6, (8.5.13.16), (8.2.16), and (8.2.17), the left-hand side of (8.5.13.13) equals

$$\begin{aligned}&f(-q^3,q^6)\psi(-q^9)\chi(q^3)-q\psi^2(-q^9)\chi(q^3)\\&=\psi(-q^9)\chi(q^3)\left\{f(-q^3,q^6)-q\psi(-q^9)\right\}\\&=\psi(-q)\psi(-q^9)\chi(q^3)=f(-q^4)\chi(-q)\frac{f(-q^9)}{\chi(-q^{18})}\frac{\chi(-q^6)}{\chi(-q^3)}.\end{aligned}\tag{8.5.13.18}$$

We have thus completed the proof of Theorem 8.5.1. □

Theorem 8.5.2. *For* $|\, q\,|<1$,

$$\begin{aligned}&f(-q^4,-q^{16})f(-q^{18},-q^{27})-qf(-q^8,-q^{12})f(-q^9,-q^{36})\\&=f(-q,-q^4)f(-q^{72},-q^{108})-q^7f(-q^2,-q^3)f(-q^{36},-q^{144})\\&=f(-q,-q^7)f(-q^{27},-q^{45})-q^4f(-q^3,-q^5)f(-q^9,-q^{63}).\end{aligned}\tag{8.5.13.19}$$

Proof. We apply the ideas of Rogers with the three sets of parameters $\alpha_1=4$, $\beta_1=9$, $m_1=3$, $p_1=5$, $\lambda_1=9$; $\alpha_2=1$, $\beta_2=36$, $m_2=3$, $p_2=5$, $\lambda_2=9$; and $\alpha_3=2$, $\beta_3=18$, $m_3=3$, $p_3=4$, $\lambda_3=9$. The requisite conditions (8.4.9) are satisfied. Therefore, we find that

$$\begin{aligned}&q^{9/4}f(-q^8,-q^{32})f(-q^{36},-q^{54})-q^{17/4}f(-q^{16},-q^{24})f(-q^{18},-q^{72})\\&=q^{9/4}f(-q^2,-q^8)f(-q^{144},-q^{216})-q^{65/4}f(-q^4,-q^6)f(-q^{72},-q^{288})\\&=q^{9/4}f(-q^2,-q^{14})f(-q^{54},-q^{90})-q^{41/4}f(-q^6,-q^{10})f(-q^{18},-q^{126}).\end{aligned}\tag{8.5.13.20}$$

Dividing each term of (8.5.13.20) by $q^{9/4}$, and replacing q^2 by q, we are able to derive (8.5.13.19) from (8.5.13.20). □

We are now going to prove Entries 8.3.13 and 8.3.14.

Proof of Entry 8.3.13. By Theorems 8.5.1 and 8.5.2, we find that

$$\begin{aligned}&f(-q^4,-q^{16})f(-q^{18},-q^{27})-qf(-q^8,-q^{12})f(-q^9,-q^{36})\\&=(q^4;q^4)_\infty(q^9;q^9)_\infty\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}.\end{aligned}\tag{8.5.13.21}$$

Dividing both sides of (8.5.13.21) by $(q^4;q^4)_\infty(q^9;q^9)_\infty$ and using the definitions of $G(q)$ and $H(q)$, we derive Entry 8.3.13. □

Proof of Entry 8.3.14. By Theorems 8.5.1 and 8.5.2, we find that

$$f(-q,-q^4)f(-q^{72},-q^{108}) - q^7 f(-q^2,-q^3)f(-q^{36},-q^{144})$$
$$= (q^4;q^4)_\infty(q^9;q^9)_\infty \frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}. \tag{8.5.13.22}$$

Dividing both sides of (8.5.13.22) by $(q;q)_\infty(q^{36};q^{36})_\infty$ and using the definitions of $G(q)$ and $H(q)$, we find that the left-hand side of (8.5.13.22) equals $G(q^{36})H(q) - q^7G(q)H(q^{36})$, and the right-hand side of (8.5.13.22) equals

$$\frac{(q^4;q^4)_\infty(q^9;q^9)_\infty}{(q;q)_\infty(q^{36};q^{36})_\infty} \cdot \frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})} = \frac{\chi(-q^6)\chi(-q^9)}{\chi(-q^2)\chi(-q^3)},$$

which completes the proof of Entry 8.3.14. □

We offer now a second, completely different proof of Entry 8.3.13.

Second Proof of Entry 8.3.13. By (8.2.11), (8.2.8), (8.2.6), and some elementary product manipulations, Entry 8.3.13 is easily seen to be equivalent to

$$f(-q^4,-q^{16})f(-q^{18},-q^{27}) - qf(-q^8,-q^{12})f(-q^9,-q^{36}) = \psi(-q)f(q^3,q^{15}). \tag{8.5.13.23}$$

We prove (8.5.13.23).

Employing Theorem 8.4.1 with the set of parameters $a = q^3$, $b = q^{15}$, $c = q$, $d = q^2$, $\alpha = 3$, $\beta = 1$, $m = 5$, $\epsilon_1 = 0$, and $\epsilon_2 = 1$, we find that

$$\begin{aligned} f(q^3,q^{15})f(-q,-q^2) &= f(-q^{18},-q^{27})f(q^8,q^{22}) - qf(-q^9,-q^{36})f(q^{14},q^{16}) \\ &\quad + q^3 f(-q^9,-q^{36})f(q^4,q^{26}) - q^2 f(-q^{18},-q^{27})f(q^2,q^{28}). \end{aligned}$$

Upon the rearrangement of terms and the use of (8.4.28) and (8.4.29) with q replaced by q^2, we deduce that

$$\begin{aligned} f(q^3,q^{15})f(-q) &= f(-q^{18},-q^{27})\{f(q^8,q^{22}) - q^2 f(q^2,q^{28})\} \\ &\quad - qf(-q^9,-q^{36})\{f(q^{14},q^{16}) - q^2 f(q^4,q^{26})\} \\ &= f(-q^{18},-q^{27})H(q^4)f(-q^2) - qf(-q^9,-q^{36})G(q^4)f(-q^2) \\ &= \frac{f(-q^2)}{f(-q^4)}\left\{f(-q^4,-q^{16})f(-q^{18},-q^{27}) - qf(-q^8,-q^{12})f(-q^9,-q^{36})\right\}. \end{aligned} \tag{8.5.13.24}$$

But by (8.2.17),

$$\frac{f(-q)f(-q^4)}{f(-q^2)} = \psi(-q).$$

Using the last equality in (8.5.13.24), we complete the proof of (8.5.13.23) and also that of Entry 8.3.14. □

In Section 8.5.6, we promised that in the current section we would give a proof that simultaneously yields Entries 8.3.6 and 8.3.13. We show that Entry 8.3.11 implies both Entries 8.3.6 and 8.3.13.

Another Proof of Entries 8.3.6 *and* 8.3.13. In Entry 8.3.11, we employ (8.4.31) and (8.4.30) with q replaced by q^3 to find that

$$\begin{aligned}&\left\{-q^3\frac{\chi(q^{18})}{\chi(-q^{12})}H(-q^{18})+\frac{\chi(q^6)}{\chi(-q^{36})}G(q^{72})\right\}G(q^8)\\&\quad-q\left\{\frac{\chi(q^{18})}{\chi(-q^{12})}G(-q^{18})-q^{15}\frac{\chi(q^6)}{\chi(-q^{36})}H(q^{72})\right\}H(q^8)\\&=\chi(q^9)\chi(-q^3)\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}.\end{aligned}\tag{8.5.13.25}$$

Upon collecting terms, we deduce from (8.5.13.25) that

$$\begin{aligned}&\frac{\chi(q^6)}{\chi(-q^{36})}\left\{G(q^8)G(q^{72})+q^{16}H(q^8)H(q^{72})\right\}\\&-q\frac{\chi(q^{18})}{\chi(-q^{12})}\left\{G(-q^{18})H(q^8)+q^2H(-q^{18})G(q^8)\right\}=\frac{\chi(-q^4)}{\chi(-q^{12})}\chi(-q)\chi(q^9).\end{aligned}\tag{8.5.13.26}$$

To equate even and odd parts on both sides of (8.5.13.26), we need the 2-dissection of $\chi(-q)\chi(q^9)$, which we obtain from Theorem 8.4.1. To that end, we employ Theorem 8.4.1 with the set of parameters $a=1$, $b=q^9$, $c=q$, $d=q^2$, $\epsilon_1=0$, $\epsilon_2=1$, $\alpha=\beta=1$, and $m=4$ to find that

$$\begin{aligned}f(1,q^9)f(-q,-q^2)&=f(-q^2,-q^{10})f(-q^{12},-q^{24})+f(-q,-q^{11})f(-q^{15},-q^{21})\\&\quad-qf(-q^4,-q^8)f(-q^6,-q^{30})-q^2f(-q^5,-q^7)f(-q^3,-q^{33}).\end{aligned}\tag{8.5.13.27}$$

We employ Theorem 8.4.1 again with the same set of parameters, except this time we take $\epsilon_1=1$ and $\epsilon_2=0$ to find that

$$\begin{aligned}f(-1,-q^9)f(q,q^2)&=f(-q^2,-q^{10})f(-q^{12},-q^{24})-f(-q,-q^{11})f(-q^{15},-q^{21})\\&\quad-qf(-q^4,-q^8)f(-q^6,-q^{30})+q^2f(-q^5,-q^7)f(-q^3,-q^{33}).\end{aligned}\tag{8.5.13.28}$$

By (8.2.4), the product on the left side of (8.5.13.28) equals 0. Recalling the definitions (8.2.8) and (8.2.9), and employing (8.2.3), (8.5.13.27), and (8.5.13.28), we conclude that

$$\begin{aligned}\psi(q^9)f(-q)&=\frac{1}{2}f(1,q^9)f(-q,-q^2)\\&=\frac{1}{2}\left\{f(1,q^9)f(-q,-q^2)+f(-1,-q^9)f(q,q^2)\right\}\\&=f(-q^2,-q^{10})f(-q^{12},-q^{24})-qf(-q^4,-q^8)f(-q^6,-q^{30})\\&=f(-q^2,-q^{10})f(-q^{12})-qf(-q^4)f(-q^6,-q^{30}).\end{aligned}\tag{8.5.13.29}$$

Next, we use (8.2.14), (8.2.17), and (8.5.7.6) twice with q replaced by q^2 and q^6, respectively, to find from (8.5.13.29) that

$$\begin{aligned}&\chi(q^9)f(-q^{36})\chi(-q)f(-q^2)\\&=f(-q^{12})\chi(-q^2)\chi(q^6)f(-q^{24})-qf(-q^4)\chi(-q^6)\chi(q^{18})f(-q^{72}),\end{aligned}$$

which after several uses of (8.2.14) simplifies to

$$\chi(-q)\chi(q^9)=\frac{f(-q^{12})\psi(q^6)}{f(-q^4)f(-q^{36})}-q\frac{\chi(-q^6)}{\chi(-q^2)\chi(-q^{18})}. \tag{8.5.13.30}$$

Returning to (8.5.13.26), we substitute the value of $\chi(-q)\chi(q^9)$ from (8.5.13.30) and equate the odd parts on both sides of the resulting equation. Hence, using (8.2.15), we conclude that

$$\begin{aligned}G(-q^{18})H(q^8)+q^2H(-q^{18})G(q^8)&=\frac{\chi(-q^{12})}{\chi(q^{18})}\frac{\chi(-q^4)}{\chi(-q^{12})}\frac{\chi(-q^6)}{\chi(-q^2)\chi(-q^{18})}\\&=\frac{\chi(q^2)\chi(-q^6)}{\chi(-q^{36})},\end{aligned}$$

which is Entry 8.3.13 with q replaced by $-q^2$. Similarly, equating the even parts in (8.5.13.26) with the use of (8.5.13.30), using (8.2.14) and (8.2.17), and replacing q^8 by q, we deduce Entry 8.3.6. ☐

8.5.14 Proof of Entry 8.3.15

Let

$$M(q):=G(q^3)G(q^7)+q^2H(q^3)H(q^7) \tag{8.5.14.1}$$

and

$$N(q):=G(q^{21})H(q)-q^4G(q)H(q^{21}). \tag{8.5.14.2}$$

Consider the system of equations

$$N(q^2)=H(q^2)G(q^{42})-q^8G(q^2)H(q^{42}),$$

$$\frac{\chi(-q^7)}{\chi(-q^{21})}=:R(q)=H(q^7)G(q^{42})-q^7G(q^7)H(q^{42}), \tag{8.5.14.3}$$

$$\frac{\chi(-q^{21})}{\chi(-q^3)}=:S(q)=G(q^3)G(q^{42})+q^9H(q^3)H(q^{42}), \tag{8.5.14.4}$$

arising from (8.5.14.2), Entry 8.3.8 with q replaced by q^7, and Entry 8.3.10 with q replaced by q^3, respectively. It follows that

$$\begin{vmatrix} H(q^2) & -q^8G(q^2) & N(q^2) \\ H(q^7) & -q^7G(q^7) & R(q) \\ G(q^3) & q^9H(q^3) & S(q) \end{vmatrix} = 0,$$

or, using (8.5.14.1), Entry 8.3.7, Entry 8.3.9, (8.5.14.3), and (8.5.14.4), we find that

$$\begin{aligned} 0 &= N(q^2)\left(q^9H(q^3)H(q^7) + q^7G(q^3)G(q^7)\right) \\ &\quad - R(q)\left(q^9H(q^2)H(q^3) + q^8G(q^2)G(q^3)\right) \\ &\quad + S(q)\left(-q^7G(q^7)H(q^2) + q^8G(q^2)H(q^7)\right) \\ &= q^7N(q^2)M(q) - q^8\frac{\chi(-q^7)}{\chi(-q^{21})}\frac{\chi(-q^3)}{\chi(-q)} - q^7\frac{\chi(-q^{21})}{\chi(-q^3)}\frac{\chi(-q)}{\chi(-q^7)}. \end{aligned}$$

Solving the equation above for $N(q^2)M(q)$, we find that if

$$T(q) := \frac{\chi(-q^3)\chi(-q^7)}{\chi(-q)\chi(-q^{21})}, \tag{8.5.14.5}$$

then

$$N(q^2)M(q) = qT(q) + \frac{1}{T(q)}. \tag{8.5.14.6}$$

Next, we derive a similar formula for $M(q^2)N(q)$. Using (8.5.14.1), Entry 8.3.10, and Entry 8.3.7 with q replaced by q^7, we find that

$$M(q^2) = G(q^6)G(q^{14}) + q^4H(q^6)H(q^{14}),$$

$$\frac{\chi(-q^7)}{\chi(-q)} =: R_1(q) = G(q)G(q^{14}) + q^3H(q)H(q^{14}), \tag{8.5.14.7}$$

$$\frac{\chi(-q^{21})}{\chi(-q^7)} =: S_1(q) = G(q^{21})G(q^{14}) + q^7H(q^{21})H(q^{14}), \tag{8.5.14.8}$$

which implies that

$$\begin{vmatrix} G(q^6) & q^4H(q^6) & M(q^2) \\ G(q) & q^3H(q) & R_1(q) \\ G(q^{21}) & q^7H(q^{21}) & S_1(q) \end{vmatrix} = 0.$$

Hence, by (8.5.14.2), Entry 8.3.9 with q replaced by q^3, Entry 8.3.8, (8.5.14.7), and (8.5.14.8),

$$\begin{aligned} 0 &= M(q^2)\left(q^7G(q)H(q^{21}) - q^3H(q)G(q^{21})\right) \\ &\quad - R_1(q)\left(q^7G(q^6)H(q^{21}) - q^4H(q^6)G(q^{21})\right) \\ &\quad + S_1(q)\left(q^3G(q^6)H(q) - q^4H(q^6)G(q)\right) \\ &= -q^3M(q^2)N(q) + q^4\frac{\chi(-q^7)}{\chi(-q)}\frac{\chi(-q^3)}{\chi(-q^{21})} + q^3\frac{\chi(-q^{21})}{\chi(-q^7)}\frac{\chi(-q)}{\chi(-q^3)}. \end{aligned}$$

Hence, solving the equation above for $M(q^2)N(q)$, we find that

$$M(q^2)N(q) = qT(q) + \frac{1}{T(q)}, \tag{8.5.14.9}$$

where $T(q)$ is defined by (8.5.14.5). Comparing (8.5.14.6) with (8.5.14.9), we find that

$$N(q^2)M(q) = M(q^2)N(q). \tag{8.5.14.10}$$

Equation (8.5.14.10) easily implies that $M(q) = N(q)$, which is what we wanted to prove, i.e., (8.3.16). To see this, let

$$M(q) := \sum_{n=0}^{\infty} a_n q^n \qquad \text{and} \qquad N(q) := \sum_{n=0}^{\infty} b_n q^n.$$

From the definitions (8.5.14.1) and (8.5.14.2), we see that $a_0 = b_0$. Then by an easy inductive argument, we find that $a_n = b_n$, for every positive integer n. Hence, $M(q) = N(q)$, as we wanted to demonstrate.

As an immediate consequence of our main identity, $M(q) = N(q)$, and (8.5.14.9), we derive the following curious corollary.

Corollary 8.5.1. *If $T(q)$ is defined by (8.5.14.5), then*

$$M(q)M(q^2) = N(q)N(q^2) = qT(q) + \frac{1}{T(q)}.$$

Next, we prove the second part of Entry 8.3.15, i.e., (8.3.17). Let $J(q)$ denote the right-hand side of (8.3.17), so that

$$J(q^2) = \frac{1}{2q}\left\{\chi(q)\chi(-q^3)\chi(q^7)\chi(-q^{21}) - \chi(-q)\chi(q^3)\chi(-q^7)\chi(q^{21})\right\}. \tag{8.5.14.11}$$

Recall that $M(q)$ and $N(q)$ are defined by (8.5.14.1) and (8.5.14.2), respectively. Using the previously established fact $M(q) = N(q)$, we see that it suffices to show that $M(q^2)N(q^2) = J^2(q^2)$.

Using (8.4.41) and (8.4.42) in (8.3.10) with q replaced by q^7, we obtain

$$\begin{aligned}&\left\{a(q^7)G(q^{42}) - q^7 b(q^7)H(q^{28})\right\} H(q^2)\\ &\quad - qG(q^2)\left\{q^7 a(q^7)H(q^{42}) + b(q^7)G(q^{28})\right\} = \frac{\chi(-q)}{\chi(-q^7)}.\end{aligned}$$

Upon rearrangement and the use of (8.5.14.2) and (8.3.11) with q replaced by q^2, we find that

$$a(q^7)N(q^2) - qb(q^7)\frac{\chi(-q^{14})}{\chi(-q^2)} = \frac{\chi(-q)}{\chi(-q^7)},$$

from which, by (8.4.40), we conclude that

$$N(q^2) = \frac{1}{\chi^2(q^7)} \left\{ \frac{\chi(-q)\chi(-q^{42})}{\chi(-q^7)\chi(-q^{14})} + q\frac{\chi(-q^7)\chi(-q^{14})}{\chi(-q^2)\chi(-q^{21})} \right\}. \tag{8.5.14.12}$$

Similarly, employing Lemma 8.4.4 in (8.3.11), we find that

$$\begin{aligned}&\left\{a(q)G(q^6) - qb(q)H(q^4)\right\} G(q^{14}) \\ &\quad + q^3 \left\{qa(q)H(q^6) + b(q)G(q^4)\right\} H(q^{14}) = \frac{\chi(-q^7)}{\chi(-q)}.\end{aligned}$$

Upon rearrangement and the use of (8.5.14.1) and (8.3.10) with q replaced by q^2, we obtain

$$a(q)M(q^2) - qb(q)\frac{\chi(-q^2)}{\chi(-q^{14})} = \frac{\chi(-q^7)}{\chi(-q)},$$

from which we similarly find that

$$M(q^2) = \frac{1}{\chi^2(q)} \left\{ \frac{\chi(-q^6)\chi(-q^7)}{\chi(-q)\chi(-q^2)} + q\frac{\chi(-q)\chi(-q^2)}{\chi(-q^3)\chi(-q^{14})} \right\}. \tag{8.5.14.13}$$

Next, recall that [55, p. 124, Entries 12 (v), (vi), (vii)]

$$\chi(q) = 2^{1/6}\left(\frac{q}{\alpha(1-\alpha)}\right)^{1/24} \quad \text{and} \quad \chi(-q) = 2^{1/6}\left(\frac{(1-\alpha)^2 q}{\alpha}\right)^{1/24}. \tag{8.5.14.14}$$

Let α, β, γ, and δ be of degrees 1, 3, 7, and 21, respectively. In (8.5.14.13), we use the representations (8.5.14.14) and (8.5.5.14) and conclude, after some algebra, that

$$\begin{aligned}M(q^2) = &\frac{2^{-1/3}q^{1/3}}{\{\alpha\beta^2\gamma(1-\alpha)(1-\beta)^2(1-\gamma)\}^{1/24}} \\ &\times \{\alpha^{1/4}(1-\beta)^{1/8}(1-\gamma)^{1/8} + \beta^{1/8}\gamma^{1/8}(1-\alpha)^{1/4}\}.\end{aligned} \tag{8.5.14.15}$$

Similarly, from (8.5.14.12), we find that

$$\begin{aligned}N(q^2) = &\frac{2^{-1/3}q^{1/3}}{\{\alpha\gamma\delta^2(1-\alpha)(1-\gamma)(1-\delta)^2\}^{1/24}} \\ &\times \{\gamma^{1/4}(1-\alpha)^{1/8}(1-\delta)^{1/8} + \alpha^{1/8}\delta^{1/8}(1-\gamma)^{1/4}\}.\end{aligned} \tag{8.5.14.16}$$

Lastly, from (8.5.14.11), we conclude that

$$\begin{aligned}J(q^2) = &\frac{2^{-1/3}q^{1/3}}{\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}} \\ &\times \{(1-\beta)(1-\delta)\}^{1/8} - \{(1-\alpha)(1-\gamma)\}^{1/8}\}.\end{aligned} \tag{8.5.14.17}$$

Therefore, the equation $M(q^2)N(q^2) = J^2(q^2)$ is equivalent to the modular equation

$$\begin{aligned}&\Big\{\{(1-\beta)(1-\delta)\}^{1/8}-\{(1-\alpha)(1-\gamma)\}^{1/8}\Big\}^2 \qquad (8.5.14.18)\\&=\Big\{\alpha^{1/4}(1-\beta)^{1/8}(1-\gamma)^{1/8}+\beta^{1/8}\gamma^{1/8}(1-\alpha)^{1/4}\Big\}\\&\quad\times\Big\{\gamma^{1/4}(1-\alpha)^{1/8}(1-\delta)^{1/8}+\alpha^{1/8}\delta^{1/8}(1-\gamma)^{1/4}\Big\}\\&=\big\{\alpha^2\gamma^2(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\big\}^{1/8}+\big\{\alpha^3\delta(1-\beta)(1-\gamma)^3\big\}^{1/8}\\&\quad+\big\{\gamma^3\beta(1-\delta)(1-\alpha)^3\big\}^{1/8}+\big\{\alpha\beta\gamma\delta(1-\alpha)^2(1-\gamma)^2\big\}^{1/8}.\end{aligned}$$

To prove (8.5.14.18), we invoke two modular equations, of degrees 3 and 7, respectively. Namely, if β has degree 3 over α, then [55, p. 230, Entry 5 (i)]

$$\{\alpha^3(1-\beta)\}^{1/8}-\{\beta(1-\alpha)^3\}^{1/8}=\{\beta(1-\beta)\}^{1/8}, \qquad (8.5.14.19)$$

and if γ has degree 7 over α, then [55, p. 314, Entry 19 (i)]

$$\{(1-\alpha)(1-\gamma)\}^{1/8}+\{\alpha\gamma\}^{1/8}=1. \qquad (8.5.14.20)$$

Let

$$\begin{aligned}&u:=(\alpha\gamma)^{1/8},\;\; v:=(\beta\delta)^{1/8},\;\; x:=\{\beta(1-\alpha)^3\}^{1/8},\\&y:=\{\alpha^3(1-\beta)\}^{1/8},\;\; \bar{x}:=\{\delta(1-\gamma)^3\}^{1/8},\;\text{ and }\;\bar{y}:=\{\gamma^3(1-\delta)\}^{1/8}.\end{aligned}$$

Since γ has degree 7 over α and δ has degree 7 over β, by (8.5.14.20),

$$\{(1-\alpha)(1-\gamma)\}^{1/8}=1-u \qquad \text{and} \qquad \{(1-\beta)(1-\delta)\}^{1/8}=1-v.$$

Since β has degree 3 over α and δ has degree 3 over γ, by (8.5.14.19),

$$y-x=\{\beta(1-\beta)\}^{1/8} \qquad \text{and} \qquad \bar{y}-\bar{x}=\{\delta(1-\delta)\}^{1/8}.$$

Using the trivial identity

$$y\bar{x}+\bar{y}x=x\bar{x}+y\bar{y}-(x-y)(\bar{x}-\bar{y}),$$

we conclude that

$$y\bar{x}+\bar{y}x=v(1-u)^3+u^3(1-v)-v(1-v).$$

Returning to the equation (8.5.14.18), we see that the right-hand side of (8.5.14.18) is

$$\begin{aligned}&u^2(1-u)(1-v)+y\bar{x}+\bar{y}x+uv(1-u)^2\\&=u^2(1-u)(1-v)+v(1-u)^3+u^3(1-v)-v(1-v)+uv(1-u)^2,\end{aligned}$$

which, after some algebra, simplifies to

$$(u-v)^2=\{(1-v)-(1-u)\}^2,$$

which is exactly the far left side of (8.5.14.18). Hence, the proof of (8.3.17) is complete.

8.5.15 Proof of Entry 8.3.16

We prove that both sides of (8.3.18) are independently equal to the right-hand side of (8.3.19). For brevity of exposition, we make the following definition. Assuming that S is a subset of the rational numbers and $\sum_{n\in S} a_n q^n$ is a generic q-series, we define an operator $\mathcal{L}$ acting on $\sum_{n\in S} a_n q^n$ by $\mathcal{L}\left(\sum_{n\in S} a_n q^n\right) = \sum_{n\in S'} a_n q^n$, where $S' \subseteq S$ is the set of all integers in S.

We apply Lemma 8.5.1 with q replaced by q^2 and q^{13} to respectively deduce that

$$f(-q^2)f(-q^{2/5}) = f^2(-q^4,-q^6) - q^{4/5}f^2(-q^2,-q^8) - q^{2/5}f(-q^2)f(-q^{10}), \tag{8.5.15.1}$$

$$f(-q^{13})f(-q^{13/5}) = f^2(-q^{26},-q^{39}) - q^{26/5}f^2(-q^{13},-q^{52}) - q^{13/5}f(-q^{13})f(-q^{65}). \tag{8.5.15.2}$$

Multiplying together (8.5.15.1) and (8.5.15.2), we obtain

$$\begin{aligned} f(-q^2)f(-q^{13})f(-q^{2/5})f(-q^{13/5}) &= f^2(-q^4,-q^6)f^2(-q^{26},-q^{39}) \qquad (8.5.15.3)\\ &+ q^6 f^2(-q^2,-q^8)f^2(-q^{13},-q^{52}) + q^3 f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65})\\ &- q^{26/5}f^2(-q^4,-q^6)f^2(-q^{13},-q^{52}) - q^{13/5}f^2(-q^4,-q^6)f(-q^{13})f(-q^{65})\\ &- q^{4/5}f^2(-q^2,-q^8)f^2(-q^{26},-q^{39}) + q^{17/5}f^2(-q^2,-q^8)f(-q^{13})f(-q^{65})\\ &- q^{2/5}f^2(-q^{26},-q^{39})f(-q^2)f(-q^{10}) + q^{28/5}f^2(-q^{13},-q^{52})f(-q^2)f(-q^{10}). \end{aligned}$$

We consider terms with integral powers of q on both sides of (8.5.15.3) and observe that

$$\begin{aligned} &\mathcal{L}\left(f(-q^2)f(-q^{13})f(-q^{2/5})f(-q^{13/5})\right)\\ &= f^2(-q^4,-q^6)f^2(-q^{26},-q^{39}) + q^6 f^2(-q^2,-q^8)f^2(-q^{13},-q^{52})\\ &\quad + q^3 f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65}). \end{aligned} \tag{8.5.15.4}$$

We now derive an alternative expression for the left-hand side of (8.5.15.4) above. To this end, we first employ (8.2.19) with $a=-q$, $b=-q^2$, and $n=5$ to deduce that

$$\begin{aligned} f(-q) &= f(-q^{35},-q^{40}) - qf(-q^{50},-q^{25}) + q^5 f(-q^{65},-q^{10})\\ &\quad - q^{12}f(-q^{80},-q^{-5}) + q^{22}f(-q^{95},-q^{-20})\\ &= f(-q^{35},-q^{40}) - qf(-q^{50},-q^{25}) + q^5 f(-q^{65},-q^{10})\\ &\quad + q^7 f(-q^{70},-q^5) - q^2 f(-q^{20},-q^{55}), \end{aligned} \tag{8.5.15.5}$$

after two applications of (8.2.5). We then apply (8.5.15.5) above to obtain representations for $f(-q^{2/5})$ and $f(-q^{13/5})$ by replacing q by $q^{2/5}$ and q by $q^{13/5}$, respectively. This gives us

$$f(-q^{2/5}) = f(-q^{14},-q^{16}) - q^{2/5}f(-q^{20},-q^{10}) + q^2 f(-q^{26},-q^4) \\ + q^{14/5}f(-q^{28},-q^2) - q^{4/5}f(-q^8,-q^{22}) \tag{8.5.15.6}$$

and

$$f(-q^{13/5}) = f(-q^{91},-q^{104}) - q^{13/5}f(-q^{130},-q^{65}) + q^{13}f(-q^{169},-q^{26}) \\ + q^{91/5}f(-q^{182},-q^{13}) - q^{26/5}f(-q^{52},-q^{143}). \tag{8.5.15.7}$$

Thus, multiplying (8.5.15.6) and (8.5.15.7), we see that

$$\begin{aligned}\mathcal{L}\left(f(-q^{2/5})f(-q^{13/5})\right) &= f(-q^{14},-q^{16})f(-q^{91},-q^{104}) \\ &+ q^3 f(-q^{10},-q^{20})f(-q^{65},-q^{130}) + q^{15}f(-q^4,-q^{26})f(-q^{26},-q^{169}) \\ &+ q^{21}f(-q^2,-q^{28})f(-q^{13},-q^{182}) + q^6 f(-q^8,-q^{22})f(-q^{52},-q^{143}) \\ &+ q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169}) + q^2 f(-q^4,-q^{26})f(-q^{91},-q^{104}) \\ &- q^8 f(-q^2,-q^{28})f(-q^{52},-q^{143}) - q^{19}f(-q^8,-q^{22})f(-q^{13},-q^{182}).\end{aligned} \tag{8.5.15.8}$$

Since $f(-q^2)f(-q^{13})$ contains only integral powers of q, it follows that

$$\begin{aligned}&\mathcal{L}\left(f(-q^2)f(-q^{13})f(-q^{2/5})f(-q^{13/5})\right) \\ &= f(-q^2)f(-q^{13})\mathcal{L}\left(f(-q^{2/5})f(-q^{13/5})\right) \\ &= f(-q^2)f(-q^{13})\big\{f(-q^{14},-q^{16})f(-q^{91},-q^{104}) \\ &\quad + q^3 f(-q^{10},-q^{20})f(-q^{65},-q^{130}) + q^{15}f(-q^4,-q^{26})f(-q^{26},-q^{169}) \\ &\quad + q^{21}f(-q^2,-q^{28})f(-q^{13},-q^{182}) + q^6 f(-q^8,-q^{22})f(-q^{52},-q^{143}) \\ &\quad + q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169}) + q^2 f(-q^4,-q^{26})f(-q^{91},-q^{104}) \\ &\quad - q^8 f(-q^2,-q^{28})f(-q^{52},-q^{143}) - q^{19}f(-q^8,-q^{22})f(-q^{13},-q^{182})\big\}.\end{aligned} \tag{8.5.15.9}$$

Equating the right-hand sides of (8.5.15.4) and (8.5.15.9), we deduce that

$$\begin{aligned}&f^2(-q^4,-q^6)f^2(-q^{26},-q^{39}) + q^6 f^2(-q^2,-q^8)f^2(-q^{13},-q^{52}) \\ &\quad + q^3 f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65}) \\ &= f(-q^2)f(-q^{13})\big\{f(-q^{14},-q^{16})f(-q^{91},-q^{104}) \\ &\quad + q^3 f(-q^{10},-q^{20})f(-q^{65},-q^{130}) + q^{15}f(-q^4,-q^{26})f(-q^{26},-q^{169}) \\ &\quad + q^{21}f(-q^2,-q^{28})f(-q^{13},-q^{182}) + q^6 f(-q^8,-q^{22})f(-q^{52},-q^{143}) \\ &\quad + q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169}) + q^2 f(-q^4,-q^{26})f(-q^{91},-q^{104}) \\ &\quad - q^8 f(-q^2,-q^{28})f(-q^{52},-q^{143}) - q^{19}f(-q^8,-q^{22})f(-q^{13},-q^{182})\big\}.\end{aligned} \tag{8.5.15.10}$$

We seek to simplify the right-hand side of (8.5.15.10). Applying (8.4.13) with $\alpha = 1$, $\beta = \frac{13}{2}$, $m = 1$, and $p = 15$, we see that

$$
\begin{aligned}
q^{-1/8}\sum_{k=1}^{7} F(1,\tfrac{13}{2},1,15,\tfrac{1}{2},k) &= f(-q^{14},-q^{16})f(-q^{91},-q^{104}) \qquad (8.5.15.11)\\
&+ qf(-q^{12},-q^{18})f(-q^{78},-q^{117}) + q^{3}f(-q^{10},-q^{20})f(-q^{65},-q^{130})\\
&+ q^{6}f(-q^{8},-q^{22})f(-q^{52},-q^{143}) + q^{10}f(-q^{6},-q^{24})f(-q^{39},-q^{156})\\
&+ q^{15}f(-q^{4},-q^{26})f(-q^{26},-q^{169}) + q^{21}f(-q^{2},-q^{28})f(-q^{13},-q^{182}).
\end{aligned}
$$

We now observe that five out of the seven terms appearing on the right-hand side of (8.5.15.11) also appear on the right-hand side of (8.5.15.10). This enables us to rewrite (8.5.15.10) as

$$
\begin{aligned}
&f^{2}(-q^{4},-q^{6})f^{2}(-q^{26},-q^{39}) + q^{6}f^{2}(-q^{2},-q^{8})f^{2}(-q^{13},-q^{52}) \qquad (8.5.15.12)\\
&\quad + q^{3}f(-q^{2})f(-q^{10})f(-q^{13})f(-q^{65})\\
&= f(-q^{2})f(-q^{13})\Big\{q^{-1/8}\sum_{k=1}^{7} F(1,\tfrac{13}{2},1,15,\tfrac{1}{2},k)\\
&\quad - qf(-q^{12},-q^{18})f(-q^{78},-q^{117}) - q^{10}f(-q^{6}-q^{24})f(-q^{39},-q^{156})\\
&\quad + q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169}) + q^{2}f(-q^{4},-q^{26})f(-q^{91},-q^{104})\\
&\quad - q^{8}f(-q^{2},-q^{28})f(-q^{52},-q^{143}) - q^{19}f(-q^{8},-q^{22})f(-q^{13},-q^{182})\Big\}.
\end{aligned}
$$

We next apply (8.4.13) again with $\alpha = 1$, $\beta = \frac{13}{2}$, $m = 11$, and $p = 15$. This yields

$$
\begin{aligned}
q^{-1/8}\sum_{k=1}^{7} F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k) &= q^{2}f(-q^{26},-q^{4})f(-q^{104},-q^{91}) \qquad (8.5.15.13)\\
&+ q^{19}f(-q^{48},-q^{-18})f(-q^{117},-q^{78}) + q^{53}f(-q^{70},-q^{-40})f(-q^{130},-q^{65})\\
&+ q^{104}f(-q^{92},-q^{-62})f(-q^{143},-q^{52}) + q^{172}f(-q^{114},-q^{-84})f(-q^{156},-q^{39})\\
&+ q^{257}f(-q^{136},-q^{-106})f(-q^{169},-q^{26})\\
&+ q^{359}f(-q^{158},-q^{-128})f(-q^{182},-q^{13}).
\end{aligned}
$$

After several applications of (8.2.5), we rewrite (8.5.15.13) as

$$
\begin{aligned}
q^{-1/8}\sum_{k=1}^{7} F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k) &= q^{2}f(-q^{4},-q^{26})f(-q^{91},-q^{104}) \qquad (8.5.15.14)\\
&- qf(-q^{12},-q^{18})f(-q^{78},-q^{117}) + q^{3}f(-q^{10},-q^{20})f(-q^{65},-q^{130})\\
&- q^{8}f(-q^{2},-q^{28})f(-q^{52},-q^{143}) - q^{10}f(-q^{6},-q^{24})f(-q^{39},-q^{156})\\
&+ q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169}) - q^{19}f(-q^{8},-q^{22})f(-q^{13},-q^{182}).
\end{aligned}
$$

We now note from (8.2.9) that $q^{3}f(-q^{10},-q^{20})f(-q^{65},-q^{130}) = q^{3}f(-q^{10})f(-q^{65})$, and upon comparing the right-hand side of (8.5.15.14) with that of (8.5.15.12), we rewrite (8.5.15.12) as

$$
\begin{aligned}
&f^2(-q^4,-q^6)f^2(-q^{26},-q^{39})+q^6f^2(-q^2,-q^8)f^2(-q^{13},-q^{52})\\
&\quad+q^3f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65})\\
&=f(-q^2)f(-q^{13})\Big\{q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac12,k)\\
&\quad+q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)-q^3f(-q^{10})f(-q^{65})\Big\},
\end{aligned}\tag{8.5.15.15}
$$

or equivalently upon applying the Jacobi triple product identity (8.2.6) to $f(-q^4,-q^6)$, $f(-q^{26},-q^{39})$, $f(-q^2,-q^8)$, and $f(-q^{13},-q^{52})$, we deduce that

$$
\begin{aligned}
&(f(-q^4,-q^6)f(-q^{26},-q^{39})+q^3f(-q^2,-q^8)f(-q^{13},-q^{52}))^2\\
&=f(-q^2)f(-q^{13})\Big\{q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac12,k)\\
&\qquad\qquad+q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)\Big\}.
\end{aligned}\tag{8.5.15.16}
$$

We now turn our attention to the two sums appearing on the right-hand side of (8.5.15.16). From (8.4.13), we see that

$$
\begin{aligned}
q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac12,k)&=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{\frac12\left\{u+\frac12+2t\right\}^2+13t^2}\\
&=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{\frac12\left\{u+\frac12\right\}^2+13t^2}\\
&=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}q^{\frac12\left\{u+\frac12\right\}^2}\sum_{t=-\infty}^{\infty}(-1)^tq^{13t^2}\\
&=\psi(q)\varphi(-q^{13})=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(q^{13};q^{26})^2_\infty(q^{26};q^{26})_\infty\\
&=\frac{(q^2;q^2)^2_\infty(q^{13};q^{13})^2_\infty}{(q;q)_\infty(q^{26};q^{26})_\infty}\\
&=\frac{f^2(-q^2)f^2(-q^{13})}{f(-q)f(-q^{26})},
\end{aligned}\tag{8.5.15.17}
$$

where we have utilized (8.2.7)–(8.2.9). Similarly, from (8.4.13), we find that

$$
\begin{aligned}
&q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)\\
&=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{\frac{17}{2}\left\{u+\frac12+\frac{22}{17}t\right\}^2+\frac{13}{17}t^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q^{-1/8}}{2} \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{17}{2}\left\{u+\frac{1}{2}+\frac{5}{17}t\right\}^2+\frac{13}{17}t^2} \\
&= \frac{1}{2} \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{2+\frac{3}{2}t^2+\frac{5}{2}t+5tu+\frac{17}{2}u(u+1)} \\
&= -\frac{q}{2} \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}, \qquad (8.5.15.18)
\end{aligned}
$$

where in the last equality we replaced t by $t-1$.

We now claim that

$$
\frac{1}{2} \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u} = f(-q)f(-q^{26}). \qquad (8.5.15.19)
$$

To this end, we dissect the series according as $u \equiv 0, 1, -1 \pmod 3$, respectively. We consider each of the three sums separately. If we replace u by $3u$ and t by $-t-5u$, we find that

$$
\begin{aligned}
&\sum_{\substack{u=-\infty \\ u\equiv 0 \pmod 3}}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u} \\
&= \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^{t+u} q^{\frac{3}{2}t^2+\frac{1}{2}t+39u^2+13u} \\
&= \sum_{u=-\infty}^{\infty} (-1)^u q^{39u^2+13u} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2+\frac{1}{2}t} = f(-q^{26})f(-q), \qquad (8.5.15.20)
\end{aligned}
$$

by (8.2.9). Next, if we replace u by $3u+1$ and t by $-t-5u$ in the series in (8.5.15.19), we find that

$$
\begin{aligned}
&\sum_{\substack{u=-\infty \\ u\equiv 1 \pmod 3}}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u} \\
&= \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^{t+u} q^{12+\frac{3}{2}t^2-\frac{9}{2}t+39u^2+39u} \\
&= \sum_{u=-\infty}^{\infty} (-1)^u q^{39u^2+39u} \sum_{t=-\infty}^{\infty} (-1)^t q^{12+\frac{3}{2}t^2-\frac{9}{2}t} = 0, \qquad (8.5.15.21)
\end{aligned}
$$

by (8.2.4). Finally, if we replace u by $3u-1$ and t by $-t-5u+2$ in the series in (8.5.15.19), we see that

$$
\sum_{\substack{u=-\infty \\ u\equiv -1 \pmod 3}}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}
$$

$$= \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^{t+u} q^{\frac{3}{2}t^2-\frac{1}{2}t+39u^2-13u}$$

$$= \sum_{u=-\infty}^{\infty} (-1)^u q^{39u^2-13u} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t} = f(-q^{26})f(-q), \quad (8.5.15.22)$$

by (8.2.9). Thus, in (8.5.15.20)–(8.5.15.22), we have shown that

$$\frac{1}{2} \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}$$
$$= \frac{1}{2}(f(-q)f(-q^{26}) + f(-q)f(-q^{26})) = f(-q)f(-q^{26}), \quad (8.5.15.23)$$

as claimed in (8.5.15.19). Thus, from (8.5.15.18) and (8.5.15.23), we deduce that

$$q^{-1/8} \sum_{k=1}^{7} F(1, \tfrac{13}{2}, 11, 15, \tfrac{17}{2}, k) = -qf(-q)f(-q^{26}). \quad (8.5.15.24)$$

Finally, we insert (8.5.15.17) and (8.5.15.24) into (8.5.15.16) to arrive at

$$(f(-q^4,-q^6)f(-q^{26},-q^{39}) + q^3 f(-q^2,-q^8)f(-q^{13},-q^{52}))^2$$
$$= f(-q^2)f(-q^{13}) \left\{ \frac{f^2(-q^2)f^2(-q^{13})}{f(-q)f(-q^{26})} - qf(-q)f(-q^{26}) \right\}. \quad (8.5.15.25)$$

Dividing both sides of (8.5.15.25) by $f^2(-q^2)f^2(-q^{13})$ and taking square roots, we obtain

$$\frac{f(-q^4,-q^6)f(-q^{26},-q^{39})}{f(-q^2)f(-q^{13})} + q^3 \frac{f(-q^2,-q^8)f(-q^{13},-q^{52})}{f(-q^2)f(-q^{13})}$$
$$= \sqrt{\frac{f(-q^2)f(-q^{13})}{f(-q)f(-q^{26})} - q\frac{f(-q)f(-q^{26})}{f(-q^2)f(-q^{13})}}$$
$$= \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}, \quad (8.5.15.26)$$

by (8.2.14). Using (8.2.11), we see that we have shown that the left side in (8.3.18) is equal to (8.3.19).

We now turn to the right-hand side of (8.3.18) and show that it equals the expression in (8.3.19). Our argument is brief, since the proof is similar to the previous proof above. We apply Lemma 8.5.1, and then apply it a second time with q replaced by q^{26}. Then multiply the two resulting equalities together to obtain

$$
\begin{aligned}
f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5}) &= f^2(-q^2,-q^3)f^2(-q^{52},-q^{78}) \qquad (8.5.15.27)\\
&- q^{2/5}f^2(-q,-q^4)f^2(-q^{52},-q^{78}) - q^{1/5}f(-q)f(-q^5)f^2(-q^{52},-q^{78})\\
&- q^{52/5}f^2(-q^2,-q^3)f^2(-q^{26},-q^{104}) + q^{54/5}f^2(-q,-q^4)f^2(-q^{26},-q^{104})\\
&+ q^{53/5}f(-q)f(-q^5)f^2(-q^{26},-q^{104}) - q^{26/5}f^2(-q^2,-q^3)f(-q^{26})f(-q^{130})\\
&+ q^{28/5}f^2(-q,-q^4)f(-q^{26})f(-q^{130}) + q^{27/5}f(-q)f(-q^5)f(-q^{26})f(-q^{130}).
\end{aligned}
$$

Recalling the definition of $\mathcal{L}$ at the beginning of this section, we have

$$
\begin{aligned}
\mathcal{L}\left(q^{-2/5}f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5})\right) &= -f^2(-q,-q^4)f^2(-q^{52},-q^{78})\\
&- q^{10}f^2(-q^2,-q^3)f^2(-q^{26},-q^{104}) + q^5f(-q)f(-q^5)f(-q^{26})f(-q^{130}).
\end{aligned} \tag{8.5.15.28}
$$

We then apply (8.5.15.5) to obtain representations for $f(-q^{1/5})$ and $f(-q^{26/5})$ by replacing q by $q^{1/5}$ and q by $q^{26/5}$, respectively. This gives us

$$
\begin{aligned}
f(-q^{1/5}) = f(-q^7,-q^8) &- q^{1/5}f(-q^{10},-q^5) + qf(-q^{13},-q^2)\\
&+ q^{7/5}f(-q^{14},-q) - q^{2/5}f(-q^4,-q^{11})
\end{aligned} \tag{8.5.15.29}
$$

and

$$
\begin{aligned}
f(-q^{26/5}) = f(-q^{182},-q^{208}) &- q^{26/5}f(-q^{260},-q^{130}) + q^{26}f(-q^{338},-q^{52})\\
&+ q^{182/5}f(-q^{364},-q^{26}) - q^{52/5}f(-q^{104},-q^{286}).
\end{aligned} \tag{8.5.15.30}
$$

Thus, since $f(-q)f(-q^{26})$ contains only terms with integral powers, we find, upon using (8.5.15.29) and (8.5.15.30), that

$$
\begin{aligned}
&\mathcal{L}\left(q^{-2/5}f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5})\right) \qquad (8.5.15.31)\\
&= f(-q)f(-q^{26})\mathcal{L}\left(q^{-2/5}f(-q^{1/5})f(-q^{26/5})\right)\\
&= f(-q)f(-q^{26})\big\{ - f(-q^4,-q^{11})f(-q^{182},-q^{208})\\
&\quad + q^{37}f(-q^2,-q^{13})f(-q^{26},-q^{364}) + q^{36}f(-q^7,-q^8)f(-q^{26},-q^{364})\\
&\quad + q^5f(-q^5,-q^{10})f(-q^{130},-q^{260}) - q^{10}f(-q^7,-q^8)f(-q^{104},-q^{286})\\
&\quad - q^{11}f(-q^2,-q^{13})f(-q^{104},-q^{286}) + qf(-q,-q^{14})f(-q^{182},-q^{208})\\
&\quad + q^{27}f(-q,-q^{14})f(-q^{52},-q^{338}) - q^{26}f(-q^4,-q^{11})f(-q^{52},-q^{338})\big\}.
\end{aligned}
$$

Now from (8.4.13) and several applications of (8.2.5), we see that

$$
\begin{aligned}
&q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,7,15,\tfrac{5}{2},k) = f(-q^4,-q^{11})f(-q^{182},-q^{208}) \qquad (8.5.15.32)\\
&- q^2f(-q^3,-q^{12})f(-q^{156},-q^{234}) - q^5f(-q^5,-q^{10})f(-q^{130},-q^{260})\\
&+ q^{11}f(-q^2,-q^{13})f(-q^{104},-q^{286}) + q^{17}f(-q^6,-q^9)f(-q^{78},-q^{312})
\end{aligned}
$$

$$- q^{27}f(-q,-q^{14})f(-q^{52},-q^{338}) - q^{36}f(-q^7,-q^8)f(-q^{26},-q^{364})$$

and

$$q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,13,15,\tfrac{13}{2},k) = qf(-q,-q^{14})f(-q^{182},-q^{208}) \tag{8.5.15.33}$$
$$- q^2f(-q^3,-q^{12})f(-q^{156},-q^{234}) + q^5f(-q^5,-q^{10})f(-q^{130},-q^{260})$$
$$- q^{10}f(-q^7,-q^8)f(-q^{104},-q^{286}) + q^{17}f(-q^6,-q^9)f(-q^{78},-q^{312})$$
$$- q^{26}f(-q^4,-q^{11})f(-q^{52},-q^{338}) + q^{37}f(-q^2,-q^{13})f(-q^{26},-q^{364}).$$

Comparing the right-hand sides of (8.5.15.31), (8.5.15.32), and (8.5.15.33), we see that

$$\mathcal{L}\left(q^{-2/5}f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5})\right)$$
$$= f(-q)f(-q^{26})\Big\{-q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,7,15,\tfrac{5}{2},k)$$
$$+ q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,13,15,\tfrac{13}{2},k) - q^5f(-q^5)f(-q^{130})\Big\}. \tag{8.5.15.34}$$

Now, combining the right-hand sides of (8.5.15.28) and (8.5.15.34), applying the Jacobi triple product identity (8.2.6) to $f(-q,-q^4)$, $f(-q^{52},-q^{78})$, $f(-q^2,-q^3)$, and $f(-q^{26},-q^{104})$, and simplifying, we deduce that

$$\left(f(-q,-q^4)f(-q^{52},-q^{78}) - q^5f(-q^2,-q^3)f(-q^{26},-q^{104})\right)^2$$
$$= -f(-q)f(-q^{26})\left\{-q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,7,15,\tfrac{5}{2},k)\right.$$
$$\left.+q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,13,15,\tfrac{13}{2},k)\right\}. \tag{8.5.15.35}$$

We now concentrate on the two sums arising on the right-hand side of (8.5.15.35) above. From (8.4.13), we have

$$q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,13,15,\tfrac{13}{2},k) = \frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{13}{2}\left\{u+\frac{1}{2}+t\right\}^2+t^2}$$
$$= \frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{13}{2}\left\{u+\frac{1}{2}\right\}^2+t^2} = \frac{q}{2}\sum_{u=-\infty}^{\infty}q^{\frac{13}{2}u(u+1)}\sum_{t=-\infty}^{\infty}(-1)^t q^{t^2}$$
$$= q\psi(q^{13})\varphi(-q) = q\frac{(q^{26};q^{26})_\infty}{(q^{13};q^{26})_\infty}(q;q^2)_\infty^2(q^2;q^2)_\infty$$
$$= q\frac{(q^{26};q^{26})_\infty^2(q;q)_\infty^2}{(q^{13};q^{13})_\infty(q^2;q^2)_\infty} = q\frac{f^2(-q^{26})f^2(-q)}{f(-q^{13})f(-q^2)}, \tag{8.5.15.36}$$

where we have used (8.2.7)–(8.2.9). Similarly, from (8.4.13), we find that

$$\begin{aligned} q^{-5/8}\sum_{k=1}^{7} F(\tfrac{1}{2},13,7,15,\tfrac{5}{2},k) &= \frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{5}{2}\left\{u+\frac{1}{2}+\frac{7}{5}t\right\}^2+\frac{13}{5}t^2} \\ &= \frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{5}{2}\left\{u+\frac{1}{2}+\frac{2}{5}t\right\}^2+\frac{13}{5}t^2} \\ &= \frac{1}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{3t^2+t+2tu+\frac{5}{2}u(u+1)}. \end{aligned} \tag{8.5.15.37}$$

As in the previous proof, we dissect the series above according as $u \equiv 0, 1, -1 \pmod 3$. Assuming that $u \equiv 0 \pmod 3$, we replace u by $3u$ and t by $-t-u$ to find that

$$\begin{aligned} &\sum_{\substack{u=-\infty \\ u\equiv 0 \pmod 3}}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{3t^2+t+2tu+\frac{5}{2}u(u+1)} \\ &= \sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t+u} q^{3t^2-t+\frac{39}{2}u^2+\frac{13}{2}u} \\ &= \sum_{u=-\infty}^{\infty}(-1)^u q^{\frac{13}{2}u(3u+1)}\sum_{t=-\infty}^{\infty}(-1)^t q^{t(3t-1)} = f(-q^{13})f(-q^2), \end{aligned} \tag{8.5.15.38}$$

by (8.2.9). Next, if we replace u by $3u+1$ and t by $t-u$ in the series in (8.5.15.19), we find that

$$\begin{aligned} &\sum_{\substack{u=-\infty \\ u\equiv 1 \pmod 3}}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{3t^2+t+2tu+\frac{5}{2}u(u+1)} \\ &= \sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t+u} q^{3t^2+3t+\frac{39}{2}u^2+\frac{39}{2}u+5} \\ &= \sum_{u=-\infty}^{\infty}(-1)^u q^{\frac{39}{2}u^2+\frac{39}{2}u+5}\sum_{t=-\infty}^{\infty}(-1)^t q^{3t^2+3t} = 0, \end{aligned} \tag{8.5.15.39}$$

by (8.2.4). Finally, if we replace u by $3u-1$ and t by $t-u$, we obtain

$$\begin{aligned} &\sum_{\substack{u=-\infty \\ u\equiv -1 \pmod 3}}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{3t^2+t+2tu+\frac{5}{2}u(u+1)} \\ &= \sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t+u} q^{3t^2-t+\frac{39}{2}u^2-\frac{13}{2}u} \\ &= f(-q^{13})f(-q^2), \end{aligned} \tag{8.5.15.40}$$

by (8.2.9). Thus, from (8.5.15.37)–(8.5.15.40), we have shown that

$$q^{-5/8}\sum_{k=1}^{7}F(\tfrac{1}{2},13,7,15,\tfrac{5}{2},k)=\frac{1}{2}\left(f(-q^{13})f(-q^{2})+f(-q^{13})f(-q^{2})\right)$$
$$=f(-q^{13})f(-q^{2}). \tag{8.5.15.41}$$

Finally, we insert (8.5.15.36) and (8.5.15.41) into (8.5.15.35) and conclude that

$$\begin{aligned}&\left(f(-q,-q^{4})f(-q^{52},-q^{78})-q^{5}f(-q^{2},-q^{3})f(-q^{26},-q^{104})\right)^{2}\\&=-f(-q)f(-q^{26})\left\{-f(-q^{13})f(-q^{2})+q\frac{f^{2}(-q^{26})f^{2}(-q)}{f(-q^{13})f(-q^{2})}\right\}.\end{aligned} \tag{8.5.15.42}$$

Dividing both sides of (8.5.15.42) by $f^{2}(-q)f^{2}(-q^{26})$ and taking square roots, we arrive at

$$\begin{aligned}&\frac{f(-q,-q^{4})f(-q^{52},-q^{78})-q^{5}f(-q^{2},-q^{3})f(-q^{26},-q^{104})}{f(-q)f(-q^{26})}\\&=\sqrt{\frac{f(-q^{2})f(-q^{13})}{f(-q)f(-q^{26})}-q\frac{f(-q)f(-q^{26})}{f(-q^{2})f(-q^{13})}}\\&=\sqrt{\frac{\chi(-q^{13})}{\chi(-q)}-q\frac{\chi(-q)}{\chi(-q^{13})}},\end{aligned} \tag{8.5.15.43}$$

which completes the proof of the second part of Entry 8.3.16.

8.5.16 Proof of Entry 8.3.17

Using the product representations of $\chi(q)$ and $f(-q)$ given in (8.2.9) and (8.2.10), respectively, together with (8.2.7) and (8.2.8), we find that

$$\varphi(q)=f(q,q)=(-q;q^{2})_{\infty}^{2}(q^{2};q^{2})_{\infty}=\chi^{2}(q)f(-q^{2}) \tag{8.5.16.1}$$

and

$$\psi(q)=f(q,q^{3})=\frac{(q^{2};q^{2})_{\infty}}{(q;q^{2})_{\infty}}=\frac{(q;q)_{\infty}}{(q;q^{2})_{\infty}}\frac{1}{(q;q^{2})_{\infty}}=\frac{f(-q)}{\chi^{2}(-q)}. \tag{8.5.16.2}$$

By (8.2.11), (8.5.16.2), and (8.5.16.1) with q replaced by $q^{1/2}$ and $-q^{1/2}$, respectively, we find that Entry 8.3.17 is equivalent to the identity

$$\begin{aligned}&f(-q^{2},-q^{3})f(-q^{38},-q^{57})+q^{4}f(-q,-q^{4})f(-q^{19},-q^{76})\\&=f(-q)f(-q^{19})\left\{\frac{\chi^{2}(q^{1/2})\chi^{2}(q^{19/2})}{4\sqrt{q}}-\frac{\chi^{2}(-q^{1/2})\chi^{2}(-q^{19/2})}{4\sqrt{q}}\right.\\&\qquad\left.-\frac{q^{2}}{\chi^{2}(-q)\chi^{2}(-q^{19})}\right\}\\&=\frac{1}{4\sqrt{q}}\left(\varphi(q^{1/2})\varphi(q^{19/2})-\varphi(-q^{1/2})\varphi(-q^{19/2})\right)-q^{2}\psi(q)\psi(q^{19}).\end{aligned} \tag{8.5.16.3}$$

We now apply Theorem 8.4.1 with the parameters $\epsilon_1 = \epsilon_2 = 0$, $a = b = q$, $c = d = q^{19}$, $\alpha = 1$, $\beta = 19$, and $m = 20$. Accordingly, we deduce that

$$\begin{aligned}
\varphi(q)\varphi(q^{19}) &= f(q^{20}, q^{20})f(q^{380}, q^{380}) + q^{19}f(q^{-18}, q^{58})f(q^{342}, q^{418}) \\
&\quad + q^{76}f(q^{-56}, q^{96})f(q^{304}, q^{456}) + q^{171}f(q^{-94}, q^{134})f(q^{266}, q^{494}) \\
&\quad + q^{304}f(q^{-132}, q^{172})f(q^{228}, q^{532}) + q^{475}f(q^{-170}, q^{210})f(q^{190}, q^{570}) \\
&\quad + q^{684}f(q^{-208}, q^{248})f(q^{152}, q^{608}) + q^{931}f(q^{-246}, q^{286})f(q^{114}, q^{646}) \\
&\quad + q^{1216}f(q^{-284}, q^{324})f(q^{76}, q^{684}) + q^{1539}f(q^{-322}, q^{362})f(q^{38}, q^{722}) \\
&\quad + q^{1900}f(q^{-360}, q^{400})f(1, q^{760}) + q^{2299}f(q^{-398}, q^{438})f(q^{-38}, q^{798}) \\
&\quad + q^{2736}f(q^{-436}, q^{476})f(q^{-76}, q^{836}) + q^{3211}f(q^{-474}, q^{514})f(q^{-114}, q^{874}) \\
&\quad + q^{3724}f(q^{-512}, q^{552})f(q^{-152}, q^{912}) + q^{4275}f(q^{-550}, q^{590})f(q^{-190}, q^{950}) \\
&\quad + q^{4864}f(q^{-588}, q^{628})f(q^{-228}, q^{988}) + q^{5491}f(q^{-626}, q^{666})f(q^{-266}, q^{1026}) \\
&\quad + q^{6156}f(q^{-664}, q^{704})f(q^{-304}, q^{1064}) + q^{6859}f(q^{-702}, q^{742})f(q^{-342}, q^{1102}) \\
&= f(q^{20}, q^{20})f(q^{380}, q^{380}) + 2\,qf(q^{18}, q^{22})f(q^{342}, q^{418}) \\
&\quad + 2\,q^{4}f(q^{16}, q^{24})f(q^{304}, q^{456}) + 2\,q^{9}f(q^{14}, q^{26})f(q^{266}, q^{494}) \\
&\quad + 2\,q^{16}f(q^{12}, q^{28})f(q^{228}, q^{532}) + 2\,q^{25}f(q^{10}, q^{30})f(q^{190}, q^{570}) \\
&\quad + 2\,q^{36}f(q^{8}, q^{32})f(q^{152}, q^{608}) + 2\,q^{49}f(q^{6}, q^{34})f(q^{114}, q^{646}) \\
&\quad + 2\,q^{64}f(q^{4}, q^{36})f(q^{76}, q^{684}) + 2\,q^{81}f(q^{2}, q^{38})f(q^{38}, q^{722}) \\
&\quad + q^{100}f(1, q^{40})f(1, q^{760}),
\end{aligned} \tag{8.5.16.4}$$

after several applications of (8.2.5). Upon replacing q by $-q$ in (8.5.16.4), we conclude that

$$\begin{aligned}
&\frac{1}{4q}\left(\varphi(q)\varphi(q^{19}) - \varphi(-q)\varphi(-q^{19})\right) \\
&= f(q^{18}, q^{22})f(q^{342}, q^{418}) + q^{8}f(q^{14}, q^{26})f(q^{266}, q^{494}) \\
&\quad + q^{24}f(q^{10}, q^{30})f(q^{190}, q^{570}) + q^{48}f(q^{6}, q^{34})f(q^{114}, q^{646}) \\
&\quad + q^{80}f(q^{2}, q^{38})f(q^{38}, q^{722}).
\end{aligned} \tag{8.5.16.5}$$

Next, we employ (8.4.14) with the two sets of parameters $\alpha_1 = 1$, $\beta_1 = 19$, $m_1 = 1$, $p_1 = 2$, $\lambda_1 = 10$ and $\alpha_2 = 1$, $\beta_2 = 19$, $m_2 = 9$, $p_2 = 10$, $\lambda_2 = 10$. We find that the conditions in (8.4.9) are satisfied. Hence, using (8.4.18) and (8.4.14), we find that

$$\begin{aligned}
&q^{5/2}\psi(-q)\psi(-q^{19}) = q^{5/2}f(-q^{19}, -q)f(-q^{209}, -q^{171}) \\
&+ q^{45/2}f(-q^{37}, -q^{-17})f(-q^{247}, -q^{133}) + q^{125/2}f(-q^{55}, -q^{-35})f(-q^{285}, -q^{95}) \\
&+ q^{245/2}f(-q^{73}, -q^{-53})f(-q^{323}, -q^{57}) + q^{405/2}f(-q^{91}, -q^{-71})f(-q^{361}, -q^{19}).
\end{aligned}$$

After several applications of (8.2.5), we obtain the identity

$$\begin{aligned}\psi(-q)\psi(-q^{19}) &= f(-q,-q^{19})f(-q^{171},-q^{209}) \\ &\quad - q^3 f(-q^3,-q^{17})f(-q^{133},-q^{247}) + q^{10} f(-q^5,-q^{15})f(-q^{95},-q^{285}) \\ &\quad - q^{21} f(-q^7,-q^{13})f(-q^{57},-q^{323}) + q^{36} f(-q^9,-q^{11})f(-q^{19},-q^{361}).\end{aligned} \tag{8.5.16.6}$$

By (8.2.19), with $a=-q^2$, $b=q^3$, and $n=2$, and with $a=q$, $b=-q^4$, and $n=2$, respectively,

$$f(-q^2,q^3) = f(-q^9,-q^{11}) - q^2 f(-q^{19},-q), \tag{8.5.16.7}$$

$$f(q,-q^4) = f(-q^7,-q^{13}) + q f(-q^{17},-q^3). \tag{8.5.16.8}$$

Using (8.5.16.5) with q replaced by $q^{1/2}$ and (8.5.16.6) with q replaced by $-q$, we conclude that

$$\begin{aligned}&\frac{1}{4\sqrt{q}}\Big(\varphi(q^{1/2})\varphi(q^{19/2}) - \varphi(-q^{1/2})\varphi(-q^{19/2})\Big) - q^2\psi(q)\psi(q^{19}) \\ &= f(q^9,q^{11})f(q^{171},q^{209}) + q^4 f(q^7,q^{13})f(q^{133},q^{247}) \\ &\quad + q^{24} f(q^3,q^{17})f(q^{57},q^{323}) + q^{40} f(q,q^{19})f(q^{19},q^{361}) \\ &\quad - q^2 f(q,q^{19})f(q^{171},q^{209}) - q^5 f(q^3,q^{17})f(q^{133},q^{247}) \\ &\quad - q^{23} f(q^7,q^{13})f(q^{57},q^{323}) - q^{38} f(q^9,q^{11})f(q^{19},q^{361}) \\ &= \left(f(q^9,q^{11}) - q^2 f(q,q^{19})\right)\left(f(q^{171},q^{209}) - q^{38} f(q^{19},q^{361})\right) \\ &\quad + q^4\left(f(q^7,q^{13}) - q f(q^3,q^{17})\right)\left(f(q^{133},q^{247}) - q^{19} f(q^{57},q^{323})\right) \\ &= f(-q^2,-q^3)f(-q^{38},-q^{57}) + q^4 f(-q,-q^4)f(-q^{19},-q^{76}),\end{aligned}$$

where in the last step we used (8.5.16.7) and (8.5.16.8) with q replaced by $-q$ and $-q^{19}$, respectively. This completes the proof of Entry 8.3.17.

8.5.17 Proof of Entry 8.3.18

The following proof of Entry 8.3.18 is due to Bressoud [81].

By (8.2.11), (8.2.7), (8.2.8), and (8.2.14), it is easy to see that (8.3.21) is equivalent to the identity

$$\begin{aligned}&f(-q,-q^4)f(-q^{62},-q^{93}) - q^6 f(-q^2,-q^3)f(-q^{31},-q^{124}) \\ &= \frac{1}{2q}\varphi(-q^2)\varphi(-q^{62}) - \frac{1}{2q}\varphi(-q)\varphi(-q^{31}) + q^3\psi(-q)\psi(-q^{31}).\end{aligned} \tag{8.5.17.1}$$

By (8.4.16) and (8.4.13), with the set of parameters $\alpha=\frac{1}{2}$, $\beta=\frac{31}{2}$, $m=3$, $p=5$, and $\lambda=4$, we find that

$$\begin{aligned}&q f(-q,-q^4)f(-q^{62},-q^{93}) - q^7 f(-q^2,-q^3)f(-q^{31},-q^{124}) \\ &= \sum_{k=1}^{2} F(\tfrac{1}{2},\tfrac{31}{2},3,5,4,k) = \frac{1}{2}\sum_{u,\,t=-\infty}^{\infty}(-1)^t q^I =: \frac{1}{2}R,\end{aligned} \tag{8.5.17.2}$$

where, by (8.4.7), I is given by

$$I = 4\left\{u + \frac{1}{2} + \frac{3t}{8}\right\}^2 + \frac{31}{16}t^2 = (2u+1)^2 + \frac{3}{2}(2u+1)t + \frac{5}{2}t^2. \tag{8.5.17.3}$$

Therefore, by (8.5.17.1)–(8.5.17.3), it suffices to prove that

$$\begin{aligned} R &= \sum_{u,\,t=-\infty}^{\infty} (-1)^t q^{(2u+1)^2 + \frac{3}{2}(2u+1)t + \frac{5}{2}t^2} \\ &= \varphi(-q^2)\varphi(-q^{62}) - \varphi(-q)\varphi(-q^{31}) + 2q^4\psi(-q)\psi(-q^{31}). \end{aligned} \tag{8.5.17.4}$$

We establish (8.5.17.4) by a series of changes of the indices of summation. To that end

$$\begin{aligned} R &= \sum_{u,\,t=-\infty}^{\infty} (-1)^t q^{(2u+1)^2 + \frac{3}{2}(2u+1)t + \frac{5}{2}t^2} \\ &= \sum_{j=0}^{1} \sum_{u,\,r=-\infty}^{\infty} (-1)^{2r+j} q^{(2u+1)^2 + \frac{3}{2}(2u+1)(2r+j) + \frac{5}{2}(2r+j)^2} \\ &= \sum_{u,\,r=-\infty}^{\infty} q^{4u^2+10r^2+6ru+4u+3r+1} - \sum_{u,\,r=-\infty}^{\infty} q^{4u^2+10r^2+6ru+7u+13r+5} \\ &= \sum_{s,\,r=-\infty}^{\infty} q^{4(s-r)^2+10r^2+6r(s-r)+4(s-r)+3r+1} \\ &\quad - \sum_{s,\,r=-\infty}^{\infty} q^{4(s-r-1)^2+10r^2+6r(s-r-1)+7(s-r-1)+13r+5} \\ &= \sum_{s,\,r=-\infty}^{\infty} q^{(2s+1)^2-(2s+1)r+8r^2} - \sum_{s,\,r=-\infty}^{\infty} q^{4s^2-s(2r+1)+2(2r+1)^2} \\ &= \sum_{\substack{s,\,r=-\infty \\ s\,\text{odd}}}^{\infty} q^{s^2-sr+8r^2} - \sum_{\substack{s,\,r=-\infty \\ r\,\text{odd}}}^{\infty} q^{4s^2-sr+2r^2}. \end{aligned} \tag{8.5.17.5}$$

But trivially,

$$\sum_{\substack{s,\,r=-\infty \\ r\,\text{even}}}^{\infty} q^{4s^2-sr+2r^2} - \sum_{\substack{s,\,r=-\infty \\ s\,\text{even}}}^{\infty} q^{s^2-sr+8r^2} = 0. \tag{8.5.17.6}$$

Therefore, returning to (8.5.17.5), we find that

$$R = \sum_{\substack{s,\,r=-\infty \\ s\,\text{odd}}}^{\infty} q^{s^2-sr+8r^2} - \sum_{\substack{s,\,r=-\infty \\ r\,\text{odd}}}^{\infty} q^{4s^2-sr+2r^2} \tag{8.5.17.7}$$

$$
+ \sum_{\substack{s,\, r=-\infty \\ r \text{ even}}}^{\infty} q^{4s^2-sr+2r^2} - \sum_{\substack{s,\, r=-\infty \\ s \text{ even}}}^{\infty} q^{s^2-sr+8r^2}
$$
$$
= \sum_{s,\, r=-\infty}^{\infty} (-1)^r q^{4s^2-sr+2r^2} - \sum_{s,\, r=-\infty}^{\infty} (-1)^s q^{s^2-sr+8r^2} =: R_1 - R_2.
$$

Next, we evaluate R_1 and R_2 separately. First,

$$
\begin{aligned}
R_1 &= \sum_{s,\, r=-\infty}^{\infty} (-1)^r q^{4s^2-sr+2r^2} = \sum_{j=0}^{3} \sum_{t,\, r=-\infty}^{\infty} (-1)^r q^{4(4t+j)^2-(4t+j)r+2r^2} \\
&= \sum_{j=0}^{3} \sum_{t,\, r=-\infty}^{\infty} (-1)^r q^{64t^2+32tj+4j^2-4rt-jr+2r^2} \\
&= \sum_{j=0}^{3} \sum_{t,\, r=-\infty}^{\infty} (-1)^r q^{62(t+\frac{j}{4})^2+2(t-r+\frac{j}{4})^2} \\
&= \sum_{j=0}^{3} \sum_{t,\, s=-\infty}^{\infty} (-1)^{t-s} q^{62(t+\frac{j}{4})^2+2(s+\frac{j}{4})^2} \\
&= \sum_{j=0}^{3} \left\{ \sum_{t=-\infty}^{\infty} (-1)^t q^{62(t+\frac{j}{4})^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{2(s+\frac{j}{4})^2} \right\}.
\end{aligned} \tag{8.5.17.8}
$$

Observe that

$$
\begin{aligned}
\sum_{s=-\infty}^{\infty} (-1)^s q^{(s+\frac{1}{4})^2} &= \sum_{s=-\infty}^{\infty} (-1)^s q^{(s+1-\frac{3}{4})^2} = \sum_{s=-\infty}^{\infty} (-1)^{s-1} q^{(s-\frac{3}{4})^2} \\
&= -\sum_{s=-\infty}^{\infty} (-1)^{-s} q^{(-s-\frac{3}{4})^2} = -\sum_{s=-\infty}^{\infty} (-1)^s q^{(s+\frac{3}{4})^2}.
\end{aligned} \tag{8.5.17.9}
$$

Thus, in (8.5.17.8), the contributions from the terms when $j=1$ and $j=3$ are the same. By (8.2.4), the contribution from the term $j=2$ is 0. Therefore, by (8.2.1), (8.2.7), and (8.2.8), we conclude that

$$
\begin{aligned}
R_1 &= \sum_{t=-\infty}^{\infty} (-1)^t q^{62t^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{2s^2} \\
&\quad + 2q^4 \sum_{t=-\infty}^{\infty} (-1)^t q^{62t^2+31t} \sum_{s=-\infty}^{\infty} (-1)^s q^{2s^2+s} \\
&= \varphi(-q^2)\varphi(-q^{62}) + 2q^4 f(-q,-q^3) f(-q^{31},-q^{93}) \\
&= \varphi(-q^2)\varphi(-q^{62}) + 2q^4 \psi(-q)\psi(-q^{31}).
\end{aligned} \tag{8.5.17.10}
$$

Similarly,

$$
\begin{aligned}
R_2 &= \sum_{s,\,r=-\infty}^{\infty}(-1)^s q^{s^2-sr+8r^2} = \sum_{j=0}^{1}\sum_{s,\,t=-\infty}^{\infty}(-1)^s q^{s^2-s(2t+j)+8(2t+j)^2}\\
&= \sum_{j=0}^{1}\sum_{s,\,t=-\infty}^{\infty}(-1)^s q^{s^2-sj-2st+32t^2+32tj+8j^2}\\
&= \sum_{j=0}^{1}\sum_{s,\,t=-\infty}^{\infty}(-1)^s q^{31(t+\frac{j}{2})^2+(t-s+\frac{j}{2})^2}\\
&= \sum_{j=0}^{1}\sum_{t,\,r=-\infty}^{\infty}(-1)^{t-r} q^{31(t+\frac{j}{2})^2+(r+\frac{j}{2})^2}\\
&= \sum_{j=0}^{1}\left\{\sum_{t=-\infty}^{\infty}(-1)^t q^{31(t+\frac{j}{2})^2}\sum_{r=-\infty}^{\infty}(-1)^r q^{(r+\frac{j}{2})^2}\right\}\\
&= \sum_{t=-\infty}^{\infty}(-1)^t q^{31t^2}\sum_{r=-\infty}^{\infty}(-1)^r q^{r^2} = \varphi(-q)\varphi(-q^{31}),
\end{aligned}
\tag{8.5.17.11}
$$

where we used (8.2.4) again. By (8.5.17.7), (8.5.17.10), and (8.5.17.11), we conclude that

$$R = \varphi(-q^2)\varphi(-q^{62}) + 2q^4\psi(-q)\psi(-q^{31}) - \varphi(-q)\varphi(-q^{31}),$$

which is (8.5.17.4). Hence, the proof of Entry 8.3.18 is complete.

8.5.18 Proof of Entry 8.3.19

In the proof below, we actually provide two variations. Like Bressoud [81], we begin with an application of Rogers's ideas, but then the proofs diverge.

Proof. By (8.2.11), the first part of Entry 8.3.19 can be put in the form

$$
\begin{aligned}
&f(-q^2,-q^3)f(-q^{78},-q^{117}) + q^8 f(-q,-q^4)f(-q^{39},-q^{156})\\
&\quad = f(-q^{26},-q^{39})f(-q^3,-q^{12}) - q^2 f(-q^6,-q^9)f(-q^{13},-q^{52}).
\end{aligned}
\tag{8.5.18.1}
$$

We apply Rogers's method first with $\alpha_1 = \frac{1}{2}$, $\beta_1 = \frac{39}{2}$, $p_1 = 5$, $m_1 = 1$, and $\lambda_1 = 4$, and secondly with $\alpha_2 = \frac{3}{2}$, $\beta_2 = \frac{13}{2}$, $p_2 = 5$, $m_2 = 3$, and $\lambda_2 = 4$. Then both sets of parameters satisfy (8.4.9). By (8.4.15) and (8.4.16), respectively, we find that

$$
\begin{aligned}
\sum_{k=1}^{2} F\left(\tfrac{1}{2},\tfrac{39}{2},1,5,4,k\right) &= qf(-q^2,-q^3)f(-q^{78},-q^{117})\\
&\qquad + q^9 f(-q,-q^4)f(-q^{39},-q^{156})
\end{aligned}
\tag{8.5.18.2}
$$

and

$$\sum_{k=1}^{2} F\left(\tfrac{3}{2}, \tfrac{13}{2}, 3, 5, 4, k\right) = qf(-q^3, -q^{12})f(-q^{26}, -q^{39}) \\ - q^3 f(-q^6, -q^9)f(-q^{13}, -q^{52}). \qquad (8.5.18.3)$$

Combining (8.5.18.2) and (8.5.18.3), we deduce (8.5.18.1) to complete the proof.

Next, we prove (8.3.23). Let us define, by (8.2.11),

$$g(q) := f(-q)G(q) = f(-q^2, -q^3) \quad \text{and} \quad h(q) := f(-q)H(q) = f(-q, -q^4). \qquad (8.5.18.4)$$

Therefore, by (8.5.18.1), we can define $N(q)$ by

$$N(q) := g(q)g(q^{39}) + q^8 h(q)h(q^{39}) = g(q^{13})h(q^3) - q^2 g(q^3)h(q^{13}). \quad (8.5.18.5)$$

Let us also define

$$M(q) := g(q^2)g(q^{13}) + q^3 h(q^2)h(q^{13}), \qquad (8.5.18.6)$$

$$L(q) := g(q^{26})h(q) - q^5 g(q)h(q^{26}). \qquad (8.5.18.7)$$

Lemma 8.5.7. *We have*

$$g(q) = \frac{1}{\varphi(-q^9)}\left\{-q^2\varphi(-q)h(q^9) + \varphi(-q^3)\chi(-q^3)g(q^6)\right\},$$

$$h(q) = \frac{1}{\varphi(-q^9)}\left\{\varphi(-q)g(q^9) + q\varphi(-q^3)\chi(-q^3)h(q^6)\right\}. \qquad (8.5.18.8)$$

Proof. To prove (8.5.18.8) we employ (8.2.19) with $a = -q^2$, $b = -q^3$, and $n = 3$ to find that

$$f(-q^2, -q^3) = f(-q^{21}, -q^{24}) - q^2 f(-q^{36}, -q^9) + q^9 f(-q^{51}, -q^{-6}) \\ = f(-q^{21}, -q^{24}) - q^2 f(-q^9, -q^{36}) - q^3 f(-q^6, -q^{39}), \quad (8.5.18.9)$$

where in the last step (8.2.5) is used. Similarly, with the choice of parameters $a = -q$, $b = -q^4$, and $n = 3$, one obtains

$$f(-q, -q^4) = f(-q^{18}, -q^{27}) - qf(-q^{12}, -q^{33}) - q^4 f(-q^3, -q^{42}). \quad (8.5.18.10)$$

Therefore, in the notation of (8.5.18.4), we obtain

$$g(q) = A(q^3) - q^2 h(q^9) \qquad \text{and} \qquad h(q) = g(q^9) - qB(q^3), \qquad (8.5.18.11)$$

where

$$A(q) = f(-q^7, -q^8) - qf(-q^2, -q^{13})$$

and

$$B(q) = f(-q^4, -q^{11}) + qf(-q, -q^{14}).$$

Next, we use Entries 8.3.6–8.3.8. By (8.2.14), we can rewrite them in their equivalent forms

$$g(q)g(q^9) + q^2h(q)h(q^9) = f^2(-q^3), \tag{8.5.18.12}$$

$$g(q^2)g(q^3) + qh(q^2)h(q^3) = \psi(q)\varphi(-q^3), \tag{8.5.18.13}$$

$$h(q)g(q^6) - qg(q)h(q^6) = \psi(q^3)\varphi(-q). \tag{8.5.18.14}$$

Starting from (8.5.18.12), we have, by (8.5.18.11),

$$\begin{aligned} f^2(-q^3) &= g(q)g(q^9) + q^2h(q)h(q^9) \\ &= \left(A(q^3) - q^2h(q^9)\right)g(q^9) + q^2\left(g(q^9) - qB(q^3)\right)h(q^9) \\ &= A(q^3)g(q^9) - q^3B(q^3)h(q^9). \end{aligned} \tag{8.5.18.15}$$

Similarly, starting from (8.5.18.14) and using (8.5.18.11) and (8.5.18.13) with q replaced by q^3, we deduce that

$$\begin{aligned} \psi(q^3)\varphi(-q) &= h(q)g(q^6) - qg(q)h(q^6) \\ &= g(q^6)g(q^9) + q^3h(q^6)h(q^9) - q\left(A(q^3)h(q^6) + B(q^3)g(q^6)\right) \\ &= \psi(q^3)\varphi(-q^9) - q\left(A(q^3)h(q^6) + B(q^3)g(q^6)\right). \end{aligned} \tag{8.5.18.16}$$

Solving for $A(q)$ from the last two equations, we find that

$$\begin{aligned} &\left(g(q^6)g(q^9) + q^3h(q^6)h(q^9)\right)A(q^3) \\ &= f^2(-q^3)g(q^6) + q^2\psi(q^3)\left(\varphi(-q^9) - \varphi(-q)\right)h(q^9). \end{aligned}$$

Using (8.5.18.13) again with q replaced by q^3, we conclude that

$$\varphi(-q^9)A(q^3) = \frac{f^2(-q^3)}{\psi(q^3)}g(q^6) + q^2\left(\varphi(-q^9) - \varphi(-q)\right)h(q^9). \tag{8.5.18.17}$$

Substituting this value of $A(q)$ in (8.5.18.11) yields the first identity of (8.5.18.8) after observing that

$$\frac{f^2(-q^3)}{\psi(q^3)} = \varphi(-q^3)\chi(-q^3).$$

Similarly, one solves for $B(q)$ and obtains the analogous identity for $h(q)$. $\square$

Using (8.5.18.8) in the first equation of (8.5.18.5), we find that

$$\begin{aligned} N(q) &= \frac{1}{\varphi(-q^9)}\left\{-q^2\varphi(-q)h(q^9) + \varphi(-q^3)\chi(-q^3)g(q^6)\right\}g(q^{39}) \\ &\quad + \frac{q^8}{\varphi(-q^9)}\left\{\varphi(-q)g(q^9) + q\varphi(-q^3)\chi(-q^3)h(q^6)\right\}h(q^{39}) \end{aligned}$$

$$
\begin{aligned}
&= -q^2\frac{\varphi(-q)}{\varphi(-q^9)}\left\{g(q^{39})h(q^9) - q^6 g(q^9)h(q^{39})\right\} \\
&\quad + \frac{\varphi(-q^3)\chi(-q^3)}{\varphi(-q^9)}\left\{g(q^6)g(q^{39}) + q^9 h(q^6)h(q^{39})\right\} \\
&= -q^2\frac{\varphi(-q)}{\varphi(-q^9)}N(q^3) + \frac{\varphi(-q^3)\chi(-q^3)}{\varphi(-q^9)}M(q^3). \qquad (8.5.18.18)
\end{aligned}
$$

Employing (8.5.18.8) again, this time with q replaced by q^{13} in the second equation of (8.5.18.5), we find that

$$
\begin{aligned}
N(q) &= \frac{1}{\varphi(-q^{117})}\left\{-q^{26}\varphi(-q^{13})h(q^{117}) + \varphi(-q^{39})\chi(-q^{39})g(q^{78})\right\}h(q^3) \\
&\quad - \frac{q^2}{\varphi(-q^{117})}\left\{\varphi(-q^{13})g(q^{117}) + q^{13}\varphi(-q^{39})\chi(-q^{39})h(q^{78})\right\}g(q^3) \\
&= -q^2\frac{\varphi(-q^{13})}{\varphi(-q^{117})}\left\{g(q^3)g(q^{117}) + q^{24}h(q^3)h(q^{117})\right\} \\
&\quad + \frac{\varphi(-q^{39})\chi(-q^{39})}{\varphi(-q^{117})}\left\{h(q^3)g(q^{78}) - q^{15}g(q^3)h(q^{78})\right\} \\
&= -q^2\frac{\varphi(-q^{13})}{\varphi(-q^{117})}N(q^3) + \frac{\varphi(-q^{39})\chi(-q^{39})}{\varphi(-q^{117})}L(q^3). \qquad (8.5.18.19)
\end{aligned}
$$

From (8.3.18), (8.2.11), and (8.2.14), we deduce that

$$
\frac{L(q)}{M(q)} = \frac{f(-q)f(-q^{26})}{f(-q^2)f(-q^{13})} = \frac{\chi(-q)}{\chi(-q^{13})}. \qquad (8.5.18.20)
$$

Thus, by (8.5.18.18)–(8.5.18.20), we conclude that

$$
\begin{aligned}
&q^2\left\{\frac{\varphi(-q^{13})}{\varphi(-q^{117})} - \frac{\varphi(-q)}{\varphi(-q^9)}\right\}N(q^3) \\
&\qquad = \left\{\frac{\varphi(-q^{39})\chi(-q^3)}{\varphi(-q^{117})} - \frac{\varphi(-q^3)\chi(-q^3)}{\varphi(-q^9)}\right\}M(q^3). \qquad (8.5.18.21)
\end{aligned}
$$

By (8.5.30.23), with q replaced by q^{13} and q, respectively, we find that

$$
\begin{aligned}
&\varphi(-q^9)\varphi(-q^{13}) - \varphi(-q)\varphi(-q^{117}) \\
&= \varphi(-q^9)\left\{\varphi(-q^{117}) - 2q^{13}f(-q^{39},-q^{195})\right\} \\
&\quad - \left\{\varphi(-q^9) - 2qf(-q^3,-q^{15})\right\}\varphi(-q^{117}) \\
&= 2q\left\{f(-q^3,-q^{15})\varphi(-q^{117}) - q^{12}\varphi(-q^9)f(-q^{39},-q^{195})\right\}. \qquad (8.5.18.22)
\end{aligned}
$$

Using (8.5.18.22) in (8.5.18.21) and replacing q^3 by q, we arrive at

$$
\begin{aligned}
&2q\left\{f(-q,-q^5)\varphi(-q^{39}) - q^4\varphi(-q^3)f(-q^{13},-q^{65})\right\}N(q) \\
&= \chi(-q)\left\{\varphi(-q^3)\varphi(-q^{13}) - \varphi(-q)\varphi(-q^{39})\right\}M(q). \qquad (8.5.18.23)
\end{aligned}
$$

Comparing (8.5.18.23) with (8.3.23), we see that it suffices to prove that

$$\begin{aligned} M(q) &= \frac{1}{\chi(-q)} \left\{ f(-q,-q^5)\varphi(-q^{39}) - q^4\varphi(-q^3) f(-q^{13},-q^{65}) \right\} \\ &= \psi(q^3)\varphi(-q^{39}) - q^4 f(q,q^2) f(-q^{13},-q^{65}), \end{aligned} \tag{8.5.18.24}$$

where in the last step we used (8.5.7.6) and (8.5.7.8). To verify (8.5.18.24), we employ Theorem 8.4.1 with the parameters $a = b = q^{39}$, $c = 1$, $d = q^3$, $\epsilon_1 = 1$, $\epsilon_2 = 0$, $\alpha = 2$, $\beta = 1$, and $m = 15$, to find that

$$\begin{aligned} f(1,q^3)\varphi(-q^{39}) &= 2\, f(-q^{42},-q^{48}) f(-q^{273},-q^{312}) \\ &+ 2\,q^3 f(-q^{36},-q^{54}) f(-q^{234},-q^{351}) + 2\,q^9 f(-q^{30},-q^{60}) f(-q^{195},-q^{390}) \\ &+ 2\,q^{18} f(-q^{24},-q^{66}) f(-q^{156},-q^{429}) + 2\,q^{30} f(-q^{18},-q^{72}) f(-q^{117},-q^{468}) \\ &+ 2\,q^{45} f(-q^{12},-q^{78}) f(-q^{78},-q^{507}) + 2\,q^{63} f(-q^{6},-q^{84}) f(-q^{39},-q^{546}). \end{aligned} \tag{8.5.18.25}$$

Employing Theorem 8.4.1 again, this time with the parameters $a = q^{13}$, $b = q^{65}$, $c = q$, $d = q^2$, $\epsilon_1 = 1$, $\epsilon_2 = 0$, $\alpha = 13$, $\beta = 1$, and $m = 15$, we find that

$$\begin{aligned} f(-q^{13},-q^{65}) f(q,q^2) &= f(-q^{273},-q^{312}) f(-q^{18},-q^{72}) \\ &+ q f(-q^{234},-q^{351}) f(-q^{24},-q^{66}) + q^5 f(-q^{195},-q^{390}) f(-q^{30},-q^{60}) \\ &+ q^{12} f(-q^{156},-q^{429}) f(-q^{36},-q^{54}) + q^{22} f(-q^{117},-q^{468}) f(-q^{42},-q^{48}) \\ &+ q^{35} f(-q^{78},-q^{507}) f(-q^{42},-q^{48}) + q^{51} f(-q^{39},-q^{546}) f(-q^{36},-q^{54}) \\ &- q^{53} f(-q^{39},-q^{546}) f(-q^{24},-q^{66}) - q^{39} f(-q^{78},-q^{507}) f(-q^{18},-q^{72}) \\ &- q^{28} f(-q^{117},-q^{468}) f(-q^{12},-q^{78}) - q^{20} f(-q^{156},-q^{429}) f(-q^{6},-q^{84}) \\ &+ q^{7} f(-q^{234},-q^{351}) f(-q^{6},-q^{84}) + q^{2} f(-q^{273},-q^{312}) f(-q^{12},-q^{78}). \end{aligned} \tag{8.5.18.26}$$

Now by (8.2.3), (8.5.18.25), and (8.5.18.26), we conclude that

$$\begin{aligned} &\psi(q^3)\varphi(-q^{39}) - q^4 f(q,q^2) f(-q^{13},-q^{65}) \\ &= \left\{ f(-q^{42},-q^{48}) - q^4 f(-q^{18},-q^{72}) - q^6 f(-q^{12},-q^{78}) \right\} \\ &\quad \times \left\{ f(-q^{273},-q^{312}) - q^{26} f(-q^{117},-q^{468}) - q^{39} f(-q^{78},-q^{507}) \right\} \\ &\quad + q^3 \left\{ f(-q^{36},-q^{54}) - q^2 f(-q^{24},-q^{66}) - q^8 f(-q^{6},-q^{84}) \right\} \\ &\quad \times \left\{ f(-q^{234},-q^{351}) - q^{13} f(-q^{156},-q^{429}) - q^{52} f(-q^{39},-q^{546}) \right\}. \end{aligned} \tag{8.5.18.27}$$

But from (8.2.19) with $n = 3$, we know that

$$g(q) = f(-q^2,-q^3) = f(-q^{21},-q^{24}) - q^2 f(-q^9,-q^{36}) - q^3 f(-q^6,-q^{39}), \tag{8.5.18.28}$$

$$h(q) = f(-q,-q^4) = f(-q^{18},-q^{27}) - q f(-q^{12},-q^{33}) - q^4 f(-q^3,-q^{42}), \tag{8.5.18.29}$$

where we used (8.2.5). Replacing q by q^2 and q^{13} in each of (8.5.18.28) and (8.5.18.29), we see that (8.5.18.24) holds, since the right-hand side of (8.5.18.27) is exactly

$$g(q^2)g(q^{13})+q^3h(q^2)h(q^{13})=M(q).$$

Hence, the proof of Entry 8.3.19 is complete. □

Next, we sketch a different proof for Entry 8.3.19, which, by (8.5.18.20), is equivalent to showing that

$$\frac{L(q)}{M(q)}=\frac{\chi(-q)}{\chi(-q^{13})}. \tag{8.5.18.30}$$

Therefore, by (8.5.18.24), one needs to prove that

$$\begin{aligned} L(q) &= \frac{\chi(-q)}{\chi(-q^{13})}\left\{\psi(q^3)\varphi(-q^{39})-q^4f(q,q^2)f(-q^{13},-q^{65})\right\} \\ &= f(-q,-q^5)f(q^{13},q^{26})-q^4\varphi(-q^3)\psi(q^{39}), \end{aligned} \tag{8.5.18.31}$$

where in the last step we used (8.5.7.6) and (8.5.7.8). The equality (8.5.18.31) is proved in the same way that we proved (8.5.18.24), and so we omit the details.

8.5.19 Proof of Entry 8.3.20

Proof. Using (8.4.23) and (8.4.24) in (8.3.3), we arrive at

$$\begin{aligned} \chi^2(q) &= G(q)G(q^4)+qH(q)H(q^4) \\ &= \frac{f(-q^8)}{f(-q^2)}\left\{G(q^4)\left(G(q^{16})+qH(-q^4)\right)+qH(q^4)\left(q^3H(q^{16})+G(-q^4)\right)\right\} \\ &= \frac{f(-q^8)}{f(-q^2)}\left\{G(q^4)G(q^{16})+q^4H(q^4)H(q^{16})\right. \\ &\qquad \left.+q\left(H(-q^4)G(q^4)+H(q^4)G(-q^4)\right)\right\}. \end{aligned} \tag{8.5.19.1}$$

Separating the even- and odd-indexed terms, we easily show that

$$\varphi(q)=\varphi(q^4)+2q\psi(q^8). \tag{8.5.19.2}$$

Using (8.2.10), (8.2.7), and (8.5.19.2), we conclude from (8.5.19.1) that

$$\begin{aligned} &G(q^4)G(q^{16})+q^4H(q^4)H(q^{16})+q\left(H(-q^4)G(q^4)+H(q^4)G(-q^4)\right) \\ &= \frac{\chi^2(q)f(-q^2)}{f(-q^8)}=\frac{\varphi(q)}{f(-q^8)}=\frac{\varphi(q^4)+2q\psi(q^8)}{f(-q^8)}. \end{aligned} \tag{8.5.19.3}$$

Equating odd parts on both sides of (8.5.19.3), we deduce that

$$H(-q^4)G(q^4)+H(q^4)G(-q^4)=2\frac{\psi(q^8)}{f(-q^8)}, \tag{8.5.19.4}$$

which is Entry 8.3.20 with q replaced by q^4. □

8.5.20 Proof of Entry 8.3.21

We shall see that Watson's proof [333] of Entry 8.3.21 follows by combining Entries 8.3.1 and 8.3.2 with some elementary identities for theta functions.

From Entries 8.3.2 and 8.3.3, we easily deduce that

$$G(q)=\frac{\varphi(q)+\varphi(q^5)}{2G(q^4)f(-q^2)} \qquad \text{and} \qquad qH(q)=\frac{\varphi(q)-\varphi(q^5)}{2H(q^4)f(-q^2)}. \tag{8.5.20.1}$$

Applying each of the equalities in (8.5.20.1) twice, but with q replaced by $-q$ in two instances, using (8.2.16), using Lemma 8.5.2, and invoking (8.2.12), we find that

$$\begin{aligned} qG(q)H(-q)-qG(-q)H(q)&=\frac{\varphi(q^5)\varphi(-q^5)-\varphi(q)\varphi(-q)}{2G(q^4)H(q^4)f^2(-q^2)}\\ &=\frac{\varphi^2(-q^{10})-\varphi^2(-q^2)}{2G(q^4)H(q^4)f^2(-q^2)}\\ &=\frac{2q^2\chi(-q^2)f^2(-q^{20})}{G(q^4)H(q^4)\chi(-q^{10})f^2(-q^2)}\\ &=\frac{2q^2f(-q^{20})}{\chi(-q^{10})f(-q^2)}\cdot\frac{f(-q^4)\chi(-q^2)}{f(-q^2)}\\ &=\frac{2q^2\psi(q^{10})}{f(-q^2)}, \end{aligned} \tag{8.5.20.2}$$

where we applied the elementary identities

$$\psi(q)\chi(-q)=f(-q^2)=\frac{f(-q)}{\chi(-q)}, \tag{8.5.20.3}$$

with q replaced by q^{10} and q^2, respectively. The identities in (8.5.20.3) both follow from (8.2.14). The truth of Entry 8.3.21 is readily apparent from (8.5.20.2).

8.5.21 Proof of Entry 8.3.22

First Proof of Entry 8.3.22. Using (8.4.23) and (8.4.24) in Entry 8.3.5, we find that

$$\begin{aligned} \chi(q^2)&=G(q^{16})H(q)-q^3G(q)H(q^{16})\\ &=\left\{\frac{f(-q^2)}{f(-q^8)}G(q)-qH(-q^4)\right\}H(q)-\left\{\frac{f(-q^2)}{f(-q^8)}H(q)-G(-q^4)\right\}G(q)\\ &=G(q)G(-q^4)-qH(q)H(-q^4), \end{aligned}$$

which is Entry 8.3.22 with q replaced by $-q$. ∎

Second Proof of Entry 8.3.22. Consider the system of three equations

$$G(-q)G(-q^4) + qH(-q)H(-q^4) =: T(q), \tag{8.5.21.1}$$

$$H(q^4)G(-q^4) + G(q^4)H(-q^4) = \frac{2\psi(q^8)}{f(-q^8)}, \tag{8.5.21.2}$$

$$-H(q^4)G(-q^4) + G(q^4)H(-q^4) = \frac{2q^4\psi(q^{40})}{f(-q^8)}. \tag{8.5.21.3}$$

Note that (8.5.21.1) merely gives the definition of $T(q)$, and that our goal is to show that $T(q) = \chi(q^2)$. The equality (8.5.21.2) is (8.3.24) with q replaced by q^4, and (8.5.21.3) is (8.3.25) with q replaced by q^4. We regard (8.5.21.1)–(8.5.21.3) as a system of three equations in the "variables" $G(-q^4)$, $H(-q^4)$, and -1. Thus, we have

$$\begin{vmatrix} G(-q) & qH(-q) & T(q) \\ H(q^4) & G(q^4) & \dfrac{2\psi(q^8)}{f(-q^8)} \\ -H(q^4) & G(q^4) & \dfrac{2q^4\psi(q^{40})}{f(-q^8)} \end{vmatrix} = 0. \tag{8.5.21.4}$$

Expanding (8.5.21.4) by the last column, we find that

$$\begin{aligned} 2T(q)G(q^4)H(q^4) - \frac{2\psi(q^8)}{f(-q^8)}\left\{G(-q)G(q^4) + qH(-q)H(q^4)\right\} \\ + \frac{2q^4\psi(q^{40})}{f(-q^8)}\left\{G(-q)G(q^4) - qH(-q)H(q^4)\right\} = 0. \end{aligned} \tag{8.5.21.5}$$

Using (8.2.12), (8.3.4) with q replaced by $-q$, and (8.3.3) with q replaced by $-q$, we rewrite (8.5.21.5) in the form

$$\frac{T(q)f(-q^{20})}{f(-q^4)} - \frac{\psi(q^8)\varphi(-q^5)}{f(-q^8)f(-q^2)} + \frac{q^4\psi(q^{40})\varphi(-q)}{f(-q^8)f(-q^2)} = 0, \tag{8.5.21.6}$$

or, upon rearrangement,

$$T(q) = \frac{f(-q^4)}{f(-q^2)f(-q^8)f(-q^{20})}\left\{\varphi(-q^5)\psi(q^8) - q^4\varphi(-q)\psi(q^{40})\right\}. \tag{8.5.21.7}$$

By (8.2.15), (8.5.21.7) can be rewritten as

$$T(q) = \frac{\chi(-q^2)\chi(q^2)}{f(-q^2)f(-q^{20})}\left\{\varphi(-q^5)\psi(q^8) - q^4\varphi(-q)\psi(q^{40})\right\}. \tag{8.5.21.8}$$

From (8.5.5.12), we find, after simplification, that

$$\begin{aligned}
&\varphi(-q^5)\psi(q^8) - q^4\varphi(-q)\psi(q^{40}) \\
&= \sqrt{z_5}(1-\beta)^{1/4}\frac{1}{4q}\sqrt{z_1}\left\{1-(1-\alpha)^{1/4}\right\} \\
&\quad - q^4\sqrt{z_1}(1-\alpha)^{1/4}\frac{1}{4q^5}\sqrt{z_5}\left\{1-(1-\beta)^{1/4}\right\} \\
&= \frac{\sqrt{z_1 z_5}}{4q}\left\{(1-\beta)^{1/4}-(1-\alpha)^{1/4}\right\}. \qquad (8.5.21.9)
\end{aligned}$$

Putting (8.5.5.15) and (8.5.21.9) in (8.5.21.8), we arrive at

$$T(q) = \frac{\chi(q^2)\left\{(1-\beta)^{1/4}-(1-\alpha)^{1/4}\right\}}{2^{2/3}(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}}. \qquad (8.5.21.10)$$

In comparing (8.5.21.10) with (8.3.26), we see that it remains to show that

$$\frac{(1-\beta)^{1/4}-(1-\alpha)^{1/4}}{2^{2/3}(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/24}} = 1. \qquad (8.5.21.11)$$

But (8.5.21.11) is equivalent to (8.5.5.1), and so the proof is complete. □

8.5.22 Proof of Entry 8.3.23

We first remark that we have already given one proof of Entry 8.3.23 along with one of our proofs of Entry 8.3.11. We provide a second proof here.

Using (8.2.11) and (8.2.14), we see that Entry 8.3.23 is equivalent to the identity

$$\begin{aligned}
&f(-q^4,q^6)f(-q^6,q^9) + qf(q^2,-q^8)f(q^3,-q^{12}) \\
&\qquad = \frac{\chi(q)\chi(q^6)}{\chi(q^2)\chi(q^3)}f(q^2)f(q^3) = f(-q^4)f(-q^6)\chi(q)\chi(q^6). \quad (8.5.22.1)
\end{aligned}$$

Using the product representations of $\chi(q)$ and $f(-q)$ given in (8.2.9) and (8.2.10), respectively, together with (8.2.6), we find that

$$\begin{aligned}
f(-q^4)\chi(q) &= (q^4;q^4)_\infty(-q;q^2)_\infty = (q^4;q^4)_\infty(-q;q^4)_\infty(-q^3;q^4)_\infty \\
&= f(q,q^3) = \psi(q)
\end{aligned}$$

and

$$\begin{aligned}
f(-q^6)\chi(q^6) &= (q^6;q^6)_\infty(-q^6;q^{12})_\infty = (q^6;q^{12})_\infty(q^{12};q^{12})_\infty(-q^6;q^{12})_\infty \\
&= (q^{12};q^{24})_\infty(q^{12};q^{12})_\infty \\
&= (q^{12};q^{24})_\infty(q^{12};q^{24})_\infty(q^{24};q^{24})_\infty \\
&= f(-q^{12},-q^{12}) = \varphi(-q^{12}),
\end{aligned}$$

by (8.2.7). It thus suffices to prove that

$$f(-q^4,q^6)f(-q^6,q^9)+qf(q^2,-q^8)f(q^3,-q^{12})=\varphi(-q^{12})\psi(q). \tag{8.5.22.2}$$

We now apply Theorem 8.4.1 with the parameters $\epsilon_1=1$, $\epsilon_2=0$, $a=b=q^{12}$ $c=q$, $d=q^3$, $\alpha=2$, $\beta=1$, and $m=5$. We consequently find that

$$\begin{aligned}\varphi(-q^{12})\psi(q) &= f(-q^{22},-q^{18})f(-q^{33},-q^{27})+qf(-q^{14},-q^{26})f(-q^{21},-q^{39})\\ &\quad+q^6f(-q^6,-q^{34})f(-q^9,-q^{51})+q^{15}f(-q^{-2},-q^{42})f(-q^{-3},-q^{63})\\ &\quad+q^{28}f(-q^{-10},-q^{50})f(-q^{-15},-q^{75})\\ &= f(-q^{18},-q^{22})f(-q^{33},-q^{27})+qf(-q^{14},-q^{26})f(-q^{21},-q^{39})\\ &\quad+q^6f(-q^6,-q^{34})f(-q^9,-q^{51})+q^{10}f(-q^2,-q^{38})f(-q^3,-q^{57})\\ &\quad+q^3f(-q^{10},-q^{30})f(-q^{15},-q^{45}),\end{aligned} \tag{8.5.22.3}$$

where we applied (8.2.5) four times in the last equality. Replacing q by q^2 and q^3 in each of (8.5.16.7) and (8.5.16.8), respectively, we find that

$$\begin{aligned}f(-q^4,q^6)&=f(-q^{18},-q^{22})-q^4f(-q^{38},-q^2),\\ f(-q^6,q^9)&=f(-q^{27},-q^{33})-q^6f(-q^{57},-q^3),\\ f(q^2,-q^8)&=f(-q^{14},-q^{26})+q^2f(-q^{34},-q^6),\\ f(q^3,-q^{12})&=f(-q^{21},-q^{39})+q^3f(-q^{51},-q^9).\end{aligned}$$

Return to (8.5.22.3) and substitute each of the equalities above to deduce that

$$\begin{aligned}&\varphi(-q^{12})\psi(q)-\left\{f(-q^4,q^6)f(-q^6,q^9)+qf(q^2,-q^8)f(q^3,-q^{12})\right\}\\ &=q^3f(-q^{10},-q^{30})f(-q^{15},-q^{45})+q^4f(-q^2,-q^{38})f(-q^{27},-q^{33})\\ &\quad-q^3f(-q^6,-q^{34})f(-q^{21},-q^{39})-q^4f(-q^{14},-q^{26})f(-q^9,-q^{51})\\ &\quad+q^6f(-q^{18},-q^{22})f(-q^3,-q^{57}).\end{aligned} \tag{8.5.22.4}$$

We now use Theorem 8.4.1 again, but now with the parameters $\epsilon_1=1$, $\epsilon_2=0$, $a=1$, $b=q^{24}$, $c=q$, $d=q^3$, $\alpha=2$, $\beta=1$, and $m=5$. Hence, we find that

$$\begin{aligned}q^3f(-1,-q^{24})\psi(q) &= q^3f(-q^{10},-q^{30})f(-q^{15},-q^{45})\\ &\quad+q^4f(-q^2,-q^{38})f(-q^{27},-q^{33})+q^9f(-q^{-6},-q^{46})f(-q^{39},-q^{21})\\ &\quad+q^{18}f(-q^{-14},-q^{54})f(-q^{51},-q^9)+q^{31}f(-q^{-22},-q^{62})f(-q^{63},-q^{-3})\\ &=q^3f(-q^{10},-q^{30})f(-q^{15},-q^{45})+q^4f(-q^2,-q^{38})f(-q^{27},-q^{33})\\ &\quad-q^3f(-q^6,-q^{34})f(-q^{21},-q^{39})-q^4f(-q^{14},-q^{26})f(-q^9,-q^{51})\\ &\quad+q^6f(-q^{18},-q^{22})f(-q^3,-q^{57}),\end{aligned} \tag{8.5.22.5}$$

after four applications of (8.2.5). The product on the far left side of (8.5.22.5) equals 0, by (8.2.4). Hence, since the right-hand sides of (8.5.22.5) and (8.5.22.4) are equal, we complete the proof of (8.5.22.2), and hence also of Entry 8.3.23.

8.5.23 Proof of Entry 8.3.24

We first remark that we have already given one proof of Entry 8.3.24 along with one of our proofs of Entry 8.3.12. We provide a second proof here.

This proof of Entry 8.3.24 is very similar to that given above for Entry 8.3.23. Using (8.2.11) and (8.2.14), we see that Entry 8.3.24 is equivalent to the identity

$$\begin{aligned}f(-q^{12},q^{18})f(q,-q^4)-qf(q^6,-q^{24})f(-q^2,q^3)&\\=\frac{\chi(q^2)\chi(q^3)}{\chi(q)\chi(q^6)}f(q)f(q^6)=f(-q^2)f(-q^{12})\chi(q^2)\chi(q^3).&\quad(8.5.23.1)\end{aligned}$$

Using the product representations of $\chi(q)$ and $f(-q)$ from (8.2.10) and (8.2.9), respectively, together with (8.2.6), we obtain

$$\begin{aligned}&f(-q^2)f(-q^{12})\chi(q^2)\chi(q^3)=(q^2;q^2)_\infty(q^{12};q^{12})_\infty(-q^2;q^4)_\infty(-q^3;q^6)_\infty\\&=\frac{(q^4;q^8)_\infty}{(q^2;q^4)_\infty}(-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(q^{12};q^{12})_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty\\&=f(q^3,q^9)(q^4;q^8)_\infty(q^4;q^4)_\infty\\&=f(q^3,q^9)f(-q^4,-q^4)=\psi(q^3)\varphi(-q^4).\end{aligned}$$

It therefore remains to prove that

$$f(-q^{12},q^{18})f(q,-q^4)-qf(q^6,-q^{24})f(-q^2,q^3)=\varphi(-q^4)\psi(q^3).\quad(8.5.23.2)$$

We now apply Theorem 8.4.1 with the parameters $\epsilon_1=1$, $\epsilon_2=0$, $a=b=q^4$, $c=q^3$, $d=q^9$, $\alpha=1$, $\beta=3$, and $m=5$. Accordingly, we find that

$$\begin{aligned}\varphi(-q^4)\psi(q^3)&=f(-q^{13},-q^7)f(-q^{66},-q^{54})+q^3f(-q,-q^{19})f(-q^{42},-q^{78})\\&\quad+q^{18}f(-q^{-11},-q^{31})f(-q^{18},-q^{102})+q^{45}f(-q^{-23},-q^{43})f(-q^{-6},-q^{126})\\&\quad+q^{84}f(-q^{-35},-q^{55})f(-q^{-30},-q^{150})\\&=f(-q^7,-q^{13})f(-q^{54},-q^{66})+q^3f(-q,-q^{19})f(-q^{42},-q^{78})\\&\quad-q^7f(-q^9,-q^{11})f(-q^{18},-q^{102})-q^{13}f(-q^3,-q^{17})f(-q^6,-q^{114})\\&\quad-q^4f(-q^5,-q^{15})f(-q^{30},-q^{90}),\qquad(8.5.23.3)\end{aligned}$$

where we applied (8.2.5) five times in the last equality. Recording again (8.5.16.7) and (8.5.16.8) as well as their analogues with q replaced by q^6, we find that

$$\begin{aligned}f(-q^2,q^3)&=f(-q^9,-q^{11})-q^2f(-q^{19},-q),\\f(-q^{12},q^{18})&=f(-q^{54},-q^{66})-q^{12}f(-q^{114},-q^6),\\f(q,-q^4)&=f(-q^7,-q^{13})+qf(-q^{17},-q^3),\\f(q^6,-q^{24})&=f(-q^{42},-q^{78})+q^6f(-q^{102},-q^{18}).\end{aligned}$$

Using these identities in (8.5.23.3), we find, after some elementary algebra, that

$$\begin{aligned}
&\varphi(-q^4)\psi(q^3) - \left\{f(-q^{12},q^{18})f(q,-q^4) - qf(q^6,-q^{24})f(-q^2,q^3)\right\} \\
&= qf(-q^9,-q^{11})f(-q^{42},-q^{78}) - qf(-q^3,-q^{17})f(-q^{54},-q^{66}) \\
&\quad - q^4f(-q^5,-q^{15})f(-q^{30},-q^{90}) + q^{12}f(-q^7,-q^{13})f(-q^6,-q^{114}) \\
&\quad - q^9f(-q,-q^{19})f(-q^{18},-q^{102}).
\end{aligned} \tag{8.5.23.4}$$

Next, we apply Theorem 8.4.1 again, but now with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = 1$, $b = q^8$, $c = q^3$, $d = q^9$, $\alpha = 1$, $\beta = 3$, and $m = 5$. Accordingly, we find that

$$\begin{aligned}
&qf(-1,-q^8)\psi(q^3) = qf(-q^9,-q^{11})f(-q^{78},-q^{42}) \\
&\quad + q^4f(-q^{-3},-q^{23})f(-q^{54},-q^{66}) + q^{19}f(-q^{-15},-q^{35})f(-q^{30},-q^{90}) \\
&\quad + q^{46}f(-q^{-27},-q^{47})f(-q^6,-q^{114}) + q^{85}f(-q^{-39},-q^{59})f(-q^{138},-q^{-18}) \\
&= qf(-q^9,-q^{11})f(-q^{42},-q^{78}) - qf(-q^3,-q^{17})f(-q^{54},-q^{66}) \\
&\quad - q^4f(-q^5,-q^{15})f(-q^{30},-q^{90}) + q^{12}f(-q^7,-q^{13})f(-q^6,-q^{114}) \\
&\quad - q^9f(-q,-q^{19})f(-q^{18},-q^{102}),
\end{aligned} \tag{8.5.23.5}$$

after five applications of (8.2.5). The right sides of (8.5.23.4) and (8.5.23.5) are identical. Thus, the left sides of (8.5.23.4) and (8.5.23.5) are identical. Since the left side of (8.5.23.5) equals 0 by (8.2.4), we see that (8.5.23.2) follows immediately. This completes the proof of Entry 8.3.24.

8.5.24 Proofs of Entry 8.3.25

First Proof of Entry 8.3.25. Using (8.4.23) and (8.4.24) in (8.3.14) with q replaced by q^9, we arrive at

$$\begin{aligned}
\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})} &= G(q^9)H(q^4) - qH(q^9)G(q^4) \qquad (8.5.24.1)\\
&= \frac{f(-q^{72})}{f(-q^{18})}\left\{H(q^4)\left(G(q^{144}) + q^9H(-q^{36})\right)\right. \\
&\qquad \left. -qG(q^4)\left(q^{27}H(q^{144}) + G(-q^{36})\right)\right\} \\
&= \frac{f(-q^{72})}{f(-q^{18})}\left\{H(q^4)G(q^{144}) - q^{28}G(q^4)H(q^{144})\right. \\
&\qquad \left. -q\left(G(q^4)G(-q^{36}) - q^8H(q^4)H(-q^{36})\right)\right\}.
\end{aligned}$$

Using (8.2.14), (8.2.15), and (8.5.7.10) with q replaced by $-q$, we deduce from (8.5.24.1) that

$$\begin{aligned}
&H(q^4)G(q^{144}) - q^{28}G(q^4)H(q^{144}) - q\left(G(q^4)G(-q^{36}) - q^8H(q^4)H(-q^{36})\right)\\
&= \frac{f(-q^{18})\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})f(-q^{72})} = \chi(-q)\chi(q^3)\chi(-q^{36})\\
&= \chi(-q^{36})\left\{\frac{\chi(q^{12})}{\chi(-q^8)} - q\frac{\chi(q^4)}{\chi(-q^{24})}\right\}. \qquad (8.5.24.2)
\end{aligned}$$

Equating the even and odd parts on both sides of (8.5.24.2), we readily obtain Entries 8.3.14 and 8.3.25 with q replaced by q^4 and $-q^4$, respectively. ☐

Second Proof of Entry 8.3.25. Employing Theorem 8.4.1 with the set of parameters $a = q^6$, $b = q^{12}$, $c = q$, $d = q^2$, $\alpha = 2$, $\beta = 1$, $m = 5$, $\epsilon_1 = 0$, and $\epsilon_2 = 1$, we find that

$$\begin{aligned}
f(q^6,q^{12})f(-q) &= f(q^{13},q^{17})f(-q^{18},-q^{27}) - qf(q^7,q^{23})f(-q^{18},-q^{27})\\
&\quad + q^5f(q,q^{29})f(-q^9,-q^{36}) - q^2f(q^{11},q^{19})f(-q^9,-q^{36}),
\end{aligned}$$

where we used (8.2.4) and (8.2.5) twice. Upon the rearrangement of terms and use of (8.4.26) and (8.4.27), with q replaced by $-q$, and (8.2.11), we deduce that

$$\begin{aligned}
f(q^6,q^{12})f(-q) &= f(-q^{18},-q^{27})\{f(q^{13},q^{17}) - qf(q^7,q^{23})\} \qquad (8.5.24.3)\\
&\quad - q^2f(-q^9,-q^{36})\{f(q^{11},q^{19}) - q^3f(q,q^{29})\}\\
&= f(-q^{18},-q^{27})G(-q)f(-q^2) - q^2f(-q^9,-q^{36})H(-q)f(-q^2)\\
&= f(-q^2)f(-q^9)\left\{G(q^9)G(-q) - q^2H(q^9)H(-q)\right\}.
\end{aligned}$$

By (8.5.7.8) with q replaced by q^6, (8.2.14), and (8.2.17) in the form $\chi(q)f(-q) = \varphi(-q^2)$, but with q replaced by q^9, we find that

$$\frac{f(-q)f(q^6,q^{12})}{f(-q^2)f(-q^9)} = \chi(-q)\frac{\varphi(-q^{18})}{\chi(-q^6)f(-q^9)} = \frac{\chi(-q)\chi(q^9)}{\chi(-q^6)}. \qquad (8.5.24.4)$$

Hence, by (8.5.24.3) and (8.5.24.4), the proof of Entry 8.3.25 is complete. ☐

Third Proof of Entry 8.3.25. To prove Entry 8.3.25, we need the identity [55, p. 349, Entry 2(i)]

$$\varphi(q)\varphi(q^9) - \varphi^2(q^3) = 2q\varphi(-q^2)\psi(q^9)\chi(q^3). \qquad (8.5.24.5)$$

Recall the definitions

$$g(q) = f(-q)G(q) = f(-q^2,-q^3) \quad \text{and} \quad h(q) = f(-q)H(q) = f(-q,-q^4).$$

Using (8.2.11), (8.2.7), (8.2.6), and some elementary product manipulations, we see that Entry 8.3.25 is equivalent to the identity

$$g(-q)g(q^9) - q^2h(-q)h(q^9) = \varphi(-q^2)f(q^6, q^{12}). \tag{8.5.24.6}$$

Replacing q by $-q$ in (8.5.24.6) gives

$$g(q)g(-q^9) - q^2h(q)h(-q^9) = \varphi(-q^2)f(q^6, q^{12}). \tag{8.5.24.7}$$

We prove (8.5.24.7).

Using (8.2.11), (8.2.8), (8.2.6), and some elementary product manipulations, we can express Entry 8.3.13 as

$$g(q^9)h(q^4) - qg(q^4)h(q^9) = \psi(-q)f(q^3, q^{15}). \tag{8.5.24.8}$$

It is also easily verified, using the product expansions from (8.2.8) and (8.2.9), that Entry 8.3.20 is equivalent to the identity

$$g(q)h(-q) + g(-q)h(q) = 2\psi(q)\psi(-q). \tag{8.5.24.9}$$

Consider the system of three equations

$$g(q)g(-q^9) - q^2h(q)h(-q^9) =: T(q), \tag{8.5.24.10}$$
$$g(q)g(q^9) + q^2h(q)h(q^9) = f^2(-q^3), \tag{8.5.24.11}$$
$$g(q)g(q^4) + qh(q)h(q^4) = \psi(q)\varphi(-q^2). \tag{8.5.24.12}$$

Equation (8.5.24.11) above is equation (8.5.6.1), while equation (8.5.24.12) is a variation of equation (8.3.2). We wish to show that $T(q) = \varphi(-q^2)f(q^6, q^{12})$. Regarding this system in the variables $g(q)$, $qh(q)$, and -1, we find that

$$\begin{vmatrix} g(-q^9) & -qh(-q^9) & T(q) \\ g(q^9) & qh(q^9) & f^2(-q^3) \\ g(q^4) & h(q^4) & \varphi(-q^2)\psi(q) \end{vmatrix} = 0. \tag{8.5.24.13}$$

Expanding the determinant in (8.5.24.13) along the last column, we find that

$$\begin{aligned} T(q)\left\{g(q^9)h(q^4) - qg(q^4)h(q^9)\right\} - f^2(-q^3)\left\{g(-q^9)h(q^4) + qg(q^4)h(-q^9)\right\} \\ + \varphi(-q^2)\psi(q)\left\{qg(-q^9)h(q^9) + qg(q^9)h(-q^9)\right\} = 0. \end{aligned} \tag{8.5.24.14}$$

Using (8.5.24.8), (8.5.24.8), with q replaced by $-q$, and (8.5.24.9), with q replaced by q^9, in (8.5.24.14), we find that

$$T(q)\psi(-q)f(q^3, q^{15}) = f^2(-q^3)\psi(q)f(-q^3, -q^{15}) - 2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9).$$

It suffices then to prove that

$$\begin{aligned} &\varphi(-q^2)f(q^6, q^{12})\psi(-q)f(q^3, q^{15}) \\ &= f^2(-q^3)\psi(q)f(-q^3, -q^{15}) - 2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9). \end{aligned} \tag{8.5.24.15}$$

By (8.2.6), (8.2.7), and (8.2.8), we find that

$$\begin{aligned}
&f(q^2,q^4)f(-q,-q^5)f(-q^2)\\
&=(-q^2;q^6)_\infty(-q^4;q^6)_\infty(q^6;q^6)_\infty(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty(q^2;q^2)_\infty\\
&=\frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(q^6;q^6)_\infty\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty}(q^6;q^6)_\infty(q^2;q^2)_\infty\\
&=\frac{(q^6;q^6)_\infty}{(-q^6;q^6)_\infty}(q;q)_\infty(-q^2;q^2)_\infty\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}\\
&=\varphi(-q^6)\psi(-q)\psi(q^3). \qquad (8.5.24.16)
\end{aligned}$$

Multiply both sides of (8.5.24.15) by $f(-q^6)$ and use (8.5.24.16) with q replaced by $-q^3$ to deduce that

$$\begin{aligned}
&\varphi(-q^2)\psi(-q)\varphi(-q^{18})\psi(q^3)\psi(-q^9) \qquad (8.5.24.17)\\
&=f^2(-q^3)\psi(q)f(-q^3,-q^{15})f(-q^6)-2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9)f(-q^6).
\end{aligned}$$

From (8.5.7.6) and (8.2.14), we find that

$$f(-q,-q^5)f(-q^2)=f(-q)\psi(q^3). \qquad (8.5.24.18)$$

Using (8.5.24.18), with q replaced by q^3, in (8.5.24.17), we find that

$$\begin{aligned}
&\varphi(-q^2)\psi(-q)\varphi(-q^{18})\psi(q^3)\psi(-q^9)\\
&=f^3(-q^3)\psi(q)\psi(q^9)-2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9)f(-q^6). \qquad (8.5.24.19)
\end{aligned}$$

Using (8.2.13)–(8.2.15), or using (8.2.7)–(8.2.9), we can easily verify that

$$f^3(-q)=\psi(q)\varphi^2(-q). \qquad (8.5.24.20)$$

Using (8.5.13.4) twice, with q replaced by $-q$ and $-q^9$, respectively, and (8.5.24.20), with q replaced by q^3, we deduce from (8.5.24.19) that

$$\begin{aligned}
&\varphi(-q)\psi(q)\varphi(-q^9)\psi(q^9)\psi(q^3)\\
&=\psi(q^3)\varphi^2(-q^3)\psi(q)\psi(q^9)-2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9)f(-q^6). \quad (8.5.24.21)
\end{aligned}$$

Divide both sides of (8.5.24.21) by $\psi(q)\psi(q^3)\psi(q^9)$ to conclude that

$$\begin{aligned}
\varphi(-q)\varphi(-q^9)&=\varphi^2(-q^3)-\frac{2q}{\psi(q^3)}\varphi(-q^2)\psi(-q^9)f(-q^6)\\
&=\varphi^2(-q^3)-2q\varphi(-q^2)\psi(-q^9)\chi(-q^3), \qquad (8.5.24.22)
\end{aligned}$$

where in the last step we used the extremal equality in (8.2.14) with q replaced by $-q^3$. If we replace q by $-q$, then (8.5.24.22) reduces to (8.5.24.5). Hence, the proof of Entry 8.3.25 is complete. □

8.5.25 Proofs of Entries 8.3.26 and 8.3.27

The proofs in this section are due to Watson [333].

Recall that by (8.5.19.2),

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{8.5.25.1}$$

Returning to (8.5.20.1), we use (8.5.25.1) twice. Then we apply Entries 8.3.2, 8.3.3, and 8.3.20, with q replaced by q^4, and Entry 8.3.21, with q replaced by q^2. In these resulting equalities, we solve for $\varphi(q^4)$, $\varphi(q^{20})$, $\psi(q^8)$, and $\psi(q^{40})$, respectively, and substitute them in the second equality below. Accordingly, we find that

$$\begin{aligned} G(q) &= \frac{\varphi(q)+\varphi(q^5)}{2G(q^4)f(-q^2)} \\ &= \frac{\varphi(q^4)+\varphi(q^{20})}{2G(q^4)f(-q^2)} + \frac{q\psi(q^8)+q^5\psi(q^{40})}{G(q^4)f(-q^2)} \\ &= \frac{f(-q^8)}{f(-q^2)}\left(G(q^{16})+qH(-q^4)\right). \end{aligned} \tag{8.5.25.2}$$

Performing exactly the same steps on the second equality of (8.5.20.1), we find that

$$\begin{aligned} qH(q) &= \frac{\varphi(q)-\varphi(q^5)}{2H(q^4)f(-q^2)} \\ &= \frac{\varphi(q^4)-\varphi(q^{20})}{2H(q^4)f(-q^2)} + \frac{q\psi(q^8)-q^5\psi(q^{40})}{H(q^4)f(-q^2)} \\ &= \frac{f(-q^8)}{f(-q^2)}\left(q^4H(q^{16})+qG(-q^4)\right). \end{aligned} \tag{8.5.25.3}$$

For brevity, set

$$T(q) := G(q^{11})H(-q) + q^2G(-q)H(q^{11}). \tag{8.5.25.4}$$

Next, in the definition (8.5.25.4), we substitute for each of the functions G and H their respective representations from (8.5.25.2) and (8.5.25.3). We therefore deduce that

$$\begin{aligned} \frac{f(-q^2)}{f(-q^8)}\cdot\frac{f(-q^{22})}{f(-q^{88})}T(q) &= \left\{G(q^{176})+q^{11}H(-q^{44})\right\}\left\{G(-q^4)-q^3H(q^{16})\right\} \\ &\quad + q^2\left\{G(q^{16})-qH(-q^4)\right\}\left\{G(-q^{44})+q^{33}H(q^{176})\right\} \\ &= \left\{G(-q^4)G(q^{176})-q^{36}H(-q^4)H(q^{176})\right\} \\ &\quad + q^2\left\{G(q^{16})G(-q^{44})-q^{12}H(q^{16})H(-q^{44})\right\} \\ &\quad - q^3\left\{G(q^{176})H(q^{16})-q^{32}G(q^{16})H(q^{176})\right\} \\ &\quad - q^3\left\{G(-q^{44})H(-q^4)-q^8G(-q^4)H(-q^{44})\right\}. \end{aligned} \tag{8.5.25.5}$$

Recalling the definitions of U and V in (8.3.31) and (8.3.32), respectively, recalling the definition (8.5.25.4), and using Entry 8.3.4, we find that (8.5.25.5) can be written in the form

$$\chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44})T(q) = U(-q^4) + q^2V(-q^4) - 2q^3. \quad (8.5.25.6)$$

Now replace q by $-q$ in (8.5.25.6) and subtract the two equalities to deduce that

$$\chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44})\left\{T(-q) - T(q)\right\} = 4q^3. \quad (8.5.25.7)$$

We can obtain a second equation connecting $T(q)$ and $T(-q)$ in the following manner. We record (8.3.5), (8.5.25.4), and (8.3.24), with q replaced by q^{11}. Accordingly,

$$\begin{aligned} G(q^{11})H(q) - q^2G(q)H(q^{11}) &= 1, \\ G(q^{11})H(-q) + q^2G(-q)H(q^{11}) &= T(q), \\ G(q^{11})H(-q^{11}) + G(-q^{11})H(q^{11}) &= \frac{2}{\chi^2(-q^{22})}. \end{aligned}$$

Regarding $G(q^{11})$, $H(q^{11})$, and 1 as the "variables," we conclude from this triple of equations that

$$\begin{vmatrix} H(q) & -q^2G(q) & 1 \\ H(-q) & q^2G(-q) & T(q) \\ H(-q^{11}) & G(-q^{11}) & \dfrac{2}{\chi^2(-q^{22})} \end{vmatrix} = 0. \quad (8.5.25.8)$$

Expanding this determinant (8.5.25.8) by the last column, using Entry 8.3.4, recalling the definition (8.5.25.4), and using Entry 8.3.20, we find that

$$1 - T(q)T(-q) + \frac{4q^2}{\chi^2(-q^2)\chi^2(-q^{22})} = 0. \quad (8.5.25.9)$$

We now use the theory of modular equations. Let β have degree 11 over α. The standard modular equation of degree 11, first found by Schröter and rediscovered by Ramanujan [55, p. 363, Entry 7(i)], is given by

$$(\alpha\beta)^{1/4} + \left\{(1-\alpha)(1-\beta)\right\}^{1/4} + 2^{4/3}\left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/12} = 1. \quad (8.5.25.10)$$

We also need the representations [55, p. 124, Entries 12(v), (vii)]

$$\chi(q) = 2^{1/6}\left(\frac{q}{\alpha(1-\alpha)}\right)^{1/24} \qquad \text{and} \qquad \chi(-q^2) = 2^{1/3}\left(\frac{(1-\alpha)q^2}{\alpha^2}\right)^{1/24}. \quad (8.5.25.11)$$

Lastly, we set $-q^2 = Q$. Thus, using (8.5.25.7), (8.5.25.9), and (8.5.25.11), the modular equation (8.5.25.10), and lastly (8.5.25.11), we deduce that

$$
\begin{aligned}
&\chi^2(-q^2)\chi^2(-q^4)\chi^2(-q^{22})\chi^2(-q^{44})\left\{T(q)+T(-q)\right\}^2 \\
&= 4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22}) - 16Q\chi^2(-Q^2)\chi^2(-Q^{22}) - 16Q^3 \\
&= 4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22}) \\
&\quad \times \left(1 - 16\frac{Q}{\chi^2(Q)\chi^2(Q^{11})} - 16\frac{Q^3}{\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})}\right) \\
&= 4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22}) \\
&\quad \times \left(1 - 2^{4/3}\left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/12} - (\alpha\beta)^{1/4}\right) \\
&= 4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})\left\{(1-\alpha)(1-\beta)\right\}^{1/4} \\
&= 4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})\frac{\chi^2(-Q^2)\chi^2(-Q^{22})}{\chi^4(Q)\chi^4(Q^{11})}.
\end{aligned} \tag{8.5.25.12}
$$

Changing back to the original variable q, we take the square root of both sides of (8.5.25.12) to deduce that

$$
T(q) + T(-q) = 2\frac{\chi(-q^4)\chi(-q^{44})}{\chi^2(-q^2)\chi^2(-q^{22})} = 2\frac{\chi(q^2)\chi(q^{22})}{\chi(-q^2)\chi(-q^{22})}, \tag{8.5.25.13}
$$

by (8.2.15). Now combine (8.5.25.13) with (8.5.25.7) to derive the desired formula (8.3.30).

It remains to prove (8.3.33) and (8.3.34). Return to (8.5.25.6) and insert the just proved formula for $T(q)$ in Entry 8.3.26. We thus find that

$$
U(-q^4) + q^2V(-q^4) = \chi(q^2)\chi(-q^4)\chi(q^{22})\chi(-q^{44}). \tag{8.5.25.14}
$$

Changing the sign of q^2 in (8.5.25.14), we find that

$$
U(-q^4) - q^2V(-q^4) = \chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44}). \tag{8.5.25.15}
$$

Multiplying (8.5.25.14) and (8.5.25.15) together, we arrive at

$$
\begin{aligned}
U^2(-q^4) - q^4V^2(-q^4) &= \chi(q^2)\chi(-q^2)\chi(q^{22})\chi(-q^{22})\chi^2(-q^4)\chi^2(-q^{44}) \\
&= \chi^3(-q^4)\chi^3(-q^{44}),
\end{aligned} \tag{8.5.25.16}
$$

by (8.2.15). If we replace $-q^4$ by q in (8.5.25.16), we obtain (8.3.33).

Finally, we prove (8.3.34). We record the definition (8.3.31) of $U(q)$, (8.3.5), with q replaced by q^4, and (8.3.3), with q replaced by q^{11}, in the array

$$
\begin{aligned}
G(q)G(q^{44}) + q^9H(q)H(q^{44}) &= U(q), \\
H(q^4)G(q^{44}) - q^8G(q^4)H(q^{44}) &= 1, \\
G(q^{11})G(q^{44}) + q^{11}H(q^{11})H(q^{44}) &= \chi^2(q^{11}).
\end{aligned}
$$

Regard this system of equations as three equations in the unknowns $G(q^{44})$, $q^8H(q^{44})$, and -1. It follows that

$$\begin{vmatrix} G(q) & qH(q) & U(q) \\ H(q^4) & -G(q^4) & 1 \\ G(q^{11}) & q^3H(q^{11}) & \chi^2(q^{11}) \end{vmatrix} = 0. \tag{8.5.25.17}$$

Expanding the determinant (8.5.25.17) by the last column, and then using the definition (8.3.32) of V, (8.3.5), and (8.3.3), we find that

$$U(q)V(q) + q - \chi^2(q^{11})\chi^2(q) = 0,$$

which is precisely (8.3.34).

8.5.26 Proof of Entry 8.3.28

Our argument below for the first portion of Entry 8.3.28 is the same as that of Bressoud [81].

To prove (8.3.36), we use (8.2.11) to rewrite the identity as

$$\frac{f(-q^{34},-q^{51})f(-q^2,-q^8) - q^3f(-q^4,-q^6)f(-q^{17},-q^{68})}{f(-q^2,-q^3)f(-q^{68},-q^{102}) + q^7f(-q,-q^4)f(-q^{34},-q^{136})} \\ = \frac{\chi(-q)f(-q^2)f(-q^{17})}{\chi(-q^{17})f(-q)f(-q^{34})}. \tag{8.5.26.1}$$

From (8.2.9) and (8.2.10), it is easy to see that

$$\begin{aligned} \chi(-q)f(-q^2)f(-q^{17}) &= (q;q^2)_\infty(q^2;q^2)_\infty(q^{17};q^{17})_\infty \\ &= (q;q)_\infty(q^{17};q^{34})_\infty(q^{34};q^{34})_\infty \\ &= f(-q)\chi(-q^{17})f(-q^{34}). \end{aligned}$$

Thus, the right-hand side of (8.5.26.1) equals 1, and so (8.3.36) is equivalent to

$$\begin{aligned} &f(-q^{34},-q^{51})f(-q^2,-q^8) - q^3f(-q^4,-q^6)f(-q^{17},-q^{68}) \\ &\quad = f(-q^2,-q^3)f(-q^{68},-q^{102}) + q^7f(-q,-q^4)f(-q^{34},-q^{136}). \end{aligned} \tag{8.5.26.2}$$

We now apply (8.4.16) with $\alpha = 1$ and $\beta = \frac{17}{2}$ to obtain

$$\begin{aligned} &\sum_{k=1}^{2} F(1, \tfrac{17}{2}, 3, 5, \tfrac{7}{2}, k) \\ &\quad = q^{7/8}(f(-q^{34},-q^{51})f(-q^2,-q^8) - q^3f(-q^4,-q^6)f(-q^{17},-q^{68})). \end{aligned} \tag{8.5.26.3}$$

Similarly, letting $\alpha = \frac{1}{2}$ and $\beta = 17$ in (8.4.15), we deduce that

$$\begin{aligned} &\sum_{k=1}^{2} F(\tfrac{1}{2}, 17, 1, 5, \tfrac{7}{2}, k) \\ &\quad = q^{7/8}(f(-q^2,-q^3)f(-q^{68},-q^{102}) + q^7f(-q,-q^4)f(-q^{34},-q^{136})). \end{aligned} \tag{8.5.26.4}$$

The two sets of parameters $\{1, \frac{17}{2}, 3, 5, \frac{7}{2}\}$ and $\{\frac{1}{2}, 17, 1, 5, \frac{7}{2}\}$ give rise to the same series on the right-hand side of (8.4.8), since the parameters satisfy the conditions in (8.4.9). Hence, the right-hand sides of (8.5.26.3) and (8.5.26.4) are equal. This completes the proof of (8.5.26.2) and so also of the first part of Entry 8.3.28.

The proof of the second portion of Entry 8.3.28 is due to Yesilyurt [348]. Let

$$S(q) := U(q)V(q) \quad \text{and} \quad T(q) := \chi^2(-q)\chi^2(-Q), \quad Q := q^{17}. \tag{8.5.26.5}$$

The proof of (8.3.37) will follow from a series of identities given below. The last identity, (8.5.26.13), is clearly equivalent to (8.3.37). We have

$$\chi(-Q)U(q) = \frac{\chi(Q)}{\chi(-q^2)} - q^2\frac{\chi(q)}{\chi(-Q^2)}, \tag{8.5.26.6}$$

$$2qV(q^2) = \chi^2(-Q^2)\left(\frac{\chi(q)}{\chi(Q)} - \frac{\chi(-q)}{\chi(-Q)}\right), \tag{8.5.26.7}$$

$$\chi(-Q^2)U(q)U(-q) = \frac{\chi(Q^2)}{\chi(-q^4)} + q^4\frac{\chi(q^2)}{\chi(-Q^4)}, \tag{8.5.26.8}$$

$$2U(-q^2)V(q^4) = \chi^2(-Q^2)\left(\frac{\chi(q)}{\chi(Q)} + \frac{\chi(-q)}{\chi(-Q)}\right), \tag{8.5.26.9}$$

$$S(q)S(-q) - S(q^2) = \frac{4q^4}{T(q^2)}, \tag{8.5.26.10}$$

$$S(-q)S(q^2) - qS(q) = T(q), \tag{8.5.26.11}$$

$$S^3(q) - 5qS(q) = T(q) + \frac{4q^3}{T(q)}, \tag{8.5.26.12}$$

$$S^4(q) - qS^2(q) = \frac{\chi^3(-q^{17})}{\chi^3(-q)}\left(1 + q^2\frac{\chi^3(-q)}{\chi^3(-q^{17})}\right)^2. \tag{8.5.26.13}$$

We commence by proving (8.5.26.6). By (8.4.65) with the set of parameters $z = 1$, $\epsilon = 1$, $\delta = 1$, $l = t = 0$, $\alpha = 17$, $\beta = 3$, $m = 1$, and $p = 4$ ($\lambda = 5$), we find that

$$R_1(1, 1, 1, 0, 0, 17, 3, 1, 4) = R_1(1, 1, 1, 7, -2, 1, 51, 17, 5). \tag{8.5.26.14}$$

By Lemma 8.4.2, we also find that

$$R_1(1, 1, 1, 0, 0, 17, 3, 1, 4) = q^{1/3}f(-q^8)\left(\varphi(-Q^4) - q^4\frac{\tau(-q^4)\psi(-Q^2)}{\psi(-q^2)}\right). \tag{8.5.26.15}$$

By several applications of (8.2.6) together with (8.2.11), we find that

$$\frac{f(-q^2, -q^3)}{f(q, q^4)} = \frac{f(-q)}{f(-q^5)}G(q^2) \quad \text{and} \quad \frac{f(-q, -q^4)}{f(q^2, q^3)} = \frac{f(-q)}{f(-q^5)}H(q^2). \tag{8.5.26.16}$$

Employing Lemma 8.4.2 again, (8.2.5) with $n = 7$ and $n = 10$, (8.5.26.16) with q replaced by Q^2, and (8.2.11), we see that

$$\begin{aligned}
&R_1(1,1,1,7,-2,1,51,17,5)\\
&= q^{1/3} f(-Q^{10}) \left(\frac{f(-q^4,-q^6)f(-Q^4,-Q^6)}{f(Q^2,Q^8)} + q^{14}\frac{f(-q^2,-q^8)f(-Q^2,-Q^8)}{f(Q^4,Q^6)} \right)\\
&= q^{1/3} f(-Q^2)f(-q^2)\left(G(q^2)G(Q^4) + q^{14}H(q^2)H(Q^4)\right)\\
&= q^{1/3} f(-q^2)f(-Q^2)V(q^2). \qquad (8.5.26.17)
\end{aligned}$$

Therefore, by (8.5.26.14)–(8.5.26.17), after replacing q^2 by q, and by (8.2.7)–(8.2.9), we arrive at

$$\begin{aligned}
V(q) &= \frac{f(-q^4)}{f(-q)f(-Q)} \left(\varphi(-Q^2) - q^2 \frac{\tau(-q^2)\psi(-Q)}{\psi(-q)} \right)\\
&= \frac{1}{\chi(-q)} \left(\frac{\chi(Q)}{\chi(-q^2)} - q^2 \frac{\chi(q)}{\chi(-Q^2)} \right), \qquad (8.5.26.18)
\end{aligned}$$

which, by (8.3.36), is equivalent to (8.5.26.6).

Next, we prove (8.5.26.7). Keeping in mind the notation of Entry 8.3.28, recall that

$$\begin{aligned}
G(Q^2)H(q^4) - q^6 H(Q^2)G(q^4) &= U(q^2),\\
G(Q^2)G(q) + q^7 H(Q^2)H(q) &= V(q),\\
G(Q^2)G(-q) - q^7 H(Q^2)H(-q) &= V(-q).
\end{aligned}$$

Regarding $G(Q^2)$, $q^6H(Q^2)$, and 1 as the "variables," we conclude from this triple of equations that

$$\begin{vmatrix} H(q^4) & -G(q^4) & U(q^2) \\ G(q) & qH(q) & V(q) \\ G(-q) & -qH(-q) & V(-q) \end{vmatrix} = 0. \qquad (8.5.26.19)$$

Expanding this determinant (8.5.26.19) by the last column and using Entries 8.3.20 and 8.3.2, we deduce that

$$-2q\frac{U(q^2)}{\chi^2(-q^2)} - V(q)\chi^2(-q) + V(-q)\chi^2(q) = 0. \qquad (8.5.26.20)$$

We should remark that by (8.3.36), the identity (8.5.26.20) is equivalent to

$$\chi(q)\chi(Q)U(-q) - \chi(-q)\chi(-Q)U(q) = 2q\frac{U(q^2)}{\chi^2(-q^2)}. \qquad (8.5.26.21)$$

Therefore, by (8.5.26.20) and two applications of (8.5.26.18) with q replaced by $-q$ in the first application, we find that

$$2q\frac{U(q^2)}{\chi^2(-q^2)} = \chi^2(q)\left(\frac{1}{\chi(q)}\left(\frac{\chi(-Q)}{\chi(-q^2)} - q^2\frac{\chi(-q)}{\chi(-Q^2)}\right)\right) - \chi^2(-q)\left(\frac{1}{\chi(-q)}\left(\frac{\chi(Q)}{\chi(-q^2)} - q^2\frac{\chi(q)}{\chi(-Q^2)}\right)\right)$$
$$= \frac{\chi(-Q)}{\chi(-q)} - \frac{\chi(Q)}{\chi(q)}$$
$$= \frac{\chi(-Q^2)}{\chi(-q^2)}\left(\frac{\chi(q)}{\chi(Q)} - \frac{\chi(-q)}{\chi(-Q)}\right),$$

which, by (8.3.36), is equivalent to (8.5.26.7). We remark that a proof of (8.5.26.7) similar to the proof of (8.5.26.6) can be given.

Next, we prove (8.5.26.8). By (8.4.65) with the set of parameters $z = 1$, $\epsilon = 0$, $\delta = 1$, $l = t = 0$, $\alpha = 17$, $\beta = 3$, $m = 1$, and $p = 4$ ($\lambda = 5$), we find that

$$R_1(1,0,1,0,0,17,3,1,4) = R_1(1,1,0,7,-2,1,51,17,5). \tag{8.5.26.22}$$

By Lemma 8.4.2, we find that, since $\lambda = 5$,

$$R_1(1,0,1,0,0,17,3,1,4) = q^{1/3}f(-q^8)\left(\varphi(-Q^4) + q^4\frac{\tau(-q^4)\psi(-Q^2)}{\psi(-q^2)}\right). \tag{8.5.26.23}$$

Using (8.1.2), (8.2.6), and some elementary product manipulations, we can show that

$$G(q)G(-q) = \frac{f(q^4,q^6)}{f(-q^2)} \qquad \text{and} \qquad H(q)H(-q) = \frac{f(q^2,q^8)}{f(-q^2)}. \tag{8.5.26.24}$$

By Lemma 8.4.2, (8.2.5) with $n = 7$ and $n = 10$, (8.2.11), (8.5.31.1), and Entry 8.3.21, we also find that since $\lambda = 68$,

$$R_1(1,1,0,7,-2,1,51,17,5)$$
$$= q^{1/3}f(-Q^{10})\left(\frac{f(q^4,q^6)f(-Q^4,-Q^6)}{f(-Q^2,-Q^8)} - q^{14}\frac{f(q^2,q^8)f(-Q^2,-Q^8)}{f(-Q^4,-Q^6)} - 2q^8\psi(q^{10})\right)$$
$$= q^{1/3}\frac{f(-Q^{10})}{f(-Q^2,-Q^8)f(-Q^4,-Q^6)}\big(f(q^4,q^6)f(-Q^4,-Q^6)^2 - q^{14}f(q^2,q^8)f(-Q^2,-Q^8)^2 - 2q^8f(-Q^2,-Q^8)f(-Q^4,-Q^6)\psi(q^{10})\big)$$
$$= q^{1/3}\frac{f(-Q^{10})f^2(-Q^2)f(-q^2)}{f(-Q^2,-Q^8)f(-Q^4,-Q^6)}\bigg(G(q)G(-q)G^2(Q^2) - q^{14}H(q)H(-q)H^2(Q^2) - 2q^8G(Q^2)H(Q^2)\frac{\psi(q^{10})}{f(-q^2)}\bigg)$$
$$= q^{1/3}f(-q^2)f(-Q^2)((G(q)G(-q)G^2(Q^2) - q^{14}H(q)H(-q)H^2(Q^2)$$

$$
\begin{aligned}
&\quad - q^7 G(Q^2)H(Q^2)\left(G(q)H(-q) - G(-q)H(q)\right)) \\
&= q^{1/3} f(-q^2) f(-Q^2)\left(G(q)G(Q^2) + q^7 H(q)H(Q^2)\right) \\
&\quad \times \left(G(-q)G(Q^2) - q^7 H(-q)H(Q^2)\right) \\
&= q^{1/3} f(-q^2) f(-Q^2) V(q)V(-q). \qquad (8.5.26.25)
\end{aligned}
$$

Therefore, by (8.5.26.22)–(8.5.26.25), we conclude that

$$
V(q)V(-q) = \frac{f(-q^8)}{f(-q^2)f(-Q^2)}\left(\varphi(-Q^4) + q^4 \frac{\tau(-q^4)\psi(-Q^2)}{\psi(-q^2)}\right). \qquad (8.5.26.26)
$$

Now (8.5.26.8) follows by considerations similar to those in (8.5.26.18). In particular, one uses the product representations in (8.2.7) and (8.2.8).

Next, we prove (8.5.26.9). First, a special case of (8.2.19) is given by

$$
\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \qquad (8.5.26.27)
$$

From (8.2.7)–(8.2.9) and (8.5.26.27), we see that

$$
\begin{aligned}
\chi^2(q) = \frac{\tau(q)}{f(-q^2)} &= \frac{\tau(q^4) + 2q\psi(q^8)}{f(-q^2)} \\
&= \frac{\chi^2(q^4)}{\chi(-q^2)\chi(-q^4)} + 2q \frac{1}{\chi^2(-q^8)\chi(-q^2)\chi(-q^4)}, \qquad (8.5.26.28)
\end{aligned}
$$

and, with the replacement of q by $-q$,

$$
\chi^2(-q) = \frac{\chi^2(q^4)}{\chi(-q^2)\chi(-q^4)} - 2q \frac{1}{\chi^2(-q^8)\chi(-q^2)\chi(-q^4)}. \qquad (8.5.26.29)
$$

Therefore, adding (8.5.26.28) and (8.5.26.29), and next subtracting (8.5.26.29) from (8.5.26.28), we find that, respectively,

$$
2\chi^2(q^4) = \chi(-q^2)\chi(-q^4)\left(\chi(q) + \chi(-q)\right), \qquad (8.5.26.30)
$$

$$
\frac{4q}{\chi^2(-q^8)} = \chi(-q^2)\chi(-q^4)\left(\chi(q) - \chi(-q)\right). \qquad (8.5.26.31)
$$

By (8.5.26.6) with q replaced by q^4, (8.5.26.8) with q replaced by q^2, two applications of both (8.5.26.30) and (8.5.26.31) with q replaced by Q in one of the applications of each, and (8.5.26.7), we find that

$$
\begin{aligned}
&\chi^2(-Q^4)U(-q^2)U(q^2)U(q^4) \\
&= \frac{\chi^2(Q^4)}{\chi^2(-q^8)} - q^{16} \frac{\chi^2(q^4)}{\chi^2(-Q^8)} \\
&= \frac{\chi(-q^2)\chi(-q^4)\chi(-Q^2)\chi(-Q^4)}{8q}\left(\left(\chi^2(Q) + \chi^2(-Q)\right)\left(\chi^2(q) - \chi^2(-q)\right)\right. \\
&\quad \left. - \left(\chi^2(Q) - \chi^2(-Q)\right)\left(\chi^2(q) + \chi^2(-q)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\chi(-q^2)\chi(-q^4)\chi(-Q^2)\chi(-Q^4)}{4q}\left(\chi^2(q)\chi^2(-Q) - \chi^2(-q)\chi^2(Q)\right) \\
&= \frac{\chi(-q^2)\chi(-q^4)\chi^3(-Q^2)\chi(-Q^4)}{4q}\left(\frac{\chi(q)}{\chi(Q)} - \frac{\chi(-q)}{\chi(-Q)}\right)\left(\frac{\chi(q)}{\chi(Q)} + \frac{\chi(-q)}{\chi(-Q)}\right) \\
&= \frac{\chi(-q^2)\chi(-q^4)\chi(-Q^2)\chi(-Q^4)}{2}V(q^2)\left(\frac{\chi(q)}{\chi(Q)} + \frac{\chi(-q)}{\chi(-Q)}\right). \qquad (8.5.26.32)
\end{aligned}
$$

By two applications of (8.3.36), we observe that

$$
\chi^2(-Q^4)U(-q^2)U(q^2)U(q^4) = \chi^2(-Q^4)U(-q^2)V(q^2)\frac{\chi(-q^2)}{\chi(-Q^2)}V(q^4)\frac{\chi(-q^4)}{\chi(-Q^4)}. \qquad (8.5.26.33)
$$

Finally, we use (8.5.26.33) on the leftmost side of (8.5.26.32) to complete the proof of (8.5.26.9).

Next, we prove (8.5.26.10). By (8.5.26.8) and (8.5.26.6) with q replaced by q^2, we find that

$$
\chi^2(-Q^2)U^2(q)U^2(-q) - \chi^2(-Q^2)U^2(q^2) = 4q^4\frac{1}{\chi(-q^2)\chi(-Q^2)}. \qquad (8.5.26.34)
$$

From (8.5.26.34) and (8.3.36), we deduce that

$$
\begin{aligned}
\chi^2(-Q^2)S(q)S(-q)\frac{\chi(-q)}{\chi(-Q)}\frac{\chi(q)}{\chi(Q)} - \chi^2(-Q^2)S(q^2)\frac{\chi(-q^2)}{\chi(-Q^2)} \\
= 4q^4\frac{1}{\chi(-q^2)\chi(-Q^2)},
\end{aligned}
$$

from which (8.5.26.10) readily follows.

Next, we prove (8.5.26.11). Adding (8.5.26.7) and (8.5.26.9), we find that

$$
\chi^2(-Q^2)\frac{\chi(q)}{\chi(Q)} = U(-q^2)V(q^4) + qV(q^2). \qquad (8.5.26.35)
$$

In (8.5.26.35), we replace q by $-q$ and multiply the resulting identity by (8.5.26.35) to arrive at

$$
\chi^4(-Q^2)\frac{\chi(q)}{\chi(Q)}\frac{\chi(-q)}{\chi(-Q)} = U^2(-q^2)V^2(q^4) - q^2V^2(q^2). \qquad (8.5.26.36)
$$

Lastly, replacing q^2 by q in (8.5.26.36) and employing (8.3.36) several times, we deduce (8.5.26.11).

Now we prove (8.5.26.12). In (8.5.26.11), we replace q by $-q$ and multiply the resulting identity by (8.5.26.11) to find that

$$
S(q)S(-q)\left(S^2(q^2) - q^2\right) - qS(q^2)\left(S^2(q) - S^2(-q)\right) = T(q)T(-q) = T(q^2). \qquad (8.5.26.37)
$$

Taking (8.5.26.10) and (8.5.26.11), with q replaced by q^2, and multiplying them together, we find that

$$\begin{aligned} S(q)S(-q)S(-q^2)S(q^4) &= \left(S(q^2)+\frac{4q^4}{T(q^2)}\right)\left(q^2S(q^2)+T(q^2)\right) \\ &= q^2S^2(q^2)+S(q^2)\left(T(q^2)+\frac{4q^6}{T(q^2)}\right)+4q^4. \end{aligned} \tag{8.5.26.38}$$

Remembering the definitions of U and V in (8.3.35) and starting with the relations

$$\begin{aligned} G(q^2)G(Q^4)+q^{14}H(q^2)H(Q^4) &= V(q^2), \\ -q^3G(q^2)H(Q)+H(q^2)G(Q) &= U(q), \\ q^3G(q^2)H(-Q)+H(q^2)G(-Q) &= U(-q), \end{aligned}$$

we argue as in (8.5.26.19)–(8.5.26.21). Thus, considering the triple of equations above as equations in $G(q^2)$, $H(q^2)$, and 1, we find that

$$\begin{aligned} 0 &= \begin{vmatrix} G(Q^4) & q^{14}H(Q^4) & V(q^2) \\ -q^3H(Q) & G(Q) & U(q) \\ q^3H(-Q) & G(-Q) & U(-q) \end{vmatrix} \\ &= -q^3V(q^2)\left\{H(Q)G(-Q)+H(-Q)G(Q)\right\} \\ &\quad -U(q)\left\{G(Q^4)G(-Q)-QH(Q^4)H(-Q)\right\} \\ &\quad +U(-q)\left\{G(Q)G(Q^4)+QH(Q)H(Q^4)\right\} \\ &= -q^3V(q^2)\frac{2}{\chi^2(-Q^2)}-U(q)\chi^2(-Q)+U(-q)\chi^2(Q), \end{aligned}$$

where we used Entry 8.3.20 with q replaced by $-Q$ and Entry 8.3.2 with q replaced by $-Q$ and Q, respectively. Using (8.3.36), we deduce that

$$\chi(q)\chi(Q)V(-q)-\chi(-q)\chi(-Q)V(q)=2q^3\frac{V(q^2)}{\chi^2(-Q^2)}. \tag{8.5.26.39}$$

Next, we multiply (8.5.26.21) and (8.5.26.39) together to find that

$$\begin{aligned} &T(-q)S(-q)+T(q)S(q)-\chi(-q^2)\chi(-Q^2)\left(U(q)V(-q)+U(-q)V(q)\right) \\ &\qquad = 4q^4\frac{S(q^2)}{T(q^2)}. \end{aligned} \tag{8.5.26.40}$$

By (8.3.36) and (8.5.26.9), we observe that

$$\begin{aligned} U(q)V(-q)+U(-q)V(q) &= V(q)V(-q)\left(\frac{U(q)}{V(q)}+\frac{U(-q)}{V(-q)}\right) \\ &= 2V(q)V(-q)\frac{U(-q^2)V(q^4)}{\chi^2(-Q^2)}. \end{aligned} \tag{8.5.26.41}$$

In (8.5.26.40), we use (8.5.26.41) and the values of $T(q)$ and $T(-q)$ given in (8.5.26.11) to find that

$$2S(q)S(-q)S(q^2) - q\big(S^2(q) - S^2(-q)\big) \\ - 2\frac{\chi(-q^2)}{\chi(-Q^2)}V(q)V(-q)U(-q^2)V(q^4) = 4q^4\frac{S(q^2)}{T(q^2)}. \qquad (8.5.26.42)$$

Observe from (8.3.36) and (8.5.26.5) that

$$\frac{\chi(-q^2)}{\chi(-Q^2)}V(q)V(-q)U(-q^2)V(q^4) = \sqrt{S(q)S(-q)S(-q^2)S(q^4)}. \qquad (8.5.26.43)$$

Therefore (8.5.26.42) can be written in the form

$$2S(q)S(-q)S(q^2) - q\left(S^2(q) - S^2(-q)\right) \\ - 2\sqrt{S(q)S(-q)S(-q^2)S(q^4)} = 4q^4\frac{S(q^2)}{T(q^2)}. \qquad (8.5.26.44)$$

Next, we multiply both sides of (8.5.26.44) by $S(q^2)$ and substitute the value of $S(q^2)\left(S^2(q) - S^2(-q)\right)$ from (8.5.26.37) to deduce that

$$S(q)S(-q)S^2(q^2) + T(q^2) + q^2S(q)S(-q) \\ - 2S(q^2)\sqrt{S(q)S(-q)S(-q^2)S(q^4)} = 4q^4\frac{S^2(q^2)}{T(q^2)}. \qquad (8.5.26.45)$$

Now in (8.5.26.45), we substitute the value of $S(q)S(-q)$ from (8.5.26.10) to find that

$$S^3(q^2) + T(q^2) + \frac{4q^6}{T(q^2)} + q^2S(q^2) = 2S(q^2)\sqrt{S(q)S(-q)S(-q^2)S(q^4)}. \qquad (8.5.26.46)$$

Next, we return to (8.5.26.38) and use (8.5.26.46) to deduce that

$$S^4(q^2) - 2S^2(q^2)\sqrt{S(q)S(-q)S(-q^2)S(q^4)} + S(q)S(-q)S(-q^2)S(q^4) = 4q^4. \qquad (8.5.26.47)$$

From (8.5.26.47), we conclude that

$$\sqrt{S(q)S(-q)S(-q^2)S(q^4)} = S^2(q^2) - 2q^2. \qquad (8.5.26.48)$$

Using (8.5.26.48) in (8.5.26.38), we finally conclude that

$$S^3(q^2) - 5q^2S(q^2) = T(q^2) + \frac{4q^6}{T(q^2)}, \qquad (8.5.26.49)$$

which is (8.5.26.12) with q replaced by q^2.

Lastly, we prove (8.5.26.13). By two applications of (8.5.26.6), with q replaced by $-q$ in the second, and by (8.5.26.9), we find that

$$
\begin{aligned}
\chi(-Q^2)U(q)U(-q) &= \left(\frac{\chi(Q)}{\chi(-q^2)} - q^2\frac{\chi(q)}{\chi(-Q^2)}\right)\left(\frac{\chi(-Q)}{\chi(-q^2)} - q^2\frac{\chi(-q)}{\chi(-Q^2)}\right)\\
&= \frac{\chi(-Q^2)}{\chi^2(-q^2)} - q^2\left(\frac{1}{\chi(q)\chi(-Q)} + \frac{1}{\chi(-q)\chi(Q)}\right) + q^4\frac{\chi(-q^2)}{\chi^2(-Q^2)}\\
&= \frac{\chi(-Q^2)}{\chi^2(-q^2)}\left(1 + q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right) - q^2\frac{1}{\chi(-q^2)}\left(\frac{\chi(q)}{\chi(Q)} + \frac{\chi(-q)}{\chi(-Q)}\right)\\
&= \frac{\chi(-Q^2)}{\chi^2(-q^2)}\left(1 + q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right) - 2q^2\frac{1}{\chi(-q^2)\chi^2(-Q^2)}U(-q^2)V(q^4).
\end{aligned}
\tag{8.5.26.50}
$$

Squaring (8.5.26.50), we deduce that

$$
\begin{aligned}
&\frac{\chi^2(-Q^2)}{\chi^4(-q^2)}\left(1 + q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)^2\\
&= \left(\chi(-Q^2)U(q)U(-q) + 2q^2\frac{1}{\chi(-q^2)\chi^2(-Q^2)}U(-q^2)V(q^4)\right)^2\\
&= \chi^2(-Q^2)U^2(q)U^2(-q) + 4q^2\frac{1}{\chi(-q^2)\chi(-Q^2)}U(-q)U(q)U(-q^2)V(q^4)\\
&\quad + 4q^4\frac{1}{\chi^2(-q^2)\chi^4(-Q^2)}U^2(-q^2)V^2(q^4).
\end{aligned}
\tag{8.5.26.51}
$$

Multiply both sides of (8.5.26.51) by $\chi(-q^2)\chi(-Q^2)$, make several applications of (8.3.36), and employ (8.5.26.43) to arrive at

$$
\begin{aligned}
&\frac{\chi^3(-Q^2)}{\chi^3(-q^2)}\left(1 + q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)^2\\
&= S(q)S(-q)T(q^2) + 4q^2\sqrt{S(q)S(-q)S(-q^2)S(q^4)} + 4q^4\frac{S(-q^2)S(q^4)}{T(q^2)}.
\end{aligned}
\tag{8.5.26.52}
$$

Next, we employ (8.5.26.10), (8.5.26.48), (8.5.26.11), and (8.5.26.12), where in the last two applications q is replaced by q^2, to find that

$$
\begin{aligned}
&\frac{\chi^3(-Q^2)}{\chi^3(-q^2)}\left(1 + q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)^2\\
&= \left(S(q^2) + 4q^4\frac{1}{T(q^2)}\right)T(q^2) + 4q^2(S^2(q^2) - 2q^2) + 4q^4\left(\frac{q^2S(q^2) + T(q^2)}{T(q^2)}\right)\\
&= S(q^2)\left(T(q^2) + 4q^6\frac{1}{T(q^2)}\right) + 4q^2S^2(q^2)\\
&= S(q^2)\left(S^3(q^2) - 5q^2S(q^2)\right) + 4q^2S^2(q^2)\\
&= S^4(q^2) - q^2S^2(q^2),
\end{aligned}
$$

which is (8.5.26.13) with q replaced by q^2.

8.5.27 Proof of Entry 8.3.29

Let $Q := q^{23}$ and set

$$A(q) := G(q^2)G(Q)+q^5H(q^2)H(Q) \text{ and } B(q) := H(q)G(Q^2)-q^9G(q)H(Q^2). \tag{8.5.27.1}$$

Using (8.4.63) and noting that $\lambda = 7$, we find that

$$R_2(1,0,0,1,3,23,2,5) = R_2(1,-9,0,4,69,1,1,10). \tag{8.5.27.2}$$

We employ Lemma 8.4.3 and argue similarly as in (8.5.26.17). Noting that $\lambda = 7$ and using (8.2.5) with $n = 1$, (8.2.11), and (8.5.26.16), we find that

$$\begin{aligned} &R_2(1,0,0,1,3,23,2,5)\\ &= q^{7/6}f(-q^{10})\left(\frac{f(-q^4,-q^6)f(-Q^4,-Q^6)}{f(q^2,q^8)} + q^{10}\frac{f(-q^2,-q^8)f(-Q^2,-Q^8)}{f(q^4,q^6)}\right)\\ &= q^{7/6}f(-q^2)f(-Q^2)A(q^2). \end{aligned} \tag{8.5.27.3}$$

Next, we employ Lemma 8.4.3 to find that

$$\begin{aligned} &R_2(1,-9,0,4,69,1,1,10)\\ &= q^{7/6}f(-Q^{20})\left\{-q^2\frac{f(-Q^4,-Q^{16})f(-q^8,-q^{12})}{f(-Q^2,-Q^{18})}\right.\\ &\quad + \frac{f(-Q^8,-Q^{12})f(-q^6,-q^{14})}{f(-Q^4,-Q^{16})} - q^{12}\frac{f(-Q^8,-Q^{12})f(-q^4,-q^{16})}{f(-Q^6,-Q^{14})}\\ &\quad \left. +q^{38}\frac{f(-Q^4,-Q^{16})f(-q^2,-q^{18})}{f(-Q^8,-Q^{12})} + q^{18}\varphi(-q^{10})\right\}. \end{aligned} \tag{8.5.27.4}$$

Using (8.1.2), (8.2.6), and some elementary product manipulations, we can show that

$$\frac{f(-q^2,-q^8)}{f(-q,-q^9)} = \frac{f(-q^2)}{f(-q^{10})}G(q), \quad \frac{f(-q^4,-q^6)}{f(-q^3,-q^7)} = \frac{f(-q^2)}{f(-q^{10})}H(q). \tag{8.5.27.5}$$

By (8.5.27.5) with q replaced by Q^2 and by (8.1.2), we deduce that

$$\begin{aligned} &-q^2\frac{f(-Q^4,-Q^{16})f(-q^8,-q^{12})}{f(-Q^2,-Q^{18})} - q^{12}\frac{f(-Q^8,-Q^{12})f(-q^4,-q^{16})}{f(-Q^6,-Q^{14})}\\ &\quad = -q^2\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\left(G(q^4)G(Q^2) + q^{10}H(q^4)H(Q^2)\right)\\ &\quad = -q^2\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}A(q^2). \end{aligned} \tag{8.5.27.6}$$

With the use of (8.2.6), it is easy to verify that

$$f(-q^3,-q^7)=f(-q^2)H(-q)G(q^4) \quad \text{and} \quad f(-q,-q^9)=f(-q^2)G(-q)H(q^4). \tag{8.5.27.7}$$

Furthermore, from (8.2.11), we easily see that

$$\frac{f(-q,-q^4)}{f(-q^2,-q^3)}=\frac{f(-q)}{f(-q^5)}H^2(q) \quad \text{and} \quad \frac{f(-q^2,-q^3)}{f(-q,-q^4)}=\frac{f(-q)}{f(-q^5)}G^2(q). \tag{8.5.27.8}$$

Using (8.5.27.7) with q replaced by q^2 and (8.5.27.8) with q replaced by Q^4, we find that

$$\begin{aligned}
&\frac{f(-Q^8,-Q^{12})f(-q^6,-q^{14})}{f(-Q^4,-Q^{16})}+q^{38}\frac{f(-Q^4,-Q^{16})f(-q^2,-q^{18})}{f(-Q^8,-Q^{12})}+q^{18}\varphi(-q^{10})\\
&=\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\biggl(G^2(Q^4)H(-q^2)G(q^8)+q^{38}H^2(Q^4)G(-q^2)H(q^8)\\
&\qquad\qquad+q^{18}\frac{\varphi(-q^{10})}{f(-q^4)}\frac{f(-Q^{20})}{f(-Q^4)}\biggr).
\end{aligned} \tag{8.5.27.9}$$

Next, by Entry 8.3.3 with q replaced by $-q^2$ and by (8.2.12) with q replaced by Q^4, we deduce that

$$\frac{\varphi(-q^{10})}{f(-q^4)}\frac{f(-Q^{20})}{f(-Q^4)}=\left(G(-q^2)G(q^8)+q^2H(-q^2)H(q^8)\right)G(Q^4)H(Q^4). \tag{8.5.27.10}$$

Utilizing (8.5.27.10) in (8.5.27.9) and recalling the notation (8.5.27.1), we arrive at

$$\begin{aligned}
&\frac{f(-Q^8,-Q^{12})f(-q^6,-q^{14})}{f(-Q^4,-Q^{16})}+q^{38}\frac{f(-Q^4,-Q^{16})f(-q^2,-q^{18})}{f(-Q^8,-Q^{12})}+q^{18}\varphi(-q^{10})\\
&=\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\left(G^2(Q^4)H(-q^2)G(q^8)+q^{38}H^2(Q^4)G(-q^2)H(q^8)\right.\\
&\quad\left.+q^{18}\left(G(-q^2)G(q^8)+q^2H(-q^2)H(q^8)\right)G(Q^4)H(Q^4)\right)\\
&=\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\left(G(q^8)G(Q^4)+q^{20}H(q^8)H(Q^4)\right)\\
&\quad\times\left(H(-q^2)G(Q^4)+q^{18}G(-q^2)H(Q^4)\right)\\
&=\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}A(q^4)B(-q^2).
\end{aligned} \tag{8.5.27.11}$$

From (8.5.27.4), (8.5.27.6), and (8.5.27.11), we conclude that

$$R_2(1,-9,0,4,69,1,1,10)=q^{7/6}f(-q^4)f(-Q^4)\left(-q^2A(q^2)+A(q^4)B(-q^2)\right). \tag{8.5.27.12}$$

Therefore, by (8.5.27.2), (8.5.27.3), and (8.5.27.12), we deduce that

$$f(-q^2)f(-Q^2)A(q^2)=f(-q^4)f(-Q^4)\left(-q^2A(q^2)+A(q^4)B(-q^2)\right). \tag{8.5.27.13}$$

Lastly, replacing q^2 by q, we conclude that

$$(\chi(-q)\chi(-Q)+q)\,A(q)=B(-q)A(q^2). \tag{8.5.27.14}$$

Next, we prove the companion formula

$$(\chi(-q)\chi(-Q)+q)\,B(q)=A(-q)B(q^2). \tag{8.5.27.15}$$

By (8.4.63), we find that with $\lambda=7$,

$$R_2(1,1,1,1,3,23,2,5)=R_2(1,-8,1,-5,69,1,1,10) \tag{8.5.27.16}$$

and

$$R_2(1,1,0,0,3,23,2,5)=R_2(1,-8,0,3,69,1,1,10). \tag{8.5.27.17}$$

With applications of Lemma 8.4.3 and (8.2.5), we find that since again $\lambda=7$,

$$\begin{aligned}
&R_2(1,-8,1,-5,69,1,1,10)\\
&=q^{5/12}f(-Q^{20})\left\{q^{11}\frac{f(-Q^2,-Q^{18})f(q,q^{19})}{f(Q,Q^{19})}-q\frac{f(-Q^6,-Q^{14})f(q^3,q^{17})}{f(Q^3,Q^{17})}\right.\\
&\qquad+q^5\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)}-q^{23}\frac{f(-Q^6,-Q^{14})f(q^7,q^{13})}{f(Q^7,Q^{13})}\\
&\qquad\left.+q^{55}\frac{f(-Q^2,-Q^{18})f(q^9,q^{11})}{f(Q^9,Q^{11})}\right\}.
\end{aligned} \tag{8.5.27.18}$$

Employing Lemma 8.4.3, again with the observation that $\lambda=7$, we also find that

$$\begin{aligned}
&R_2(1,-8,0,3,69,1,1,10)\\
&=q^{5/12}f(-Q^{20})\left\{-q^9\frac{f(-Q^2,-Q^{18})f(q^9,q^{11})}{f(Q,Q^{19})}+\frac{f(-Q^6,-Q^{14})f(q^7,q^{13})}{f(Q^3,Q^{17})}\right.\\
&\qquad-q^5\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)}+q^{24}\frac{f(-Q^6,-Q^{14})f(q^3,q^{17})}{f(Q^7,Q^{13})}\\
&\qquad\left.-q^{57}\frac{f(-Q^2,-Q^{18})f(q,q^{19})}{f(Q^9,Q^{11})}\right\}.
\end{aligned} \tag{8.5.27.19}$$

Employing (8.2.19) twice with $a=-q^2$, $b=-q^3$, $n=2$, and $a=-q$, $b=-q^4$, and $n=2$, we easily find that

$$f(-q^2,-q^3)=f(q^9,q^{11})-q^2f(q,q^{19}) \text{ and } f(-q,-q^4)=f(q^7,q^{13})-qf(q^3,q^{17}). \tag{8.5.27.20}$$

Next, we add (8.5.27.18) and (8.5.27.19), collect terms, and use (8.5.27.20) to find that

$$\begin{aligned}
&R_2(1,-8,1,-5,69,1,1,10)+R_2(1,-8,0,3,69,1,1,10)\\
&=q^{5/12}f(-Q^{20})\left\{-q^9\frac{f(-Q^2,-Q^{18})f(-q^2,-q^3)}{f(Q,Q^{19})}+\frac{f(-Q^6,-Q^{14})f(-q,-q^4)}{f(Q^3,Q^{17})}\right.\\
&\quad\left.-q^{23}\frac{f(-Q^6,-Q^{14})f(-q,-q^4)}{f(Q^7,Q^{13})}+q^{55}\frac{f(-Q^2,-Q^{18})f(-q^2,-q^3)}{f(Q^9,Q^{11})}\right\}.
\end{aligned} \tag{8.5.27.21}$$

Using (8.5.27.20) again, this time with q replaced by Q, we find from (8.5.27.21) that

$$\begin{aligned}
&R_2(1,-8,1,-5,69,1,1,10)+R_2(1,-8,0,3,69,1,1,10)\\
&=q^{5/12}f(-Q^{20})\left\{-q^9\frac{f(-Q^2,-Q^{18})f(-q^2,-q^3)f(-Q^2,-Q^3)}{f(Q,Q^{19})f(Q^9,Q^{11})}\right.\\
&\quad\left.+\frac{f(-Q^6,-Q^{14})f(-q,-q^4)f(-Q,-Q^4)}{f(Q^3,Q^{17})f(Q^7,Q^{13})}\right\}.
\end{aligned} \tag{8.5.27.22}$$

With several applications of (8.2.6), we can verify that

$$\frac{f(-q^2,-q^{18})f(-q^2,-q^3)}{f(q,q^{19})f(q^9,q^{11})}=\frac{f(-q)H(q^2)}{f(-q^{20})}, \tag{8.5.27.23}$$

$$\frac{f(-q^6,-q^{14})f(-q,-q^4)}{f(q^3,q^{17})f(q^7,q^{13})}=\frac{f(-q)G(q^2)}{f(-q^{20})}. \tag{8.5.27.24}$$

Returning to (8.5.27.22) and using (8.2.11) along with (8.5.27.23) and (8.5.27.24) with q replaced by Q, we conclude that

$$\begin{aligned}
&R_2(1,-8,1,-5,69,1,1,10)+R_2(1,-8,0,3,69,1,1,10)\\
&=q^{5/12}f(-q)f(-Q)\left(H(q)G(Q^2)-q^9G(q)H(Q^2)\right)\\
&=q^{5/12}f(-q)f(-Q)B(q).
\end{aligned} \tag{8.5.27.25}$$

By Lemma 8.4.3 with the observation that again $\lambda=7$, (8.2.5), and (8.5.27.5), we find that

$$\begin{aligned}
&R_2(1,1,1,1,3,23,2,5)=-q^{17/12}f(-q^{10})\\
&\quad\times\left(\frac{f(-q^4,-q^6)f(-Q^4,-Q^6)}{f(-q^3,-q^7)}-q^9\frac{f(-q^2,-q^8)f(-Q^2,-Q^8)}{f(-q,-q^9)}\right)\\
&=-q^{17/12}f(-q^2)f(-Q^2)\left(H(q)G(Q^2)-q^9G(q)H(Q^2)\right)\\
&=-q^{17/12}f(-q^2)f(-Q^2)B(q),
\end{aligned} \tag{8.5.27.26}$$

where we once again recall the notation (8.5.27.1).

We employ Lemma 8.4.3 again and see again that $\lambda=7$, and argue similarly as in (8.5.27.9)–(8.5.27.11). Using (8.2.5), (8.5.27.7), (8.5.27.8), (8.5.27.10), (8.2.12), Entry 8.3.3, and (8.5.27.1), we deduce that

$$
\begin{aligned}
&R_2(1,1,0,0,3,23,2,5) = q^{5/12} f(-q^{10}) \\
&\times \left(-q^5 \frac{f(-q^2,-q^8) f(-Q^3,-Q^7)}{f(-q^4,-q^6)} - q^{18} \frac{f(-q^4,-q^6) f(-Q,-Q^9)}{f(-q^2,-q^8)} + \varphi(-Q^5) \right) \\
&= q^{5/12} f(-q^2) f(-Q^2) \left(H(q^2) G(Q^4) - q^{18} G(q^2) H(Q^4) \right) \\
&\quad \times \left(G(q^2) G(-Q) - q^5 H(q^2) H(-Q) \right) \\
&= q^{5/12} f(-q^2) f(-Q^2) B(q^2) A(-q). \qquad (8.5.27.27)
\end{aligned}
$$

Therefore, by (8.5.27.16), (8.5.27.17), (8.5.27.25), (8.5.27.26), and (8.5.27.27), we finally deduce that

$$
-q f(-q^2) f(-Q^2) B(q) + f(-q^2) f(-Q^2) B(q^2) A(-q) = f(-q) f(-Q) B(q), \qquad (8.5.27.28)
$$

which is clearly equivalent to (8.5.27.15).

In (8.5.27.15) replace q by $-q$ and multiply the resulting identity by (8.5.27.14) to conclude that

$$
A(q^2) B(q^2) = \left(\chi(q)\chi(Q) + q\right) \left(\chi(-q)\chi(-Q) - q\right). \qquad (8.5.27.29)
$$

Therefore, by (8.5.27.29) and by (8.3.38), it remains to prove that

$$
\chi(-q^2)\chi(-Q^2) + q^2 + \frac{2q^4}{\chi(-q^2)\chi(-Q^2)} = \left(\chi(q)\chi(Q) - q\right) \left(\chi(-q)\chi(-Q) + q\right). \qquad (8.5.27.30)
$$

However, (8.5.27.30) is equivalent to a modular equation of degree 23, first discovered by Schröter, and rediscovered by Ramanujan [282], namely,

$$
(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1, \qquad (8.5.27.31)
$$

where β is of degree 23 over α. A proof of (8.5.27.31) can be found in [55, p. 411, Chapter 20, Entry 15(i)]. Since the identity (8.5.27.30) cannot be found in either [282] or [55], we briefly indicate how we can establish the equivalence of (8.5.27.30) and (8.5.27.31). If $q = \exp\left(-K(\sqrt{1-\alpha})/K(\sqrt{\alpha})\right)$, where K denotes the complete elliptic integral of the first kind, then [55, p. 124, Entries 12(v)–(vii)]

$$
\begin{aligned}
\chi(q) &= 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}, \\
\chi(-q) &= 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}, \\
\chi(-q^2) &= 2^{1/3}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}.
\end{aligned}
$$

If q is replaced by q^{23} in any of the equalities above, then α is to be replaced by β on the corresponding right-hand side. If all six of these identities are substituted in (8.5.27.30), then after a lengthy, but straightforward, dose of elementary algebra, we obtain (8.5.27.31). These steps are of course reversible, and so this establishes the equivalence of (8.5.27.30) and (8.5.27.31).

8.5.28 Proof of Entry 8.3.30

We begin by recording several pairs of easily verified identities. Note that for future convenience, we have recorded the identity in (8.5.28.3) in two different ways. By (8.2.6),

$$\frac{f(-q^2,-q^8)}{f(-q,-q^9)}=\frac{f(-q^2,-q^3)}{\chi(-q)f(-q^{10})},\qquad \frac{f(-q^4,-q^6)}{f(-q^3,-q^7)}=\frac{f(-q,-q^4)}{\chi(-q)f(-q^{10})}, \tag{8.5.28.1}$$

$$\frac{f(-q^2,-q^3)}{f(q,q^4)}=\frac{\chi(-q)f(-q^4,-q^6)}{f(-q^5)},\qquad \frac{f(-q,-q^4)}{f(q^2,q^3)}=\frac{\chi(-q)f(-q^2,-q^8)}{f(-q^5)}, \tag{8.5.28.2}$$

$$\frac{f(-q^2,-q^6)}{f(q,q^7)}=\frac{\chi(-q)}{f(-q^8)}f(q^3,q^5),\qquad \frac{f(-q^2,-q^6)}{f(q^3,q^5)}=\frac{\chi(-q)}{f(-q^8)}f(q,q^7). \tag{8.5.28.3}$$

By (8.4.64), with the sets of parameters $\epsilon=1$, $\delta=0$, $l=0$, $t=1$, $\alpha=6$, $\beta=76$, $m=3$, $p=5$, so that $\lambda=26$, and $\alpha_1=4$, $\beta_1=114$, $m_1=-2$, $p_1=5$ (with corresponding values of $z=1$, $\epsilon=\delta=1$, $l=t=-1$), we find that

$$R_2(1,0,0,1,6,76,3,5)=R_1(1,1,1,-1,-1,4,114,-2,5). \tag{8.5.28.4}$$

Let $Q=q^{38}$. By Lemma 8.4.2, (8.5.28.1) with q replaced by Q, and (8.2.11), we find that

$$\begin{aligned}
R_1(1,1,1,-1,-1,4,114,-2,5)&=q^{13/3}f(-Q^{10})\\
&\quad\times\left(\frac{f(-Q^2,-Q^8)f(-q^8,-q^{32})}{f(-Q,-Q^9)}-q^6\frac{f(-Q^4,-Q^6)f(-q^{16},-q^{24})}{f(-Q^3,-Q^7)}\right)\\
&=q^{13/3}f(-q^8)f(-Q^2)\left(G(Q)H(q^8)-q^6H(Q)G(q^8)\right).
\end{aligned} \tag{8.5.28.5}$$

By Lemma 8.4.3, (8.5.28.1), with q replaced by $-q^2$, and (8.2.11), and by noting that $\lambda=26$, we similarly find that

$$\begin{aligned}
R_2(1,0,0,1,6,76,3,5)&=q^{13/3}f(-q^{20})\\
&\quad\times\left(\frac{f(-q^8,-q^{12})f(-Q^8,-Q^{12})}{f(q^6,q^{14})}+q^{30}\frac{f(-q^4,-q^{16})f(-Q^4,-Q^{16})}{f(q^2,q^{18})}\right)\\
&=f(-q^4)f(-Q^4)\left(H(-q^2)G(Q^4)+q^{30}G(-q^2)H(Q^4)\right).
\end{aligned} \tag{8.5.28.6}$$

By (8.5.28.4)–(8.5.28.6), we conclude that

$$\frac{G(Q)H(q^8)-q^6H(Q)G(q^8)}{H(-q^2)G(Q^4)+q^{30}G(-q^2)H(Q^4)}=\frac{f(-q^4)f(-Q^4)}{f(-q^8)f(-Q^2)}=\frac{\chi(-q^4)}{\chi(-Q^2)},$$

which is (8.3.39) with q replaced by q^2.

8.5.29 Proofs of Entries 8.3.31 and 8.3.32

We give two approaches. We first provide the proofs of these two identities from Yesilyurt's paper [347]. Secondly, we provide a more elementary argument from [65], but this approach only shows that Entries 8.3.31 are 8.3.32 are equivalent, i.e., that it suffices to prove just one of the identities.

We simultaneously prove Entries 8.3.31 and 8.3.32. By (8.4.65) with the set of parameters $z=1$, $\epsilon=1$, $\delta=0$, $l=t=1$, $\alpha=11$, $\beta=9$, $m=1$, and $p=4$ ($\lambda=5$), we find that

$$R_1(1,1,0,1,1,11,9,1,4)=R_1(1,0,1,6,-1,1,99,11,5). \tag{8.5.29.1}$$

Throughout the remainder of this section, set $Q=q^{33}$. By Lemma 8.4.2, by the use of (8.2.5) with $n=4,6$, by (8.5.27.5) with q replaced by q^{33}, and by (8.2.11), we find that since $\lambda=44$,

$$\begin{aligned} R_1(1,0,1,6,-1,1,99,11,5) &= q^{9/4}f\left(-Q^{10}\right) \\ &\times\left(\frac{f\left(-Q^2,-Q^8\right)f\left(-q^4,-q^6\right)}{f\left(-Q,-Q^9\right)}+q^7\frac{f\left(-Q^4,-Q^6\right)f\left(-q^2,-q^8\right)}{f\left(-Q^3,-Q^7\right)}\right) \\ &= q^{9/4}\frac{1}{\chi(-Q)}\left(g(q^2)g(Q)+q^7g(q^2)h(Q)\right). \end{aligned} \tag{8.5.29.2}$$

From (8.2.19), we easily find that

$$\psi(q)=f(q^6,q^{10})+qf(q^2,q^{14}). \tag{8.5.29.3}$$

By Lemma 8.4.2, by (8.2.5) with $n=1$, and by (8.5.28.3) with q replaced by q^3, we similarly find that

$$\begin{aligned} &R_1(1,1,0,1,1,11,9,1,4) \\ &= q^{9/4}f(-q^{24})\left(\frac{f\left(-q^6,-q^{18}\right)f\left(q^{33},q^{55}\right)}{f\left(q^9,q^{15}\right)}-q^4\frac{f\left(-q^6,-q^{18}\right)f\left(q^{11},q^{77}\right)}{f\left(q^3,q^{21}\right)}\right) \\ &= q^{9/4}\chi(-q^3)\Big\{f(q^3,q^{21})f(q^{33},q^{55})-q^4f(q^9,q^{15})f(q^{11},q^{77})\Big\}. \end{aligned} \tag{8.5.29.4}$$

Using (8.5.29.3), we easily find that

$$\begin{aligned} &\psi(q^3)\psi(-q^{11})-\psi(-q^3)\psi(q^{11}) \\ &\qquad= 2q^3\left\{f(q^6,q^{42})f(q^{66},q^{110})-q^8f(q^{18},q^{30})f(q^{22},q^{154})\right\}. \end{aligned} \tag{8.5.29.5}$$

Now in (8.5.29.2) and (8.5.29.4), we replace q by q^2 and use (8.5.29.1) and (8.5.29.5) to conclude that

$$\begin{aligned} &2q^3\left(g(q^4)g(Q^2)+q^{14}h(q^4)h(Q^2)\right) \\ &\qquad= \chi(-q^6)\chi(-q^{66})\left\{\psi(q^3)\psi(-q^{11})-\psi(-q^3)\psi(q^{11})\right\}. \end{aligned} \tag{8.5.29.6}$$

In what follows, $J(q)$ denotes an arbitrary power series, usually not the same with each appearance. By (8.2.19) with $n = 3$ in each instance,

$$\begin{aligned} g(q) &= f(-q^2, -q^3) = f(-q^{21}, -q^{24}) - q^2 f(-q^9, -q^{36}) - q^3 f(-q^6, -q^{39}) \\ &= J(q^3) - q^2 h(q^9), \qquad (8.5.29.7) \\ h(q) &= f(-q, -q^4) = f(-q^{18}, -q^{27}) - q f(-q^{12}, -q^{33}) - q^4 f(-q^3, -q^{42}) \\ &= g(q^9) - qJ(q^3), \qquad (8.5.29.8) \\ \psi(q) &= f(q^{15}, q^{21}) + q^3 f(q^3, q^{33}) + q\psi(q^9) = J(q^3) + q\psi(q^9). \qquad (8.5.29.9) \end{aligned}$$

By (8.5.29.9),

$$\begin{aligned} &\psi(q^3)\psi(-q^{11}) - \psi(-q^3)\psi(q^{11}) \\ &\quad = \left\{J(q^3) - q^{11}\psi(-q^{99})\right\}\psi(q^3) - \left\{J(q^3) + q^{11}\psi(q^{99})\right\}\psi(-q^3) \\ &\quad = J(q^3) - q^{11}\left\{\psi(q^3)\psi(-q^{99}) + \psi(-q^3)\psi(q^{99})\right\}. \qquad (8.5.29.10) \end{aligned}$$

Similarly, by (8.5.29.7) and (8.5.29.8) with q replaced by q^4, we find that

$$g(q^4)g(q^{66}) + q^{14}h(q^4)h(q^{66}) = J(q^3) - q^8\left\{h(q^{36})g(q^{66}) - q^6 g(q^{36})h(q^{66})\right\}. \qquad (8.5.29.11)$$

From these last two equalities and (8.5.29.6), we conclude that

$$\begin{aligned} &2\left\{h(q^{12})g(q^{22}) - q^2 g(q^{12})h(q^{22})\right\} \\ &\quad = \chi(-q^2)\chi(-q^{22})\left(\psi(q)\psi(-q^{33}) + \psi(-q)\psi(q^{33})\right). \qquad (8.5.29.12) \end{aligned}$$

Next, by (8.4.65) with the set of parameters $z = -1$, $\epsilon = 0$, $\delta = 1$, $l = 0$, $t = 1$, $\alpha = 11$, $\beta = 9$, $m = 1$, and $p = 5$ ($\lambda = 4$), we find that

$$R_1(-1, 0, 1, 0, 1, 11, 9, 1, 5) = R_1(1, 1, 0, 3, -1, 1, 99, 11, 4). \qquad (8.5.29.13)$$

By Lemma 8.4.2, by (8.2.5) with $n = 3$ and $n = 6$, and by (8.5.28.3) with q replaced by Q, we find that, upon noting that $\lambda = 54$,

$$\begin{aligned} &R_1(1, 1, 0, 3, -1, 1, 99, 11, 4) \\ &= q^{3/4} f\left(-Q^8\right)\left(\frac{f\left(-Q^2, -Q^6\right) f\left(q^3, q^5\right)}{f\left(Q, Q^7\right)} - q^{17}\frac{f\left(-Q^2, -Q^6\right) f\left(q, q^7\right)}{f\left(Q^3, Q^5\right)}\right) \\ &= q^{3/4}\chi(-Q)\left\{f(q^3, q^5)f(Q^3, Q^5) - q^{17} f(q, q^7) f(Q, Q^7)\right\}. \qquad (8.5.29.14) \end{aligned}$$

By Lemma 8.4.2, and by (8.5.27.5) with q replaced by q^3, we similarly find that with $\lambda = 4$,

$$\begin{aligned} &R_1(-1, 0, 1, 0, 1, 11, 9, 1, 5) \\ &\quad = q^{3/4} f\left(-q^{30}\right)\left(\frac{f\left(-q^6, -q^{24}\right) f\left(-q^{44}, -q^{66}\right)}{f\left(-q^3, -q^{27}\right)}\right. \end{aligned}$$

$$+q^5 \frac{f\left(-q^{12},-q^{18}\right) f\left(-q^{22},-q^{88}\right)}{f\left(-q^9,-q^{21}\right)}\Bigg)$$
$$= q^{3/4} \frac{f(-q^3)f(-q^{22})}{\chi(-q^3)} \left\{g(q^3)g(q^{22}) + q^5 h(q^3)h(q^{22})\right\}. \tag{8.5.29.15}$$

We now argue as before. In (8.5.29.3), we replace q by $-q$, q^{33}, and $-q^{33}$, and multiplying them in pairs, we find that

$$\psi(q) = 2\left\{f(q^6,q^{10})f(q^{198},q^{330}) - q^{34} f(q^2,q^{14})f(q^{66},q^{462})\right\}. \tag{8.5.29.16}$$

Next, we replace q by q^2 in both (8.5.29.14) and (8.5.29.15). Using these identities in (8.5.29.13) and then employing (8.5.29.16), we conclude that

$$\begin{aligned} &2\left\{g(q^6)g(q^{44}) + q^{10}h(q^6)h(q^{44})\right\} \\ &\quad = \chi(-q^6)\chi(-q^{66})\left(\psi(q)\psi(-q^{33}) + \psi(-q)\psi(q^{33})\right). \end{aligned} \tag{8.5.29.17}$$

We again appeal to the dissections (8.5.29.7)–(8.5.29.9). Using (8.5.29.7)–(8.5.29.9) with q replaced by q^{44}, we find that

$$2\left\{g(q^6)g(q^{44}) + q^{10}h(q^6)h(q^{44})\right\} = 2q^{10}\left\{h(q^6)g(q^{396}) - q^{78} g(q^6)h(q^{396})\right\} \tag{8.5.29.18}$$

and

$$\psi(q)\psi(-q^{33}) + \psi(-q)\psi(q^{33}) = q\left\{\psi(q^9)\psi(-q^{33}) - \psi(-q^9)\psi(q^{33})\right\}. \tag{8.5.29.19}$$

Replacing q^3 by q in (8.5.29.17), (8.5.29.18), and (8.5.29.19), we conclude that

$$\begin{aligned} &2q^3\left(h(q^2)g(q^{132}) - q^{26} g(q^2)h(q^{132})\right) \\ &\quad = \chi(-q^2)\chi(-q^{22})\big(\psi(q^3)\psi(-q^{11}) - \psi(-q^3)\psi(q^{11})\big). \end{aligned} \tag{8.5.29.20}$$

Comparing (8.5.29.5) and (8.5.29.20), and using (8.2.11), we see that Entry 8.3.31 has been proved. Similarly, the identities (8.5.29.17) and (8.5.29.12) imply Entry 8.3.32.

We now give a more elementary argument showing that Entries 8.3.31 and 8.3.32 are equivalent. For brevity, define

$$M(q) := G(q^2)G(q^{33}) + q^7 H(q^2)H(q^{33}), \tag{8.5.29.21}$$
$$N(q) := G(q^{66})H(q) - q^{13} H(q^{66})G(q), \tag{8.5.29.22}$$
$$R(q) := G(q^{66})H(q^{11}) - q^{11} H(q^{66})G(q^{11})$$
$$T(q) := G(q^3)G(q^{22}) + q^5 H(q^3)H(q^{22}), \tag{8.5.29.23}$$
$$U(q) := G(q^{11})H(q^6) - qH(q^{11})G(q^6). \tag{8.5.29.24}$$

Using (8.5.29.22), Entry 8.3.4 with q replaced by q^6, and Entry 8.3.8 with q replaced by q^{11}, we consider the system of three equations

$$\begin{aligned}N(q) &= G(q^{66})H(q) - q^{13}H(q^{66})G(q),\\ 1 &= G(q^{66})H(q^{6}) - q^{12}H(q^{66})G(q^{6}),\\ \frac{\chi(-q^{11})}{\chi(-q^{33})} &= R(q) = G(q^{66})H(q^{11}) - q^{11}H(q^{66})G(q^{11}).\end{aligned} \tag{8.5.29.25}$$

It follows that

$$\begin{vmatrix} H(q) & -q^{13}G(q) & N(q) \\ H(q^6) & -q^{12}G(q^6) & 1 \\ H(q^{11}) & -q^{11}G(q^{11}) & R(q) \end{vmatrix} = 0.$$

Expanding this determinant along the last column and using (8.5.29.24), Entry 8.3.4, (8.5.29.25), and Entry 8.3.8, we deduce that

$$\begin{aligned}0 &= N(q)\left(-q^{11}H(q^6)G(q^{11}) + q^{12}G(q^6)H(q^{11})\right) + q^{11}H(q)G(q^{11})\\ &\quad - q^{13}G(q)H(q^{11}) + R(q)\left(-q^{12}G(q^6)H(q) + q^{13}G(q)H(q^6)\right)\\ &= -N(q)q^{11}U(q) + q^{11} - q^{12}\frac{\chi(-q^{11})}{\chi(-q^{33})}\frac{\chi(-q)}{\chi(-q^3)}.\end{aligned} \tag{8.5.29.26}$$

Hence, if we define

$$W(q) := \frac{\chi(-q)\chi(-q^{11})}{\chi(-q^3)\chi(-q^{33})}, \tag{8.5.29.27}$$

then, from (8.5.29.26), we deduce that

$$N(q)U(q) = 1 - qW(q). \tag{8.5.29.28}$$

Next, using (8.5.29.21), Entry 8.3.4 with q replaced by q^3, and Entry 8.3.7 with q replaced by q^{11}, we consider the system of equations

$$\begin{aligned}M(q) &= G(q^2)G(q^{33}) + q^7H(q^2)H(q^{33}),\\ 1 &= H(q^3)G(q^{33}) - q^6G(q^3)H(q^{33}),\\ \frac{\chi(-q^{33})}{\chi(-q^{11})} &=: S(q) = G(q^{22})G(q^{33}) + q^{11}H(q^{22})H(q^{33}).\end{aligned} \tag{8.5.29.29}$$

It follows that

$$\begin{vmatrix} G(q^2) & q^7H(q^2) & M(q) \\ H(q^3) & -q^6G(q^3) & 1 \\ G(q^{22}) & q^{11}H(q^{22}) & S(q) \end{vmatrix} = 0.$$

Expanding the determinant above along the last column and employing (8.5.29.23), Entry 8.3.4 with q replaced by q^2, (8.5.29.29), and Entry 8.3.7, we find that

$$\begin{aligned}0 &= M(q)\left(q^{11}H(q^3)H(q^{22}) + q^6G(q^3)G(q^{22})\right) - q^{11}G(q^2)H(q^{22})\\ &\quad + q^7H(q^2)G(q^{22}) + S(q)\left(-q^6G(q^2)G(q^3) - q^7H(q^2)H(q^3)\right)\\ &= M(q)q^6T(q) + q^7 - q^6\frac{\chi(-q^{33})}{\chi(-q^{11})}\frac{\chi(-q^3)}{\chi(-q)}.\end{aligned} \tag{8.5.29.30}$$

Hence, using the definition of $W(q)$ in (8.5.29.27), we deduce from (8.5.29.30) that

$$M(q)T(q) = -q + \frac{1}{W(q)}. \tag{8.5.29.31}$$

Hence, dividing (8.5.29.31) by (8.5.29.28), we conclude that

$$\frac{M(q)T(q)}{N(q)U(q)} = \frac{1}{W(q)}. \tag{8.5.29.32}$$

Examining Entries 8.3.31 and 8.3.32, we see that it suffices to prove just one of them, for then the other one will follow immediately from (8.5.29.32).

8.5.30 Proof of Entry 8.3.33

We provide two proofs.

First Proof of Entry 8.3.33. Let us define $K(q)$ and $L(q)$ by

$$K(q) := G(q)G(q^{54}) + q^{11}H(q)H(q^{54}), \tag{8.5.30.1}$$

$$L(q) := H(q^2)G(q^{27}) - q^5G(q^2)H(q^{27}), \tag{8.5.30.2}$$

so that Entry 8.3.33 reads

$$\frac{K(q)}{L(q)} = \frac{\chi(-q^3)\chi(-q^{27})}{\chi(-q)\chi(-q^9)}. \tag{8.5.30.3}$$

Starting from (8.3.15) and arguing as in (8.4.32), we find that

$$\frac{\chi(-q^6)\chi(-q^9)}{\chi(-q^2)\chi(-q^3)}G(-q) + \frac{\chi(-q^6)\chi(q^9)}{\chi(-q^2)\chi(q^3)}G(q) = 2\frac{G(q^{36})}{\chi^2(-q^2)}. \tag{8.5.30.4}$$

By (8.2.15), we see that (8.5.30.4) simplifies to

$$\chi(q^3)\chi(-q^9)G(-q) + \chi(-q^3)\chi(q^9)G(q) = 2\frac{G(q^{36})}{\chi(-q^2)}. \tag{8.5.30.5}$$

Similarly, we find that

$$\chi(-q^3)\chi(q^9)H(q) - \chi(q^3)\chi(-q^9)H(-q) = 2q^7\frac{H(q^{36})}{\chi(-q^2)}. \tag{8.5.30.6}$$

In (8.5.30.1), we replace q by q^2 and employ (8.5.30.5) and (8.5.30.6) with q replaced by q^3 to find that

$$\begin{aligned}
2\frac{K(q^2)}{\chi(-q^6)} &= \frac{2}{\chi(-q^6)}\left\{G(q^2)G(q^{108}) + q^{22}H(q^2)H(q^{108})\right\} \\
&= G(q^2)\left\{\chi(q^9)\chi(-q^{27})G(-q^3) + \chi(-q^9)\chi(q^{27})G(q^3)\right\} \\
&\quad + qH(q^2)\left\{\chi(-q^9)\chi(q^{27})H(q^3) - \chi(q^9)\chi(-q^{27})H(-q^3)\right\} \\
&= \chi(q^9)\chi(-q^{27})\left\{G(q^2)G(-q^3) - qH(q^2)H(-q^3)\right\} \\
&\quad + \chi(-q^9)\chi(q^{27})\left\{G(q^2)G(q^3) + qH(q^2)H(q^3)\right\}.
\end{aligned} \tag{8.5.30.7}$$

Using (8.3.8) twice, once with q replaced by $-q$, we see that (8.5.30.7) can be put in the form

$$2\frac{K(q^2)}{\chi(-q^6)} = \chi(q^9)\chi(-q^{27})\frac{\chi(q^3)}{\chi(q)} + \chi(-q^9)\chi(q^{27})\frac{\chi(-q^3)}{\chi(-q)}.$$

Using (8.2.15), we conclude that

$$2K(q^2) = \frac{\chi(-q^6)}{\chi(-q^2)}\left\{\chi(-q)\chi(q^3)\chi(q^9)\chi(-q^{27}) + \chi(q)\chi(-q^3)\chi(-q^9)\chi(q^{27})\right\}. \tag{8.5.30.8}$$

To obtain the desired expression for $L(q^2)$, we use Lemma 8.4.3. Then, in (8.5.30.2), replacing q by q^2, employing (8.4.38) and (8.4.39), and arguing as in (8.5.30.7), we find that

$$\begin{aligned}2\frac{L(q^2)}{\chi(-q^{18})} &= \chi(q)\chi(-q^3)\left\{G(q^{54})H(q^9) - q^9H(q^{54})G(q^9)\right\}\\ &\quad + \chi(-q)\chi(q^3)\left\{G(q^{54})H(-q^9) + q^9H(q^{54})G(-q^9)\right\}. \end{aligned}\tag{8.5.30.9}$$

Using (8.3.9), with q replaced by q^9 and $-q^9$, respectively, we find from (8.5.30.9) that

$$2\frac{L(q^2)}{\chi(-q^{18})} = \chi(q)\chi(-q^3)\frac{\chi(-q^9)}{\chi(-q^{27})} + \chi(-q)\chi(q^3)\frac{\chi(q^9)}{\chi(q^{27})},$$

which, by (8.2.15), implies that

$$2L(q^2) = \frac{\chi(-q^{18})}{\chi(-q^{54})}\left\{\chi(q)\chi(-q^3)\chi(-q^9)\chi(q^{27}) + \chi(-q)\chi(q^3)\chi(q^9)\chi(-q^{27})\right\}. \tag{8.5.30.10}$$

Dividing (8.5.30.8) by (8.5.30.10), we obtain (8.5.30.3) with q replaced by q^2. Hence, the proof of Entry 8.3.33 is complete. □

Second Proof of Entry 8.3.33. Recall once more the definitions

$$g(q) = f(-q)G(q) = f(-q^2,-q^3) \quad \text{and} \quad h(q) = f(-q)H(q) = f(-q,-q^4).$$

Our proof of Entry 8.3.33 uses Entries 8.3.6, 8.3.7, and 8.3.8, which we write in their equivalent forms (see (8.5.6.1), (8.5.7.3), and (8.5.8.2))

$$g(q)g(q^9) + q^2h(q)h(q^9) = f^2(-q^3), \tag{8.5.30.11}$$

$$g(q^2)g(q^3) + qh(q^2)h(q^3) = \psi(q)\varphi(-q^3), \tag{8.5.30.12}$$

$$g(q^6)h(q) - qg(q)h(q^6) = \psi(q^3)\varphi(-q). \tag{8.5.30.13}$$

Let us define $M(q)$ and $N(q)$ by

$$M(q) := h(q^2)g(q^{27}) - q^5g(q^2)h(q^{27}), \tag{8.5.30.14}$$

$$N(q) := g(q)g(q^{54}) + q^{11}h(q)h(q^{54}). \tag{8.5.30.15}$$

By (8.2.11) and (8.2.14), Entry 8.3.33 is equivalent to the identity

$$\frac{N(q)}{M(q)}=\frac{f(-q)f(-q^{54})\chi(-q^{3})\chi(-q^{27})}{f(-q^{2})f(-q^{27})\chi(-q)\chi(-q^{9})}=\frac{\chi(-q^{3})}{\chi(-q^{9})}. \tag{8.5.30.16}$$

By (8.5.30.15), (8.5.30.11), and (8.5.30.13) with q replaced by q^6 and q^9, respectively, in the latter two cases, we deduce the following system of three equations:

$$\begin{aligned}g(q)g(q^{54})+q^{11}h(q)h(q^{54})&=N(q),\\ g(q^{6})g(q^{54})+q^{12}h(q^{6})h(q^{54})&=f^{2}(-q^{18}),\\ h(q^{9})g(q^{54})-q^{9}g(q^{9})h(q^{54})&=\psi(q^{27})\varphi(-q^{9}).\end{aligned}$$

Regarding this system in the variables $G(q^{54})$, $q^9h(q^{54})$, and -1, we find that

$$\begin{vmatrix} g(q) & q^{2}h(q) & N(q)\\ g(q^{6}) & q^{3}h(q^{6}) & f^{2}(-q^{18})\\ h(q^{9}) & -g(q^{9}) & \psi(q^{27})\varphi(-q^{9})\end{vmatrix}=0. \tag{8.5.30.17}$$

Expanding the determinant in (8.5.30.17) along the last column, we find that

$$\begin{aligned}&-N(q)\left\{g(q^{6})g(q^{9})+q^{3}h(q^{6})h(q^{9})\right\}\\ &+f^{2}(-q^{18})\left\{g(q)g(q^{9})+q^{2}h(q)h(q^{9})\right\}\\ &-q^{2}\varphi(-q^{9})\psi(q^{27})\left\{h(q)g(q^{6})-qg(q)h(q^{6})\right\}=0.\end{aligned} \tag{8.5.30.18}$$

Using (8.5.30.12) with q replaced by q^3, (8.5.30.11), and (8.5.30.13) in (8.5.30.18), we find that

$$-N(q)\psi(q^{3})\varphi(-q^{9})+f^{2}(-q^{18})f^{2}(-q^{3})-q^{2}\psi(q^{27})\varphi(-q^{9})\psi(q^{3})\varphi(-q)=0.$$

Solving for $N(q)$, we deduce that

$$N(q)=\frac{f^{2}(-q^{18})f^{2}(-q^{3})}{\psi(q^{3})\varphi(-q^{9})}-q^{2}\psi(q^{27})\varphi(-q). \tag{8.5.30.19}$$

Next, we determine $M(q)$. By (8.5.30.14), (8.5.30.12), and (8.5.30.11) with q replaced by q^9 and q^3, respectively, in the latter two equalities, we find that

$$\begin{aligned}h(q^{2})g(q^{27})-q^{5}g(q^{2})h(q^{27})&=M(q),\\ g(q^{18})g(q^{27})+q^{9}h(q^{18})h(q^{27})&=\psi(q^{9})\varphi(-q^{27}),\\ g(q^{3})g(q^{27})+q^{6}h(q^{3})h(q^{27})&=f^{2}(-q^{9}).\end{aligned}$$

Regarding this system in the variables $g(q^{27})$, $q^5h(q^{27})$, and -1 and arguing as we did above, we find that

$$qM(q)\psi(q^{9})\varphi(-q^{3})-\psi(q^{9})\varphi(-q^{27})\psi(q)\varphi(-q^{3})+f^{2}(-q^{9})f^{2}(-q^{6})=0.$$

Solving for $qM(q)$, we arrive at

$$qM(q) = \psi(q)\varphi(-q^{27}) - \frac{f^2(-q^9)f^2(-q^6)}{\psi(q^9)\varphi(-q^3)}. \tag{8.5.30.20}$$

Recall that by (8.5.7.6) and (8.5.7.8),

$$f(-q,-q^5) = \chi(-q)\psi(q^3) \tag{8.5.30.21}$$

and

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{8.5.30.22}$$

By (8.2.19) with $n = 3$ (see also [55, p. 49, Corollary]), we find that

$$\varphi(-q) = \varphi(-q^9) - 2qf(-q^3,-q^{15}), \tag{8.5.30.23}$$
$$\psi(q) = f(q^3,q^6) + q\psi(q^9). \tag{8.5.30.24}$$

Using (8.5.30.21) and (8.5.30.22) in (8.5.30.23) and (8.5.30.24) with q replaced by q^3, we obtain, respectively,

$$\varphi(-q) = \varphi(-q^9) - 2q\chi(-q^3)\psi(q^9), \tag{8.5.30.25}$$
$$\psi(q) = \frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9). \tag{8.5.30.26}$$

We deduce from (8.5.30.25) and (8.5.30.26) that

$$\chi(-q^3)\psi(q^9) = \frac{1}{2q}\left\{\varphi(-q^9) - \varphi(-q)\right\}, \tag{8.5.30.27}$$
$$\psi(q)\chi(-q^3) = \frac{1}{2}\left\{3\varphi(-q^9) - \varphi(-q)\right\}. \tag{8.5.30.28}$$

By (8.2.14), we easily find that

$$\frac{f^2(-q^2)\chi(-q)}{\varphi(-q)} = \psi(q), \tag{8.5.30.29}$$
$$\frac{f^2(-q)}{\psi(q)} = \varphi(-q)\chi(-q). \tag{8.5.30.30}$$

By (8.5.30.19), (8.5.30.29), and (8.5.30.30) with q replaced by q^9 and q^3, respectively, in the latter two equations, we find that

$$\chi(-q^9)N(q) = \chi(-q^3)\varphi(-q^3)\psi(q^9) - q^2\psi(q^{27})\chi(-q^9)\varphi(-q). \tag{8.5.30.31}$$

Using (8.5.30.27) twice in (8.5.30.31) with q replaced by q^3 in the second instance, we find that

$$\begin{aligned}\chi(-q^9)N(q) &= \frac{1}{2q}\varphi(-q^3)\left\{\varphi(-q^9)-\varphi(-q)\right\}\\ &\quad - q^2\frac{1}{2q^3}\left\{\varphi(-q^{27})-\varphi(-q^3)\right\}\varphi(-q)\\ &= \frac{1}{2q}\left\{\varphi(-q^3)\varphi(-q^9)-\varphi(-q)\varphi(-q^{27})\right\}. \qquad (8.5.30.32)\end{aligned}$$

Similarly, by (8.5.30.20), (8.5.30.29), and (8.5.30.30) with q replaced by q^3 and q^9, respectively, we find that

$$q\chi(-q^3)M(q) = \chi(-q^3)\varphi(-q^{27})\psi(q) - \psi(q^3)\chi(-q^9)\varphi(-q^9). \qquad (8.5.30.33)$$

Using (8.5.30.28) twice with q replaced by q^3 in the latter case, we find that

$$\begin{aligned}q\chi(-q^3)M(q) &= \frac{1}{2}\varphi(-q^{27})\left\{3\varphi(-q^9)-\varphi(-q)\right\}\\ &\quad - \frac{1}{2}\left\{3\varphi(-q^{27})-\varphi(-q^3)\right\}\varphi(-q^9)\\ &= \frac{1}{2}\left\{\varphi(-q^3)\varphi(-q^9)-\varphi(-q)\varphi(-q^{27})\right\}. \qquad (8.5.30.34)\end{aligned}$$

Dividing (8.5.30.34) by (8.5.30.32), we see that (8.5.30.16) is verified. Hence, the second proof of Entry 8.3.33 is complete. ∎

8.5.31 Proof of Entry 8.3.34

Our proof is a moderate modification of the proof given by Bressoud [81].

Using (8.1.2), (8.2.6), and some elementary product manipulations, we can show that

$$G(q)G(-q) = \frac{f(q^4,q^6)}{f(-q^2)} \qquad \text{and} \qquad H(q)H(-q) = \frac{f(q^2,q^8)}{f(-q^2)}. \qquad (8.5.31.1)$$

Adding Entries 8.3.20 and 8.3.21, we find that

$$G(q)H(-q) = \frac{1}{f(-q^2)}\left\{\psi(q^2)+q\psi(q^{10})\right\}. \qquad (8.5.31.2)$$

Next, we recall Entry 8.3.34:

$$\begin{aligned}&\left\{G(q)G(-q^{19})-q^4H(q)H(-q^{19})\right\}\left\{G(-q)G(q^{19})-q^4H(-q)H(q^{19})\right\}\\ &= G(q^2)G(q^{38})+q^8H(q^2)H(q^{38}). \qquad (8.5.31.3)\end{aligned}$$

Expanding the product on the left side of (8.5.31.3) and then using (8.5.31.1) and (8.5.31.2), we find that

$$\left\{G(q)G(-q^{19})-q^4H(q)H(-q^{19})\right\}\left\{G(-q)G(q^{19})-q^4H(-q)H(q^{19})\right\}$$

$$
\begin{aligned}
&= G(q)G(-q)G(q^{19})G(-q^{19}) + q^8 H(q)H(-q)H(q^{19})H(-q^{19}) \\
&\quad - q^4 G(-q)H(q)G(q^{19})H(-q^{19}) - q^4 G(q)H(-q)G(-q^{19})H(q^{19}) \\
&= \frac{1}{f(-q^2)f(-q^{38})} f(q^4,q^6)f(q^{76},q^{114}) \\
&\quad + q^8 \frac{1}{f(-q^2)f(-q^{38})} f(q^2,q^8)f(q^{38},q^{152}) \\
&\quad - q^4 \frac{1}{f(-q^2)f(-q^{38})} \left\{\psi(q^2) - q\psi(q^{10})\right\}\left\{\psi(q^{38}) + q^{19}\psi(q^{190})\right\} \\
&\quad - q^4 \frac{1}{f(-q^2)f(-q^{38})} \left\{\psi(q^2) + q\psi(q^{10})\right\}\left\{\psi(q^{38}) - q^{19}\psi(q^{190})\right\} \\
&= \frac{1}{f(-q^2)f(-q^{38})} \left\{f(q^4,q^6)f(q^{76},q^{114}) + q^8 f(q^2,q^8)f(q^{38},q^{152})\right. \\
&\quad \left. -2q^4\psi(q^2)\psi(q^{38}) + 2q^{24}\psi(q^{10})\psi(q^{190})\right\}. \qquad (8.5.31.4)
\end{aligned}
$$

Hence, from (8.5.31.3) and (8.5.31.4), we see that it suffices to prove that

$$
\begin{aligned}
&G(q^2)G(q^{38}) + q^8 H(q^2)H(q^{38}) \\
&\quad = \frac{1}{f(-q^2)f(-q^{38})}\Big\{f(q^4,q^6)f(q^{76},q^{114}) + q^8 f(q^2,q^8)f(q^{38},q^{152}) \\
&\qquad\qquad - 2q^4\psi(q^2)\psi(q^{38}) + 2q^{24}\psi(q^{10})\psi(q^{190})\Big\}. \qquad (8.5.31.5)
\end{aligned}
$$

Multiplying both sides of (8.5.31.5) by $f(-q^2)f(-q^{38})$ and using (8.2.11), we can rewrite (8.5.31.5) in its equivalent form

$$
\begin{aligned}
f(-q^4,-q^6)&f(-q^{76},-q^{114}) + q^8 f(-q^2,-q^8)f(-q^{38},-q^{152}) \\
&= f(q^4,q^6)f(q^{76},q^{114}) + q^8 f(q^2,q^8)f(q^{38},q^{152}) \\
&\quad - 2q^4\psi(q^2)\psi(q^{38}) + 2q^{24}\psi(q^{10})\psi(q^{190}). \qquad (8.5.31.6)
\end{aligned}
$$

By (8.5.16.3), with q replaced by q^2, the left-hand side of (8.5.31.6) is

$$
\frac{1}{4q}\left(\varphi(q)\varphi(q^{19}) - \varphi(-q)\varphi(-q^{19})\right) - q^4\psi(q^2)\psi(q^{38}). \qquad (8.5.31.7)
$$

Therefore, it remains to show that

$$
\begin{aligned}
&\frac{1}{4q}\left(\varphi(q)\varphi(q^{19}) - \varphi(-q)\varphi(-q^{19})\right) + q^4\psi(q^2)\psi(q^{38}) \qquad (8.5.31.8) \\
&\quad = f(q^4,q^6)f(q^{76},q^{114}) + q^8 f(q^2,q^8)f(q^{38},q^{152}) + 2q^{24}\psi(q^{10})\psi(q^{190}).
\end{aligned}
$$

By (8.5.16.5) and (8.5.16.6) with q replaced by $-q^2$, we deduce that

$$
\begin{aligned}
&\frac{1}{4q}\left(\varphi(q)\varphi(q^{19}) - \varphi(-q)\varphi(-q^{19})\right) + q^4\psi(q^2)\psi(q^{38}) \\
&\quad = f(q^{342},q^{418})f(q^{18},q^{22}) + q^8 f(q^{266},q^{494})f(q^{14},q^{26})
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\,q^{24} f(q^{190},q^{570}) f(q^{10},q^{30}) + q^{48} f(q^{114},q^{646}) f(q^{6},q^{34}) \\
&+ q^{80} f(q^{38},q^{722}) f(q^{2},q^{38}) + q^{4} f(q^{2},q^{38}) f(q^{342},q^{418}) \\
&+ q^{10} f(q^{6},q^{34}) f(q^{266},q^{494}) + q^{46} f(q^{14},q^{26}) f(q^{114},q^{646}) \\
&+ q^{76} f(q^{18},q^{22}) f(q^{38},q^{722}) \\
= &\left(f(q^{18},q^{22}) + q^{4} f(q^{2},q^{38})\right)\left(f(q^{342},q^{418}) + q^{76} f(q^{38},q^{722})\right) \\
&+ q^{8}\left(f(q^{14},q^{26}) + q^{2} f(q^{6},q^{34})\right)\left(f(q^{266},q^{494}) + q^{38} f(q^{114},q^{646})\right) \\
&+ 2q^{24}\psi(q^{10})\psi(q^{190}). \qquad (8.5.31.9)
\end{aligned}
$$

But by (8.2.19) with $n = 2$, first with $a = q$ and $b = q^4$, and second with $a = q^2$ and $b = q^3$, we have

$$
f(q,q^4) = f(q^7,q^{13}) + q f(q^3,q^{17}) \quad \text{and} \quad f(q^2,q^3) = f(q^9,q^{11}) + q^2 f(q,q^{19}). \qquad (8.5.31.10)
$$

Using both parts of (8.5.31.10) in (8.5.31.9), each with q replaced by q^2 and q^{38}, we see that (8.5.31.8) holds. Hence, the proof of Entry 8.3.34 is complete.

8.5.32 Proof of Entry 8.3.35

As indicated earlier, this proof is due to Yesilyurt [348]. Let $Q := q^{47}$, and set

$$
A(q) := H(q^2)G(Q) - q^9 G(q^2)H(Q) \text{ and } B(q) := G(q)G(Q^2) + q^{19} H(q)H(Q^2). \qquad (8.5.32.1)
$$

From (8.4.62), we find that

$$
R_2(1,0,1,1,3,47,1,5) = R_1(-1,0,1,0,4,1,141,3,10). \qquad (8.5.32.2)
$$

Employing Lemma 8.4.3, noting that $\lambda = 10$, applying (8.2.5), and arguing similarly as in (8.5.26.17), we find that

$$
\begin{aligned}
R_2&(1,0,1,1,3,47,1,5) = q^{19/6} f(-q^{10}) \\
&\times \left(-\frac{f(-q^2,-q^8) f(-Q^4,-Q^6)}{f(q^4,q^6)} + q^{18} \frac{f(-q^4,-q^6) f(-Q^2,-Q^8)}{f(q^2,q^8)} \right) \\
&= -q^{19/6} f(-q^2) f(-Q^2) A(q^2). \qquad (8.5.32.3)
\end{aligned}
$$

Next, we employ Lemma 8.4.2, note that $\lambda = 15$, and utilize (8.2.5) once again to find that

$$
\begin{aligned}
R_1&(-1,0,1,0,4,1,141,3,10) = q^{19/6} f(-Q^{20}) \\
&\times \left\{ q^4 \frac{f(-Q^4,-Q^{16}) f(-q^4,-q^{16})}{f(-Q^2,-Q^{18})} - q^{22} \frac{f(-Q^8,-Q^{12}) f(-q^8,-q^{12})}{f(-Q^6,-Q^{14})} \right. \\
&\quad - \frac{f(-Q^8,-Q^{12}) f(-q^2,-q^{18})}{f(-Q^4,-Q^{16})} - q^{74} \frac{f(-Q^4,-Q^{16}) f(-q^6,-q^{14})}{f(-Q^8,-Q^{12})}
\end{aligned}
$$

$$\left. + q^{36}\varphi(-q^{10}) \right\}. \tag{8.5.32.4}$$

By (8.5.27.5) with q replaced by Q^2 and (8.2.11), we see that

$$\begin{aligned} &q^4 \frac{f(-Q^4,-Q^{16})f(-q^4,-q^{16})}{f(-Q^2,-Q^{18})} - q^{22}\frac{f(-Q^8,-Q^{12})f(-q^8,-q^{12})}{f(-Q^6,-Q^{14})} \\ &\quad = q^4 \frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\left(G(Q^2)H(q^4) - q^{18}H(Q^2)G(q^4)\right) \\ &\quad = q^4 \frac{f(-q^4)f(-Q^4)}{f(-Q^{20})} A(q^2). \end{aligned} \tag{8.5.32.5}$$

By (8.2.6), it is easy to verify that

$$f(-q^3,-q^7) = f(-q^2)H(-q)G(q^4) \quad \text{and} \quad f(-q,-q^9) = f(-q^2)G(-q)H(q^4). \tag{8.5.32.6}$$

Also from (8.2.11), we find that

$$\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{f(-q)}{f(-q^5)}H^2(q) \quad \text{and} \quad \frac{f(-q^2,-q^3)}{f(-q,-q^4)} = \frac{f(-q)}{f(-q^5)}G^2(q). \tag{8.5.32.7}$$

Using (8.5.32.6) with q replaced by q^2 and (8.5.32.7) with q replaced by Q^4, we find that

$$\begin{aligned} &- \frac{f(-Q^8,-Q^{12})f(-q^2,-q^{18})}{f(-Q^4,-Q^{16})} - q^{74}\frac{f(-Q^4,-Q^{16})f(-q^6,-q^{14})}{f(-Q^8,-Q^{12})} \\ &\quad + q^{36}\varphi(-q^{10}) \\ &= -\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\Bigg(G^2(Q^4)G(-q^2)H(q^8) + q^{74}H^2(Q^4)H(-q^2)G(q^8) \\ &\qquad\qquad - q^{36}\frac{\varphi(-q^{10})}{f(-q^4)}\frac{f(-Q^{20})}{f(-Q^4)}\Bigg). \end{aligned} \tag{8.5.32.8}$$

Next, by Entry 8.3.3 with q replaced by $-q^2$ and (8.2.12) with q replaced by Q^4, we deduce that

$$\frac{\varphi(-q^{10})}{f(-q^4)}\frac{f(-Q^{20})}{f(-Q^4)} = \left(G(-q^2)G(q^8) + q^2H(-q^2)H(q^8)\right)G(Q^4)H(Q^4). \tag{8.5.32.9}$$

Using (8.5.32.9) in (8.5.32.8), we arrive at

$$\begin{aligned} &- \frac{f(-Q^8,-Q^{12})f(-q^2,-q^{18})}{f(-Q^4,-Q^{16})} - q^{74}\frac{f(-Q^4,-Q^{16})f(-q^6,-q^{14})}{f(-Q^8,-Q^{12})} \\ &\quad + q^{36}\varphi(-q^{10}) \\ &= -\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\left(H(q^8)G(Q^4) - q^{36}G(q^8)H(Q^4)\right) \end{aligned}$$

$$\times \left(G(-q^2)G(Q^4) - q^{38}H(-q^2)H(Q^4)\right)$$
$$= -\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}A(q^4)B(-q^2). \tag{8.5.32.10}$$

By (8.5.32.4), (8.5.32.5), and (8.5.32.10), we conclude that

$$R_1(-1,0,1,0,4,1,141,3,10)$$
$$= q^{19/6}f(-q^4)f(-Q^4)\left(q^4A(q^2) - A(q^4)B(-q^2)\right). \tag{8.5.32.11}$$

Therefore, by (8.5.32.2), (8.5.32.3), and (8.5.32.11), we arrive at

$$-f(-q^2)f(-Q^2)A(q^2) = f(-q^4)f(-Q^4)\left(q^4A(q^2) - A(q^4)B(-q^2)\right). \tag{8.5.32.12}$$

Lastly, replacing q^2 by q, we conclude that

$$\left(\chi(-q)\chi(-Q) + q^2\right)A(q) = B(-q)A(q^2). \tag{8.5.32.13}$$

Next, we prove the companion formula

$$\left(\chi(-q)\chi(-Q) + q^2\right)B(q) = A(-q)B(q^2). \tag{8.5.32.14}$$

Using (8.4.62) and noting that $\lambda = 10$, we find that

$$R_2(0,1,0,1,3,47,1,5) = R_1(-1,1,0,0,3,1,141,3,10) \tag{8.5.32.15}$$

and

$$R_2(0,1,0,0,3,47,1,5) = R_1(-1,1,0,-1,3,1,141,3,10). \tag{8.5.32.16}$$

Invoking Lemma 8.4.2, noting that $\lambda = 15$, and employing (8.2.5), we find that

$$R_1(-1,1,0,0,3,1,141,3,10)$$
$$= q^{5/12}f(-Q^{20})\left\{-q^{19}\frac{f(-Q^2,-Q^{18})f(q^7,q^{13})}{f(Q,Q^{19})} + q^2\frac{f(-Q^6,-Q^{14})f(q,q^{19})}{f(Q^3,Q^{17})}\right.$$
$$- q^{10}\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)} + q^{47}\frac{f(-Q^6,-Q^{14})f(q^9,q^{11})}{f(Q^7,Q^{13})}$$
$$\left. -q^{114}\frac{f(-Q^2,-Q^{18})f(q^3,q^{17})}{f(Q^9,Q^{11})}\right\}. \tag{8.5.32.17}$$

Employing Lemma 8.4.2, noting that $\lambda = 15$, and utilizing (8.2.5) once again, we also deduce that

$$R_1(-1,1,0,-1,3,1,141,3,10)$$
$$= q^{5/12}f(-Q^{20})\left\{-q^{20}\frac{f(-Q^2,-Q^{18})f(q^3,q^{17})}{f(Q,Q^{19})} + \frac{f(-Q^6,-Q^{14})f(q^9,q^{11})}{f(Q^3,Q^{17})}\right.$$
$$- q^{10}\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)} + q^{49}\frac{f(-Q^6,-Q^{14})f(q,q^{19})}{f(Q^7,Q^{13})}$$
$$\left. -q^{113}\frac{f(-Q^2,-Q^{18})f(q^7,q^{13})}{f(Q^9,Q^{11})}\right\}. \tag{8.5.32.18}$$

Employing (8.2.19) twice with $a=-q^2$, $b=-q^3$, $n=2$ and $a=-q$, $b=-q^4$, $n=2$, we easily find that

$$f(-q^2,-q^3)=f(q^9,q^{11})-q^2f(q,q^{19}) \text{ and } f(-q,-q^4)=f(q^7,q^{13})-qf(q^3,q^{17}). \tag{8.5.32.19}$$

Now using (8.5.32.19) and subtracting (8.5.32.17) from (8.5.32.18), we find that

$$\begin{aligned}&R_1(-1,1,0,-1,3,1,141,3,10)-R_1(-1,1,0,0,3,1,141,3,10) \\ &=q^{5/12}f(-Q^{20})\left\{q^{19}\frac{f(-Q^2,-Q^{18})f(-q,-q^4)}{f(Q,Q^{19})}+\frac{f(-Q^6,-Q^{14})f(-q^2,-q^3)}{f(Q^3,Q^{17})}\right.\\ &\quad\left.-q^{47}\frac{f(-Q^6,-Q^{14})f(-q^2,-q^3)}{f(Q^7,Q^{13})}-q^{113}\frac{f(-Q^2,-Q^{18})f(-q,-q^4)}{f(Q^9,Q^{11})}\right\}.\end{aligned} \tag{8.5.32.20}$$

Using (8.5.32.19) again, this time with q replaced by Q, we find from (8.5.32.20) that

$$\begin{aligned}&R_1(-1,1,0,-1,3,1,141,3,10)-R_1(-1,1,0,0,3,1,141,3,10)\\ &=q^{5/12}f(-Q^{20})\left\{q^{19}\frac{f(-Q^2,-Q^{18})f(-q,-q^4)f(-Q^2,-Q^3)}{f(Q,Q^{19})f(Q^9,Q^{11})}\right.\\ &\quad\left.+\frac{f(-Q^6,-Q^{14})f(-q^2,-q^3)f(-Q,-Q^4)}{f(Q^3,Q^{17})f(Q^7,Q^{13})}\right\}.\end{aligned} \tag{8.5.32.21}$$

With several applications of (8.2.6), we can verify that

$$\frac{f(-Q^2,-Q^{18})f(-Q^2,-Q^3)}{f(Q,Q^{19})f(Q^9,Q^{11})}=\frac{f(-Q)H(Q^2)}{f(-Q^{20})}, \tag{8.5.32.22}$$

$$\frac{f(-Q^6,-Q^{14})f(-Q,-Q^4)}{f(Q^3,Q^{17})f(Q^7,Q^{13})}=\frac{f(-Q)G(Q^2)}{f(-Q^{20})}. \tag{8.5.32.23}$$

Using (8.2.11), (8.5.32.22), and (8.5.32.23) in (8.5.32.21), we conclude that

$$\begin{aligned}&R_1(-1,1,0,-1,3,1,141,3,10)-R_1(-1,1,0,0,3,1,141,3,10)\\ &=q^{5/12}f(-q)f(-Q)\left(G(q)G(Q^2)+q^{19}H(q)H(Q^2)\right)\\ &=q^{5/12}f(-q)f(-Q)B(q).\end{aligned} \tag{8.5.32.24}$$

Employing Lemma 8.4.3, noting that $\lambda=10$, using (8.5.27.5), and using (8.2.11), we find that

$$\begin{aligned}&R_2(0,1,0,1,3,47,1,5)=q^{5/12}f(-q^{10})\\ &\quad\times\left(q^2\frac{f(-q^2,-q^8)f(-Q^4,-Q^6)}{f(-q,-q^9)}+q^{21}\frac{f(-q^4,-q^6)f(-Q^2,-Q^8)}{f(-q^3,-q^7)}\right)\\ &=q^{5/12}f(-q^2)f(-Q^2)\left(q^2G(q)G(Q^2)+q^{21}H(q)H(Q^2)\right)\\ &=q^{29/12}f(-q^2)f(-Q^2)B(q).\end{aligned} \tag{8.5.32.25}$$

We employ Lemma 8.4.3 again. Note that again $\lambda = 10$. Argue as we did in (8.5.32.8)–(8.5.32.10). Thus, using (8.2.11), (8.5.32.7), (8.5.32.6), and Entry 8.3.3 with q replaced by $-Q$, we find that

$$\begin{aligned}
&R_2(0,1,0,0,3,47,1,5) = q^{5/12} f(-q^{10}) \\
&\times \left(q^9 \frac{f(-q^4,-q^6)f(-Q^3,-Q^7)}{f(-q^2,-q^8)} + q^{38} \frac{f(-q^2,-q^8)f(-Q,-Q^9)}{f(-q^4,-q^6)} + \varphi(-Q^5) \right) \\
&= q^{5/12} f(-q^2) f(-Q^2) \left(G(q^2)G(Q^4) + q^{38} H(q^2)H(Q^4) \right) \\
&\quad \times \left(H(q^2)G(-Q) + q^9 G(q^2)H(-Q) \right) \\
&= q^{5/12} f(-q^2) f(-Q^2) B(q^2) A(-q). \qquad (8.5.32.26)
\end{aligned}$$

Therefore, by (8.5.32.15), (8.5.32.16), (8.5.32.24), (8.5.32.25), and (8.5.32.26), we arrive at

$$-q^2 f(-q^2)f(-Q^2)B(q) + f(-q^2)f(-Q^2)B(q^2)A(-q) = f(-q)f(-Q)B(q), \qquad (8.5.32.27)$$

which is clearly equivalent to (8.5.32.14).

Now in (8.5.32.14), we replace q by $-q$ and multiply the resulting identity by (8.5.32.13) to conclude that

$$A(q^2)B(q^2) = \left(q^2 + \chi(q)\chi(Q)\right)\left(q^2 + \chi(-q)\chi(-Q)\right). \qquad (8.5.32.28)$$

By (8.3.44) and (8.5.32.28), it remains to prove that

$$\begin{aligned}
&\left(q + \chi(q^{1/2})\chi(Q^{1/2})\right)\left(q + \chi(-q^{1/2})\chi(-Q^{1/2})\right) \\
&= \chi(-q)\chi(-Q) + 2q^2 + \frac{2q^4}{\chi(-q)\chi(-Q)} \\
&\quad + q\sqrt{4\chi(-q)\chi(-Q) + 9q^2 + \frac{8q^4}{\chi(-q)\chi(-Q)}}. \qquad (8.5.32.29)
\end{aligned}$$

To prove (8.5.32.29), we employ Ramanujan's modular equation of degree 47 [55, pp. 446–447], namely,

$$\begin{aligned}
&\frac{1}{2}\left\{\varphi(q^{1/2})\varphi(Q^{1/2}) + \varphi(-q^{1/2})\varphi(-Q^{1/2})\right\} \\
&\quad - \frac{1}{2}\left\{\varphi(q)\varphi(Q) + \varphi(-q)\varphi(-Q)\right\} - 2q^{12}\psi(q^2)\psi(Q^2) \\
&= q^2 f(q)f(Q) + q^2 f(-q)f(-Q) + 2q^8 f(-q^4)f(-Q^4). \qquad (8.5.32.30)
\end{aligned}$$

We also make use of the well-known identity [55, p. 40, Entry 25 (v), (vi)]

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \qquad (8.5.32.31)$$

Employing (8.5.32.31) twice, with q replaced by Q in the second instance, and using the elementary identity $\varphi(q)\varphi(-q) = \varphi^2(-q^2)$ [55, p. 40, Entry 25 (iii)], we find that

$$
\begin{aligned}
&\{\varphi(q)\varphi(Q)+\varphi(-q)\varphi(-Q)\}^2\\
&=\varphi^2(q)\varphi^2(Q)+\varphi^2(-q)\varphi^2(-Q)+2\varphi^2(-q^2)\varphi^2(-Q)\\
&=\left(\varphi^2(q^2)+4q\psi^2(q^4)\right)\left(\varphi^2(Q^2)+4Q\psi^2(Q^4)\right)\\
&\quad+\left(\varphi^2(q^2)-4q\psi^2(q^4)\right)\left(\varphi^2(Q^2)-4Q\psi^2(Q^4)\right)+2\varphi^2(-q^2)\varphi^2(-Q^2)\\
&=2\left(\varphi^2(q^2)\varphi^2(Q^2)+16q^{48}\psi^2(q^4)\psi^2(Q^4)+\varphi^2(-q^2)\varphi^2(-Q^2)\right).
\end{aligned}
$$

Replacing q by $q^{1/2}$ above and employing the product representations for $\varphi(\pm q)$ and $\psi(q)$ in (8.2.7) and (8.2.8), respectively, we find that

$$
\begin{aligned}
&\left\{\varphi(q^{1/2})\varphi(Q^{1/2})+\varphi(-q^{1/2})\varphi(-Q^{1/2})\right\}^2\\
&=2\left(\varphi^2(q)\varphi^2(Q)+16q^{24}\psi^2(q^2)\psi^2(Q^2)+\varphi^2(-q)\varphi^2(-Q)\right)\\
&=2f^2(-q^2)f^2(-Q^2)\Bigg(\chi^4(q)\chi^4(Q)+\chi^4(-q)\chi^4(-Q)\\
&\qquad\qquad+16q^{24}\frac{1}{\chi^4(-q^2)\chi^4(-Q^2)}\Bigg).
\end{aligned}
\tag{8.5.32.32}
$$

For simplicity, set

$$L:=\chi(q)\chi(Q)+\chi(-q)\chi(-Q)\quad\text{and}\quad T:=\chi(-q^2)\chi(-Q^2).$$

After using the aforementioned definitions and employing elementary algebra in (8.5.32.32), we find that

$$
\begin{aligned}
&\left\{\varphi(q^{1/2})\varphi(Q^{1/2})+\varphi(-q^{1/2})\varphi(-Q^{1/2})\right\}^2\\
&=2f^2(-q^2)f^2(-Q^2)\left(L^4-4TL^2+2T^2+\frac{16q^{24}}{T^4}\right).
\end{aligned}
\tag{8.5.32.33}
$$

Similarly,

$$
\begin{aligned}
&\varphi(q)\varphi(Q)+\varphi(-q)\varphi(-Q)\\
&\quad=f(-q^2)f(-Q^2)\left(\chi^2(q)\chi^2(Q)+\chi^2(-q)\chi^2(-Q)\right)\\
&\quad=f(-q^2)f(-Q^2)\left(\left(\chi(q)\chi(Q)+\chi^2(q)\chi^2(Q)\right)^2-2\chi(-q^2)\chi(-Q^4)\right)\\
&\quad=f(-q^2)f(-Q^2)\left(L^2-2T\right).
\end{aligned}
\tag{8.5.32.34}
$$

Next, in (8.5.32.30), we divide each term by $f(-q^2)f(-Q^2)$ and use (8.5.32.33) and (8.5.32.34) to conclude that

$$\frac{\sqrt{2}}{2}\sqrt{L^4-4TL^2+2T^2+\frac{16q^{24}}{T^4}}-\frac{1}{2}\left(L^2-2T\right)-\frac{2q^{12}}{T^2}=q^2L+\frac{2q^8}{T}. \tag{8.5.32.35}$$

We solve for the expression with the square root in (8.5.32.35) and then square both sides to obtain

$$\begin{aligned}&\left(-TL-2\,q^6+2\,q^2T\right)\left(TL+2\,q^6\right)\\&\times\left(4\,T^3-T^2L^2+2\,q^2T^2L+8\,T^2q^4+4\,TLq^6+4\,q^8T-4\,q^{12}\right)=0.\end{aligned}$$

It is easy to see that the first two factors do not vanish identically. Therefore, we conclude that

$$4\,T^3-T^2L^2+2\,q^2T^2L+8\,T^2q^4+4\,TLq^6+4\,q^8T-4\,q^{12}=0. \quad (8.5.32.36)$$

Divide both sides of (8.5.32.36) by $-T^2$ and then complete the square to find that

$$\left(L-q^2-\frac{2q^6}{T}\right)^2=4T+9q^4+\frac{8q^8}{T}. \quad (8.5.32.37)$$

Lastly, using (8.5.32.37), we deduce that

$$\begin{aligned}2q^4+&\frac{2q^8}{T}+T+q^2\sqrt{4T+9q^4+\frac{8q^8}{T}}\\&=2q^4+\frac{2q^8}{T}+T+q^2\left(L-q^2-\frac{2q^6}{T}\right)\\&=q^4+T+q^2L\\&=q^4+\chi(-q^2)\chi(-Q^2)+q^2\left(\chi(q)\chi(Q)+\chi(-q)\chi(-Q)\right)\\&=\left(\chi(q)\chi(Q)+q^2\right)\left(\chi(-q)\chi(-Q)+q^2\right),\end{aligned}$$

which is (8.5.32.29) with q replaced by q^2. Hence, the proof of Entry 8.3.35 is complete.

8.6 Other Identities for $G(q)$ and $H(q)$ and Final Remarks

Berndt and Yesilyurt [73] took Watson's idea, found several other results like Lemmas 8.4.2 and 8.4.3, and used them to derive many new identities for the Rogers–Ramanujan functions. An example of one of their theorems is herewith provided.

Theorem 8.6.1.

$$\begin{aligned}\frac{G(q)G(-q^{14})-q^3H(q)H(-q^{14})}{G(q^7)H(-q^2)+qH(q^7)G(-q^2)}&=\frac{G(q^{56})H(q)-q^{11}H(q^{56})G(q)}{G(q^7)G(q^8)+q^3H(q^7)H(q^8)}\\&=\frac{\chi(-q^{14})}{\chi(-q^2)}=\frac{G(q)G(q^{14})+q^3H(q)H(q^{14})}{G(-q^7)H(q^2)+qH(-q^7)G(q^2)}.\end{aligned}$$

S.-S. Huang [183] expressed

$$\{G(q^4)G(q^{21})+q^5H(q^4)H(q^{21})\}\{G(q)G(q^{84})+q^{17}H(q)H(q^{84})\}$$

in terms of two quotients, each with 14 functions of the form $f(-q^n)$.

Another source of identities for the Rogers–Ramanujan functions is the unpublished doctoral dissertation of S. Robins [302]. His 13 new identities are associated with modular equations of degrees not exceeding 7 and were proved using the theory of modular forms. Several of these have now been proved without the theory of modular forms by Gugg [161].

Another group of identities is due to M. Koike [196], who discovered them using Thompson series and a computer, but he did not prove them. However, a few years later, K. Bringmann and H. Swisher [93] used modular forms to provide proofs.

One might ask whether comparable identities hold for functions similar to the Rogers–Ramanujan functions. Indeed, Huang [183] has derived several identities of this type for the Göllnitz–Gordon functions. N.D. Baruah, J. Bora, and N. Saikia [46], S.-L. Chen and Huang [109], Gugg [161], E.X.W. Xia [344], Xia and X.M. Yao [343], Q. Yan [346], and B. Yuttanan [349] have found further identities for these functions. H. Hahn [165] has derived a large number of identities for septic analogues of the Rogers–Ramanujan functions. Baruah and Bora [45] have derived several identities for nonic analogues of the Rogers–Ramanujan functions.

Recall that in his paper [278], Ramanujan wrote that (8.1.3) "is the simplest of a large class." However, he gave no further identities of the type (8.1.3), that is, identities involving *powers* of either $G(q)$ or $H(q)$. In his doctoral thesis [302], Robins discovered two further identities involving powers of the Rogers–Ramanujan functions, namely,

$$G^2(q)H(q^2)-H^2(q)G(q^2)=2qH(q)H^2(q^2)\frac{f^2(-q^{10})}{f^2(-q^5)} \tag{8.6.1}$$

and

$$G^2(q)H(q^2)+H^2(q)G(q^2)=2G(q)G^2(q^2)\frac{f^2(-q^{10})}{f^2(-q^5)}. \tag{8.6.2}$$

They were rediscovered by B. Gordon and R.J. McIntosh [156]. Proofs have also been given by Chu [114, Example 21]. In his thesis [161] and paper [160], Gugg established not only (8.6.1) and (8.6.2) but other new identities for $G(q)$ and $H(q)$ as well.

Robins [302] also used the theory of modular forms to prove

$$G^3(q)H(q^3)-G(q^3)H^3(q)=3q\frac{f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)}. \tag{8.6.3}$$

The identity was also proved by Gugg [162], who furthermore established the companion identity

$$G^3(q^3)G(q) + q^2H^3(q^3)H(q) = \frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})}. \tag{8.6.4}$$

Combining (8.6.3) and (8.6.4), Gugg then established the identity

$$\frac{G^3(q)H(q^3) - G(q^3)H^3(q)}{G^3(q^3)G(q) + q^2H^3(q^3)H(q)} = 3q\frac{f^4(-q^{15})}{f^4(-q^5)},$$

which is in the spirit of several of Ramanujan's identities.

Gugg [159, Theorem 3.1] has also established identities for the sum of products of *three* different Rogers–Ramanujan functions, namely,

$$G^2(q)G(q^2)H(q^4) - H^2(q)H(q^2)G(q^4) = 2q\frac{\psi(q^2)\psi(-q^5)}{f(-q^2)\psi(-q)}$$

and

$$G^2(q)G(q^2)H(q^4) + H^2(q)H(q^2)G(q^4) = 2\frac{\psi(q^{10})\psi(-q^5)}{f(-q^2)\psi(-q^2)}.$$

Gugg remarks that these identities are, in fact, equivalent to Entries 8.3.20 and 8.3.21.

Further identities for $G(q)$ and $H(q)$ can be found in Gugg's paper [160].

9

Circular Summation

9.1 Introduction

On page 54 in his lost notebook [283], Ramanujan makes the following claim.

Entry 9.1.1 (p. 54). *For each positive integer n and $|ab| < 1$,*

$$\sum_{-n/2<r\leq n/2} \left(\sum_{\substack{k=-\infty \\ k\equiv r \pmod{n}}}^{\infty} a^{k(k+1)/(2n)} b^{k(k-1)/(2n)} \right)^n = f(a,b) F_n(ab), \quad (9.1.1)$$

where

$$F_n(q) := 1 + 2nq^{(n-1)/2} + \cdots, \qquad n \geq 3. \quad (9.1.2)$$

At this writing, there are four different proofs of Entry 9.1.1. The first proof was by S.S. Rangachari [286], who employed Mumford's theory of theta functions and root lattices. T. Murayama [234] and K.S. Chua [116], independently, improved the work of Rangachari by removing a condition of primality from Rangachari's work. Next, S.H. Son [320] devised a proof of (9.1.1) that is more in tune with Ramanujan's work. Son used functional equations in the spirit of q-series, and we give part of his proof in this chapter, namely, his proof of the determination of $F_n(q)$ in (9.1.2). H.H. Chan, Z.-G. Liu, and S.T. Ng [101] used the classical theory of elliptic functions to provide a proof of (9.1.1). Fourthly, P. Xu [345] devised an elementary proof of (9.1.1) that perhaps reflects Ramanujan's thinking more than previous proofs, and so we present her proof.

In examining Entry 9.1.1, we see that the interest, and the difficulty, arises from the fact that the function multiplying $f(a,b)$ on the right-hand side of (9.1.1) is a function *of ab only.* The proof of Xu brings out this fact in a more elementary manner than previous proofs. The focus in (9.1.2) is on the coefficient of $q^{(n-1)/2}$, for which two approaches have been given, the first being that of Rangachari [286] and Son [320], and the second being that of

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Chan, Liu, and Ng [101]. In this chapter, we present both approaches. Formula (9.1.2) could be extended to include the cases $n = 1, 2$, but the coefficients of $q^{(n-1)/2}$, i.e., of q^0 and $q^{1/2}$, respectively, would need to be altered. When $n = 1$, the identity (9.1.1) merely reduces to the tautology $f(a, b) = f(a, b)$. When $n = 2$, the identity (9.1.1) holds if the coefficient 2 in (9.1.2) is deleted.

On page 54, Ramanujan provides five examples to illustrate Entry 9.1.1, namely, for $n = 2, 3, 4, 5, 7$. (Of course, when $n = 2$, the identity of Entry 9.1.1 needs modification as mentioned above.) In the latter portions of this chapter, we prove each of these examples in detail. We also briefly discuss examples found by other authors to illustrate Entry 9.1.1.

The appellation *circular* was initiated by Son [320], evidently to illustrate the fact that the summation index r in Entry 9.1.1 could be replaced by any set of n consecutive integers.

So that readers may even better appreciate Entry 9.1.1, we are going to illustrate Son's observation and put the entry in a form harking back to one of Ramanujan's standard theta function identities from his earlier notebooks [282]. Set $k = r + jn$ and note that the left side of (9.1.1) is independent of the complete residue system modulo n that we use. Thus, if $G_n = G_n(a, b)$ denotes the left-hand side of (9.1.1), then

$$\begin{aligned} G_n(a,b) &= \sum_{r=0}^{n-1}\left(\sum_{j=-\infty}^{\infty} a^{(r+jn)(r+jn+1)/(2n)}b^{(r+jn)(r+jn-1)/(2n)}\right)^n \qquad (9.1.3)\\ &= \sum_{r=0}^{n-1}\left(a^{r(r+1)/(2n)}b^{r(r-1)/(2n)}\sum_{j=-\infty}^{\infty} a^{nj^2/2+j/2+rj}b^{nj^2/2-j/2+rj}\right)^n\\ &= \sum_{r=0}^{n-1}\Big(a^{r(r+1)/(2n)}b^{r(r-1)/(2n)}\\ &\qquad \times f(a^{n/2+r+1/2}b^{n/2+r-1/2}, a^{n/2-r-1/2}b^{n/2-r+1/2})\Big)^n\\ &= \sum_{r=0}^{n-1} a^{r(r+1)/2}b^{r(r-1)/2}f^n(a^{n/2+r+1/2}b^{n/2+r-1/2}, a^{n/2-r-1/2}b^{n/2-r+1/2}). \end{aligned}$$

If we set

$$U_j := U_j(n) := a^{j(j+1)/(2n)}b^{j(j-1)/(2n)}, \qquad j \geq 0, \tag{9.1.4}$$

and

$$V_j := V_j(n) := a^{j(j-1)/(2n)}b^{j(j+1)/(2n)}, \qquad j \geq 0, \tag{9.1.5}$$

then we can rewrite (9.1.3) in the form

$$G_n(a,b) = \sum_{r=0}^{n-1} U_r(1)f^n\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right) = \sum_{r=0}^{n-1}\left(U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right)\right)^n. \tag{9.1.6}$$

The proposed identity (9.1.6) should be compared with a fundamental representation for a single theta function as a sum of theta functions that can be found as Entry 31 in Chapter 16 of Ramanujan's second notebook [283], [55, p. 48] in the form

$$f(a,b)=\sum_{r=0}^{n-1}U_r f\left(\frac{U_{n+r}}{U_r},\frac{V_{n-r}}{U_r}\right), \tag{9.1.7}$$

where $|ab|<1$, n is any positive integer, and now

$$U_j := a^{j(j+1)/2}b^{j(j-1)/2}, \qquad j\geq 0, \tag{9.1.8}$$

and

$$V_j := a^{j(j-1)/2}b^{j(j+1)/2}, \qquad j\geq 0. \tag{9.1.9}$$

(See also (3.2.2) and (8.2.19).) *Note that the definitions of U_j and V_j in (9.1.8) and (9.1.9) differ from those in (9.1.4) and (9.1.5) and are, respectively, $U_j(1)$ and $V_j(1)$ in (9.1.4) and (9.1.5).* We have made this slight alteration in notation to emphasize the similarity of the forms in (9.1.6) and (9.1.7).

9.2 Proof of Entry 9.1.1

We begin by giving Xu's proof of (9.1.1) [345].

Proof. Set $a=qx$ and $b=q/x$. Using the representation on the far right side of (9.1.3), we put $G_n(a,b)=:G_n(q)$ in the form

$$G_n(q)=\sum_{r=0}^{n-1}q^{r^2}x^r\left\{\sum_{m=-\infty}^{\infty}q^{nm^2-2rm}x^{-m}\right\}^n. \tag{9.2.1}$$

Expanding G_n in a Laurent series in x, we write

$$\sum_{r=0}^{n-1}q^{r^2}x^r\left\{\sum_{m=-\infty}^{\infty}q^{nm^2-2rm}x^{-m}\right\}^n=\sum_{m=-\infty}^{\infty}F_{n,m}(q)q^{m^2}x^m. \tag{9.2.2}$$

Our objective is to show that $F_{n,m}(q)$ is independent of m, i.e., we can write $F_{n,m}(q)=F_n(ab)=F_n(q^2)$, and so reduce (9.2.2) to the form

$$\sum_{r=0}^{n-1}q^{r^2}x^r\left\{\sum_{m=-\infty}^{\infty}q^{nm^2-2rm}x^{-m}\right\}^n=F_n(q)\sum_{m=-\infty}^{\infty}q^{m^2}x^m=F_n(ab)f(a,b). \tag{9.2.3}$$

Equating coefficients of x^m on both sides of (9.2.2) and setting $F_{n,m}=F_{n,m}(q)$, we find that

$$\begin{aligned}
F_{n,m} &= \sum_{r=0}^{n-1} \sum_{\substack{m_1,\dots,m_n=-\infty \\ m_1+\cdots+m_n=r-m}}^{\infty} q^{n(m_1^2+\cdots+m_n^2)-2r(m_1+\cdots+m_n)+r^2-m^2} \\
&= \sum_{r=0}^{n-1} \sum_{\substack{m_1,\dots,m_n=-\infty \\ m_1+\cdots+m_n=r-m}}^{\infty} q^{n(m_1^2+\cdots+m_n^2)-(r-m)^2} \\
&= \sum_{r=0}^{n-1} \sum_{\substack{m_1,\dots,m_n=-\infty \\ m_1+\cdots+m_n=r-m}}^{\infty} q^{n(m_1^2+\cdots+m_n^2)-(m_1+\cdots+m_n)^2}. \qquad (9.2.4)
\end{aligned}$$

Suppose now that we replace m_j by $m_j - 1$, for each j, $1 \le j \le n$, in (9.2.4). Then, after simplifying the exponents above, we find that

$$F_{n,m} = \sum_{r=0}^{n-1} \sum_{\substack{m_1,\dots,m_n=-\infty \\ m_1+\cdots+m_n=r-m+n}}^{\infty} q^{n(m_1^2+\cdots+m_n^2)-(m_1+\cdots+m_n)^2}. \qquad (9.2.5)$$

Hence, we see that in the inner sum above, the summation condition on $m_1 + \cdots + m_n$ can be taken over any n consecutive integers. Therefore, the multiple series on the right-hand side of (9.2.5) is independent of m. Thus, by (9.2.2) and (9.2.5), we have indeed demonstrated that (9.2.3) holds to complete the proof.

We now demonstrate the truth of (9.1.2). First, we use an argument due to Son [320]. Set $x = 1$ in (9.2.1) and (9.2.3), so that

$$G_n(q) = \sum_{r=0}^{n-1} q^{r^2} f^n(q^{n+2r}, q^{n-2r}) := \sum_{r=0}^{n-1} S_r, \qquad (9.2.6)$$

where

$$S_r := S_r(q) := q^{r^2} f^n(q^{n+2r}, q^{n-2r}) = S_{-r}(q). \qquad (9.2.7)$$

Recall the fundamental property [55, p. 34, Entry 18(iv)]

$$f(a,b) = a^{k(k+1)/2} b^{k(k-1)/2} f(a(ab)^k, b(ab)^{-k}), \qquad (9.2.8)$$

where k is any integer. Applying (9.2.8) with $k = -1$, we find that

$$f(q^{n+(2n-2r)}, q^{n-(2n-2r)}) = q^{2r-n} f(q^{n+2r}, q^{n-2r}),$$

and so

$$S_{n-r} = q^{(n-r)^2} f^n(q^{n+(2n-2r)}, q^{n-(2n-2r)}) = q^{r^2} f^n(q^{n+2r}, q^{n-2r}) = S_r. \qquad (9.2.9)$$

Hence, from (9.2.6), (9.2.9), and (9.2.7), we deduce that

$$G_n(q) = \sum_{-n/2<r\le n/2} S_r = S_0 + 2 \sum_{0<r<n/2} S_r + \frac{1+(-1)^n}{2} S_{n/2}, \qquad (9.2.10)$$

where the last term is present only when n is even. By definition,

$$f(q^{n+2r}, q^{n-2r}) = 1 + q^{n-2r} + q^{n+2r} + \cdots .$$

Thus, for $0 \leq r < n/2$, by the multinomial theorem,

$$\begin{aligned} S_r &= q^{r^2}(1 + nq^{n-2r} + nq^{n+2r} + \cdots) \\ &= q^{r^2} + nq^{n-1+(r-1)^2} + nq^{n+2r+r^2} + \cdots . \end{aligned} \tag{9.2.11}$$

Excluding the squares $\{q^{r^2}\}_{r<\sqrt{n}}$, we see that the term with the lowest power of q in $G_n(q)$ arises from S_1 and is equal to nq^{n-1}. Thus, expanding $G_n(q)$ and $f(q,q) = \varphi(q)$, where $\varphi(q)$ is defined in (5.11.1), using (9.2.3), (9.2.10), and (9.2.11), and setting $r_0 = \lfloor \sqrt{n} \rfloor$, we arrive at

$$\begin{aligned} G_n(q) &= 1 + 2q + \cdots + 2q^{r_0^2} + 2nq^{n-1} + 4nq^n + \cdots + 2q^{(r_0+1)^2} + \cdots \\ &= (1 + 2q + \cdots + 2q^{r_0^2} + \cdots)F_n(q^2). \end{aligned}$$

By long division, we find that

$$F_n(q^2) = 1 + 2nq^{n-1} + \cdots ,$$

as desired, and this completes the proof.

We now give a second proof of (9.1.2) that is due to Chan, Liu, and Ng [101]. Earlier, we showed that the inner sum in (9.2.5) is independent of m. Thus, we can write

$$F_n(q) = \sum_{r=0}^{n-1} \sum_{\substack{m_1, m_2, \ldots, m_n = -\infty \\ m_1+m_2+\cdots+m_n = r}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-r^2}. \tag{9.2.12}$$

We use (9.2.12) to prove (9.1.2).

By the Cauchy–Schwarz inequality,

$$n(m_1^2 + m_2^2 + \cdots + m_n^2) \geq (m_1 + m_2 + \cdots + m_n)^2 = r^2.$$

Thus, $F_n(q)$ is a Taylor series in q. To prove (9.1.2), we need to study the number of the solutions of the Diophantine equations

$$\begin{cases} n(m_1^2 + m_2^2 + \cdots + m_n^2) - r^2 = t, \\ m_1 + m_2 + \cdots + m_n = r. \end{cases} \tag{9.2.13}$$

Let $N(t)$ denote the number of solutions of the equations above. Then,

$$F_n(q) = N(0) + N(1)q + \cdots + N(n-1)q^{n-1} + \cdots .$$

It is obvious that for any integer m, $m^2 \geq m$. Thus,

$$m_1^2 + m_2^2 + \cdots + m_n^2 \geq m_1 + m_2 + \cdots + m_n.$$

Combining this with (9.2.13), we find that

$$t \geq r(n-r), \qquad \text{for} \quad 0 \leq r \leq n-1. \tag{9.2.14}$$

When $t = 0$, this inequality holds only when $r = 0$. Then (9.2.13) becomes

$$\begin{cases} m_1^2 + m_2^2 + \cdots + m_n^2 = 0, \\ m_1 + m_2 + \cdots + m_n = 0. \end{cases}$$

The only solution of this equation is $m_1 = m_2 = \cdots = m_n = 0$, and thus $N(0) = 1$.

When $1 \leq r \leq n-1$, we find from (9.2.14) that

$$t \geq r(n-r) \geq n-1. \tag{9.2.15}$$

Hence, $N(t) = 0$ for $1 \leq t \leq n-2$. The equality in (9.2.15) holds if and only if $r = 1$ or $r = n-1$.

When $r = 1$, (9.2.13) becomes

$$\begin{cases} m_1^2 + m_2^2 + \cdots + m_n^2 = 1, \\ m_1 + m_2 + \cdots + m_n = 1. \end{cases} \tag{9.2.16}$$

The solutions of (9.2.16) are $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$, and the number of solutions is clearly n.

When $r = n-1$, (9.2.13) becomes

$$\begin{cases} m_1^2 + m_2^2 + \cdots + m_n^2 = n-1, \\ m_1 + m_2 + \cdots + m_n = n-1. \end{cases} \tag{9.2.17}$$

The solutions of the set (9.2.17) are $(0, 1, \ldots, 1), (1, 0, \ldots, 1), \ldots, (1, 1, \ldots, 0)$, since

$$m_1(m_1 - 1) + \cdots + m_n(m_n - 1) = 0,$$

and since the trivial inequality $m_i^2 \geq m_i$ implies that $m_i(m_i - 1) = 0$. It follows that $m_i = 0$ or 1. Therefore, the number of solutions in this case is n. Combining this number with the number of solutions for $r = 1$, we conclude that $N(n-1) = 2n$. This completes the proof of (9.1.2). $\square$

9.3 Reformulations and Work of H.H. Chan, Z.-G. Liu, S.T. Ng, A. Berkovich, F.G. Garvan, and H. Yesilyurt

Although the proof of Ramanujan's key theorem, Entry 9.1.1, is relatively short, it is generally not easy to calculate specific examples, i.e., to explicitly determine $F_n(ab)$. Either heavy machinery, or clever arguments, or both are needed. We find it more instructive to attempt to construct proofs in a manner

that Ramanujan might have discovered rather than to invoke the theory of modular forms. To that end, we are going to use an approach of Chan, Liu, and Ng [101] and transform Entry 9.1.1. Then we will employ ideas of Berkovich, Garvan, and Yesilyurt [53] first to reformulate the theorem of Chan, Liu, and Ng and second to devise a "constant term" method to determine individual examples.

Because we are going to apply the transformation $\tau \mapsto -1/\tau$ to Ramanujan's identity, it will be convenient to convert Ramanujan's theorem into one involving the classical theta function

$$\theta_3(z|\tau) := \sum_{m=-\infty}^{\infty} q^{m^2} e^{2miz}, \qquad q = e^{\pi i \tau}, \qquad \operatorname{Im} \tau > 0, \qquad z \in \mathbb{C}.$$

We now easily check that Entry 9.1.1 can be recast in the equivalent form given below.

Theorem 9.3.1. *For each positive integer n,*

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz} \theta_3^n(z + k\pi\tau \mid n\tau) = \theta_3(z \mid \tau) F_n(\tau), \tag{9.3.1}$$

where for $n \geq 3$,

$$F_n(\tau) = 1 + 2nq^{n-1} + \cdots . \tag{9.3.2}$$

Next, we transform Theorem 9.3.1 into an equivalent theorem that will be more convenient for us. We emphasize that Theorem 9.3.2 is equivalent to Entry 9.1.1.

Theorem 9.3.2. *For each positive integer n,*

$$\sum_{k=0}^{n-1} \theta_3^n \left(z + \frac{k\pi}{n} \middle| \tau \right) = G_n(\tau) \theta_3(nz \mid n\tau), \tag{9.3.3}$$

where

$$G_n(\tau) := \sqrt{n}(-i\tau)^{(1-n)/2} F_n\left(-\frac{1}{n\tau}\right). \tag{9.3.4}$$

Proof. Replacing τ by $-1/(n\tau)$ and then z by z/τ in (9.3.1), we find that

$$\sum_{k=0}^{n-1} e^{-\frac{\pi i k^2}{n\tau} + \frac{2ikz}{\tau}} \theta_3^n \left(\frac{z}{\tau} - \frac{k\pi}{n\tau} \middle| -\frac{1}{\tau} \right) = F_n\left(-\frac{1}{n\tau}\right) \theta_3\left(\frac{z}{\tau} \middle| -\frac{1}{n\tau}\right). \tag{9.3.5}$$

Using the transformation formula for θ_3 [339, p. 475] on both sides of (9.3.5) and simplifying, we deduce that

$$\sum_{k=0}^{n-1} \theta_3^n \left(z - \frac{k\pi}{n} \middle| \tau \right) = \sqrt{n}(-i\tau)^{(1-n)/2} F_n\left(-\frac{1}{n\tau}\right) \theta_3(nz \mid n\tau). \tag{9.3.6}$$

Replacing z by $-z$ and realizing that $\theta_3(z \mid \tau)$ is an even function of z, we complete the proof of (9.3.3). $\square$

Now set $x = e^{2iz}$ and $\omega_n = e^{2\pi i/n}$. Then, written out in summation notation, (9.3.3) assumes the shape

$$G_n(\tau) \sum_{m=-\infty}^{\infty} q^{nm^2} x^{nm} = \sum_{k=0}^{n-1} \left\{ \sum_{m=-\infty}^{\infty} q^{m^2} \omega_n^{km} x^m \right\}^n \tag{9.3.7}$$
$$= \sum_{k=0}^{n-1} \sum_{m_1,m_2,\dots,m_n=-\infty}^{\infty} q^{m_1^2+m_2^2+\cdots+m_n^2} \omega_n^{k(m_1+m_2+\cdots+m_n)} x^{m_1+m_2+\cdots+m_n}.$$

If we equate constant terms on both sides of (9.3.7), we find that

$$G_n(\tau) = n \sum_{\substack{m_1,m_2,\dots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=0}}^{\infty} q^{m_1^2+m_2^2+\cdots+m_n^2} =: nZ_n(q). \tag{9.3.8}$$

We transform (9.3.7) into Ramanujan's notation for theta functions. With $x = 1$ and ω_n as given above, (9.3.7) can be rewritten in the form

$$\sum_{k=0}^{n-1} f^n(q\omega_n^k, q\omega_n^{-k}) = G_n(\tau)\varphi(q^n) = nZ_n(q)\varphi(q^n). \tag{9.3.9}$$

We now offer the approach of Berkovich, Garvan, and Yesilyurt [53] to individual examples. Set

$$R_n(x,q) := f^n(qx, qx^{-1}), \qquad n \geq 1, \tag{9.3.10}$$

and

$$R_n(q) := [x^0]R_n(x,q), \tag{9.3.11}$$

where $[x^n]f(x)$ is the coefficient of x^n in the Laurent expansion of $f(x)$ about $x = 0$. Applying (9.2.8) with $a = qx$, $b = qx^{-1}$, and $k = 1$, we see that

$$R_n(q^2x, q) = (qx)^{-n} R_n(x,q). \tag{9.3.12}$$

It is also obvious from the definition (9.3.10) that

$$R_n(x,q) = R_n(x^{-1}, q). \tag{9.3.13}$$

Now set

$$R_n(x,q) = \sum_{j=-\infty}^{\infty} A_j(q) x^j. \tag{9.3.14}$$

Clearly, from (9.3.13),

$$A_j(q) = A_{-j}(q). \tag{9.3.15}$$

Using (9.3.12), (9.3.13), and induction on k, we can show that

$$A_{nk+j}(q) = q^{nk^2+2kj} A_j(q), \qquad 0 \leq j \leq n-1. \tag{9.3.16}$$

Hence, using (9.3.16), we find that

$$\begin{aligned} R_n(x,q) &= \sum_{j=0}^{n-1} A_j(q) \sum_{k=-\infty}^{\infty} q^{nk^2+2kj} x^{nk+j} \\ &= \sum_{j=0}^{n-1} x^j A_j(q) f(q^{n+2j}x^n, q^{n-2j}x^{-n}). \end{aligned} \tag{9.3.17}$$

For a fixed integer ℓ, $0 \le \ell \le n-1$, multiply both sides of (9.3.17) by $x^{-\ell}$. Then replace x by ω_n^r and sum on r, $0 \le r \le n-1$. Inverting the order of summation, we arrive at

$$\begin{aligned} \sum_{r=0}^{n-1} \omega_n^{-r\ell} R_n(\omega_n^r, q) &= \sum_{j=0}^{n-1} A_j(q) f(q^{n+2j}, q^{n-2j}) \sum_{r=0}^{n-1} \omega_n^{r(j-\ell)} \\ &= nA_\ell(q) f(q^{n+2\ell}, q^{n-2\ell}). \end{aligned} \tag{9.3.18}$$

In particular, taking $\ell = 0$ and noting from (9.3.11) and (9.3.14) that $R_n(q) = [x^0]R_n(x,q) = A_0(q)$, we conclude that

$$\sum_{r=0}^{n-1} R_n(\omega_n^r, q) = nR_n(q)\varphi(q^n). \tag{9.3.19}$$

If we compare (9.3.19) with (9.3.9) and note the definitions of $R_n(x,q)$, $Z_n(q)$, and $R_n(q)$ in (9.3.10), (9.3.8), and (9.3.11), respectively, we see that (9.3.19) and (9.3.9) are identical.

9.4 Special Cases

We now discuss the five identities arising from the cases $n = 2, 3, 4, 5, 7$ in Entry 9.1.1. We write Entries 9.4.1, 9.4.2, 9.4.3, and 9.4.4 in two forms, the original form of Ramanujan and the equivalent formulation under the transformation $\tau \mapsto -1/\tau$. In other words, in view of (9.3.19), we present $R_n(q)$, when $n = 2, 3, 4, 5$. In our proofs below, we either give a proof of Ramanujan's formulation or offer a proof in the transformed reformulation.

Entry 9.4.1 (p. 54). *We have*

$$f^2(a^{3/2}b^{1/2}, a^{1/2}b^{3/2}) + af^2(a^{5/2}b^{3/2}, a^{-1/2}b^{1/2}) = f(a,b)\varphi(\sqrt{ab}) \tag{9.4.1}$$

and

$$R_2(q) = \varphi(q^2). \tag{9.4.2}$$

Observe that (9.4.1) does not fit the pattern of (9.1.1), because the coefficient of $\sqrt{ab}$ in (9.4.1) equals 2, whereas if (9.1.1) were valid for $n = 2$, then the coefficient of $\sqrt{ab}$ should be 4. In passing, we note the identity [55, p. 46, Entry 30(v)]

$$f^2(\sqrt{a}, \sqrt{b}) + f^2(-\sqrt{a}, -\sqrt{b}) = 2f(a,b)\varphi(\sqrt{ab}),$$

which is more elementary than (9.4.1).

First Proof of Entry 9.4.1. This proof is due to Son [320]. First, in the addition formula (9.1.7), let $n = 2$ and replace a and b by $\sqrt{a}$ and $\sqrt{b}$, respectively. Hence,

$$f(a^{1/2}, b^{1/2}) = f(a^{3/2}b^{1/2}, a^{1/2}b^{3/2}) + a^{1/2}f(a^{5/2}b^{3/2}, a^{-1/2}b^{1/2}). \tag{9.4.3}$$

Next, replace $\sqrt{a}$ and $\sqrt{b}$ by $\pm i\sqrt{a}$ and $\pm i\sqrt{b}$ in (9.4.3) to arrive at

$$f(\pm ia^{1/2}, \pm ib^{1/2}) = f(a^{3/2}b^{1/2}, a^{1/2}b^{3/2}) \pm ia^{1/2}f(a^{5/2}b^{3/2}, a^{-1/2}b^{1/2}).$$

Multiplying the two identities above together, we deduce that

$$\begin{aligned} &f(ia^{1/2}, ib^{1/2})f(-ia^{1/2}, -ib^{1/2}) \\ &\quad = f^2(a^{3/2}b^{1/2}, a^{1/2}b^{3/2}) + af^2(a^{5/2}b^{3/2}, a^{-1/2}b^{1/2}). \end{aligned} \tag{9.4.4}$$

Comparing (9.4.4) with (9.4.1), we see that it remains to show that

$$f(ia^{1/2}, ib^{1/2})f(-ia^{1/2}, -ib^{1/2}) = f(a,b)\varphi(\sqrt{ab}). \tag{9.4.5}$$

To that end, by the Jacobi triple product identity (8.2.6) and Euler's pentagonal number theorem (8.2.9),

$$\begin{aligned} &f(ia^{1/2}, ib^{1/2})f(-ia^{1/2}, -ib^{1/2}) \\ &\quad = (-ia^{1/2}; -(ab)^{1/2})_\infty(-ib^{1/2}; -(ab)^{1/2})_\infty(-(ab)^{1/2}; -(ab)^{1/2})_\infty \\ &\qquad \times (ia^{1/2}; -(ab)^{1/2})_\infty(ib^{1/2}; -(ab)^{1/2})_\infty(-(ab)^{1/2}; -(ab)^{1/2})_\infty \\ &\quad = (-a; ab)_\infty(-b; ab)_\infty f^2(\sqrt{ab}) \\ &\quad = \frac{f(a,b)}{f(-ab)}f^2(\sqrt{ab}) \\ &\quad = f(a,b)\varphi(\sqrt{ab}), \end{aligned}$$

where in the last step we used the identity [55, p. 39, Entry 2(iii)]

$$\varphi(q) = \frac{f^2(q)}{f(-q^2)}.$$

$\square$

Second Proof of Entry 9.4.1. From [55, p. 40, Entries 30(v), (vi)],

$$f^2(a,b)+f^2(-a,-b)=2f(a^2,b^2)\varphi(ab),$$
$$f^2(a,b)-f^2(-a,-b)=4af\left(\frac{b}{a},\frac{a}{b}a^2b^2\right)\psi(a^2b^2).$$

Add the two identities above to obtain

$$f^2(a{\cdot}b)=f(a^2,b^2)\varphi(ab)+2af\left(\frac{b}{a},\frac{a}{b}a^2b^2\right)\psi(a^2b^2). \tag{9.4.6}$$

Putting $a=q/x$ and $b=qx$ in (9.4.6), we find that

$$R_2(x,q)=f^2(qx,q/x)=f(q^2x^2,q^2/x^2)\varphi(q^2)+2\frac{q}{x}f(x^2,q^4/x^2)\psi(q^4). \tag{9.4.7}$$

It is now immediate that (9.4.2) follows from (9.4.7). □

Third Proof of Entry 9.4.1. As suggested by M. Somos, in Lemma 8.5.4, set $x=-a$, $y=-\sqrt{ab}$, and $q=ab$. We then immediately obtain (9.4.1). □

Entry 9.4.2 (p. 54). *We have*

$$f^3(a^2b,ab^2)+af^3(b,a^3b^2)+bf^3(a,a^2b^3)=f(a,b)F_3(ab), \tag{9.4.8}$$

where

$$F_3(q)=\left(\frac{f^9(-q)}{f^3(-q^3)}+27q\frac{f^9(-q^3)}{f^3(-q)}\right)^{1/3}=\frac{\psi^3(q)}{\psi(q^3)}+3q\frac{\psi^3(q^3)}{\psi(q)}. \tag{9.4.9}$$

Also,

$$R_3(q)=a(q^2)=\varphi(q^2)\varphi(q^6)+4q^2\psi(q^4)\psi(q^{12}), \tag{9.4.10}$$

where $a(q)$ is the cubic theta function of J.M. and P.B. Borwein [79] *and of Ramanujan, which is defined by*

$$a(q):=\sum_{m,n=-\infty}^{\infty}q^{m^2+mn+n^2}.$$

A different representation for $F_3(q)$ can be found in Ramanujan's earlier notebooks [282, p. 321]. If $\left(\frac{n}{3}\right)$ denotes the Legendre symbol, then [57, p. 142, Entry 3]

$$f^3(a^2b,ab^2)+af^3(b,a^3b^2)+bf^3(a,a^2b^3)=f(a,b)\left(1+6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{a^nb^n}{1-a^nb^n}\right). \tag{9.4.11}$$

A proof of (9.4.11) can be found in Berndt's paper [56]. Thus, combining the identities (9.4.9) with (9.4.11), we deduce the following corollary, which, in fact, is part of the content of Entry 3(i) in Chapter 21 in Ramanujan's second notebook [282], [55, p. 460].

Corollary 9.4.1. *If* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then*

$$\left(\frac{f^9(-q)}{f^3(-q^3)}+27q\frac{f^9(-q^3)}{f^3(-q)}\right)^{1/3}=\frac{\psi^3(q)}{\psi(q^3)}+3q\frac{\psi^3(q^3)}{\psi(q)}=1+6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n}.$$

The representation (9.4.10) is given in [282, p. 328], [58, p. 93, Equation (2.7)]. Several other representations for $R_3(q)$ can be derived. For example [58, pp. 110–111],

$$R_3(q)=a(q^2)=\frac{\varphi^3(-q^6)}{\varphi(-q^2)}+4q^2\frac{\psi^3(q^6)}{\psi(q^2)}=\frac{\psi^3(q^2)}{\psi(q^6)}+3q^2\frac{\psi^3(q^6)}{\psi(q^2)}.$$

Proof. From (9.4.7),

$$\begin{aligned}R_3(q)&=[x^0]R_3(x,q) &(9.4.12)\\ &=[x^0]\left(f(qx,q/x)(f(q^2x^2,q^2/x^2)\varphi(q^2)+2\frac{q}{x}f(x^2,q^4/x^2)\psi(q^4))\right).\end{aligned}$$

We now determine the prescribed coefficient above. First, we observe that

$$\begin{aligned}[x^0](f(qx,q/x)f(q^2x^2,q^2/x^2))&=[x^0]\sum_{m,n=-\infty}^{\infty}q^{n^2+2m^2}x^{n+2m} &(9.4.13)\\ &=\sum_{\substack{m,n=-\infty\\ n+2m=0}}^{\infty}q^{n^2+2m^2}=\sum_{m=-\infty}^{\infty}q^{6m^2}=\varphi(q^6).\end{aligned}$$

A similar argument yields

$$[x^1](f(qx,q/x)f(x^2,q^4/x^2))=2q\psi(q^{12}). \tag{9.4.14}$$

Putting (9.4.13) and (9.4.14) in (9.4.12), we deduce that

$$R_3(q)=\varphi(q^2)\varphi(q^6)+4q^2\psi(q^4)\psi(q^{12}),$$

which, by (9.4.10), is what we wanted to prove. □

Entry 9.4.3 (p. 54). *We have*

$$F_4(q)=\varphi^3(q^2)+(2\sqrt{q})^3\psi^3(q^4) \tag{9.4.15}$$

and

$$R_4(q)=\frac{1}{2}\left(\varphi^3(q)+\varphi^3(-q)\right). \tag{9.4.16}$$

Proof. We use two elementary identities found in Chapter 16 of Ramanujan's second notebook [282]. From Entry 30(iv) with $a=qx$ and $b=q/x$ [55, p. 46],

$$f(qx,q/x)f(-xq,-q/x)=f(-q^2x^2,-q^2/x^2)\varphi(-q^2), \tag{9.4.17}$$

and from Entry 29(i) with $a = b = q$, $c = qx$, and $d = q/x$ [55, p. 45],

$$2f^2(q^2x, q^2/x) = \varphi(q)f(qx, q/x) + \varphi(-q)f(-qx, -q/x). \tag{9.4.18}$$

Setting $x = 1$ in (9.4.17) yields

$$\varphi^2(-q^2) = \varphi(q)\varphi(-q). \tag{9.4.19}$$

Putting $x = 1$ in (9.4.7), we find that

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \tag{9.4.20}$$

while putting $x = 1$ in (9.4.18), we find that

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2). \tag{9.4.21}$$

Square both sides of (9.4.18), with a and b as prescribed above, and employ (9.4.17), (9.4.19), (9.4.7), (9.4.20), and (9.4.21) to deduce that

$$\begin{aligned}
&4f^4(q^2x, q^2/x)\\
&\quad = \varphi^2(q)f^2(qx, q/x) + \varphi^2(-q)f^2(-qx, -q/x) + 2\varphi^3(-q^2)f(-q^2x^2, -q^2/x^2)\\
&\quad = \varphi^2(q)\left(\varphi(q^2)f(q^2x^2, q^2/x^2) + 2\frac{q}{x}\psi(q^4)f(x^2, q^4/x^2)\right)\\
&\qquad + \varphi^2(-q)\left(\varphi(q^2)f(q^2x^2, q^2/x^2) - 2\frac{q}{x}\psi(q^4)f(x^2, q^4/x^2)\right)\\
&\qquad + 2\varphi^3(-q^2)f(-q^2x^2, -q^2/x^2)\\
&\quad = \varphi(q^2)\left(\varphi^2(q) + \varphi^2(-q)\right)f(q^2x^2, q^2/x^2) + 2\frac{q}{x}\psi(q^4)\left(\varphi^2(q) - \varphi^2(-q)\right)\\
&\qquad \times f(x^2, q^4/x^2) + 2\varphi^3(-q^2)f(-q^2x^2, -q^2/x^2)\\
&\quad = 2\varphi^3(q^2)f(q^2x^2, q^2/x^2) + 16\frac{q^2}{x}\psi^3(q^4)f(x^2, q^4/x^2)\\
&\qquad + 2\varphi^3(-q^2)f(-q^2x^2, -q^2/x^2).
\end{aligned}$$

Upon dividing both sides by 2 and replacing q^2 by q, we conclude that

$$\begin{aligned}
2R_4(x, q) &= \varphi^3(q)f(qx^2, q/x^2)\\
&\qquad + 8\frac{q}{x}\psi^3(q^2)f(x^2, q^2/x^2) + \varphi^3(-q)f(-qx^2, -q/x^2),
\end{aligned} \tag{9.4.22}$$

from which (9.4.16) readily follows. □

Entry 9.4.4 (p. 54). *We have*

$$F_5(q) = \frac{f^5(-q)}{f(-q^5)} + 5q\frac{f^5(-q^5)}{f(-q)} \tag{9.4.23}$$

and

$$R_5(q) = \frac{f^5(-q^2)}{f(-q^{10})} + 25q^2\frac{f^5(-q^{10})}{f(-q^2)}. \tag{9.4.24}$$

Proof. From (9.4.22), we see that

$$\begin{aligned}2R_5(q) &= 2[x^0]\left(f(qx,q/x)R_4(q)\right)\\&= \varphi^3(q)[x^0]\left(f(qx,q/x)f(qx^2,q/x^2)\right)\\&\quad+\varphi^3(-q)[x^0]\left(f(qx,q/x)f(-qx^2,-q/x^2)\right)\\&\quad+8q\psi^3(q^2)[x^1]\left(f(xq,q/x)f(x^2,q^2/x^2)\right).\end{aligned} \tag{9.4.25}$$

Now,

$$\begin{aligned}[x^0]\left(f(qx,q/x)f(qx^2,q/x^2)\right) &= [x^0]\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2}x^{2m+n}\\&= \sum_{\substack{m,n=-\infty\\2m+n=0}}^{\infty} q^{m^2+n^2} = \sum_{m=-\infty}^{\infty} q^{5m^2} = \varphi(q^5).\end{aligned} \tag{9.4.26}$$

Similar arguments show that

$$[x^0]\left(f(qx,q/x)f(-qx^2,-q/x^2)\right) = \varphi(-q^5) \tag{9.4.27}$$

and

$$[x^1]\left(f(xq,q/x)f(x^2,q^2/x^2)\right) = 2q\psi(q^{10}). \tag{9.4.28}$$

Collecting (9.4.26)–(9.4.28) and putting our findings in (9.4.25), we conclude that

$$R_5(q) = \frac{1}{2}\left(\varphi^3(q)\varphi(q^5)+\varphi^3(-q)\varphi(-q^5)\right)+8q^2\psi^3(q^2)\psi(q^{10}). \tag{9.4.29}$$

To show that (9.4.29) and (9.4.24) are equivalent, we use an argument of Berkovich, Garvan, and Yesilyurt [53]. To do this, we need some results from a paper by Berkovich and Yesilyurt [52]. Write the factorization of a positive integer $n>1$ in the form

$$n = 5^d\prod_{i=1}^{r} p_i^{v_i}\prod_{j=1}^{s} q_j^{w_j}, \tag{9.4.30}$$

where p_i, $1\le i\le r$, and q_j, $1\le j\le s$, are primes with $p_i\equiv\pm1 \pmod 5$ and $q_j\equiv\pm2 \pmod 5$. Then [52, pp. 403–404, Equations (7.22), (7.24)]

$$\alpha(n) := [q^n]\frac{f^5(-q)}{f(-q^5)} = -5\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j} \tag{9.4.31}$$

and

$$\beta(n) := [q^n]\frac{qf^5(-q^5)}{f(-q)} = 5^d\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}(-1)^{w_j}\frac{1-(-q_j)^{w_j+1}}{1+q_j}. \tag{9.4.32}$$

We now write the factorization of n in the slightly different form

$$n = 2^g 5^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

with $p_i \equiv \pm 1 \pmod 5$, $q_j \equiv \pm 2 \pmod 5$, and q_j odd. Also, let t be the number of odd prime divisors of n, counting multiplicities, that are congruent to $\pm 2 \pmod 5$. With this reformulation, we can clearly write (9.4.31) and (9.4.32) in their equivalent forms

$$\alpha(n) = -5\frac{1-(-2)^{g+1}}{3}u(n) \tag{9.4.33}$$

and

$$\beta(n) = 5^d(-1)^{t+g}\frac{1-(-2)^{g+1}}{3}u(n), \tag{9.4.34}$$

where

$$u(n) = \prod_{i=1}^{r} \frac{1-p_i^{v_i+1}}{1-p_i} \prod_{j=1}^{s} \frac{1-(-q_j)^{w_j+1}}{1+q_j}. \tag{9.4.35}$$

It was shown in [52, Theorem 7.1] that

$$[q^n](\varphi^3(q)\varphi(q^5)) = b(n) := (-1)^{n-1}(1+5^{d+1}(-1)^{g+t})\frac{(5+(-2)^{g+1})}{3}u(n), \tag{9.4.36}$$

$$[q^n](4q\psi^3(q)\psi(q^5)) = c(n) := (-2)^g(-1+5^{d+1}(-1)^{g+t})u(n). \tag{9.4.37}$$

To demonstrate the equivalence of (9.4.29) and (9.4.24), we see, by (9.4.36), (9.4.37), (9.4.33), and (9.4.34), that it suffices to prove the identity

$$b(2n) + 2c(n) = \alpha(n) + 25\beta(n). \tag{9.4.38}$$

Observe that by (9.4.36), (9.4.37), and (9.4.35),

$$\begin{aligned}
&b(2n) + 2c(n) \\
&= (-1)^{2n-1}(1+5^{d+1}(-1)^{g+t+1})\frac{(5+(-2)^{g+2})}{3}u(n) \\
&\quad + 2(-2)^g(-1+5^{d+1}(-1)^{g+t})u(n) \\
&= -(1+5^{d+1}(-1)^{g+t+1})\frac{(5+(-2)^{g+2})}{3}u(n) \\
&\quad + (-2)^{g+1}(1+5^{d+1}(-1)^{g+t+1})u(n) \\
&= (1+5^{d+1}(-1)^{g+t+1})u(n)\left((-2)^{g+1} - \frac{(5+(-2)^{g+2})}{3}\right) \\
&= (1+5^{d+1}(-1)^{g+t+1})\left(\frac{5(-2)^{g+1}-5}{3}\right)u(n)
\end{aligned}$$

$$= (5^{d+2}(-1)^{t+g} - 5)\left(\frac{1-(-2)^{g+1}}{3}\right)u(n)$$
$$= \alpha(n) + 25\beta(n),$$

by (9.4.33), and (9.4.34). Hence, (9.4.38) has been demonstrated, and so the proof of Entry 9.4.4 is complete. □

Entry 9.4.5 (p. 54). *We have*

$$F_7(q) = \frac{f^7(-q)}{f(-q^7)} + 7qf^3(-q)f^3(-q^7) + 7q^2\frac{f^7(-q^7)}{f(-q)}. \tag{9.4.39}$$

To prove Ramanujan's formula for $F_7(q)$ given in Entry 9.4.5, we use Son's argument [320], which employs modular equations of degree 7. Thus, we first review some basic facts about modular equations; for the definition of a modular equation of degree n, see the second author's book [55, p. 4]. Set, for each positive integer n,

$$z_n = \varphi^2(q^n), \tag{9.4.40}$$

where $\varphi(q)$ is defined by (5.11.1). Recall that the multiplier m is defined by

$$m = \frac{z_1}{z_n} = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{9.4.41}$$

We require three modular equations of degree 7 from Chapter 19 of Ramanujan's second notebook [282], [55, p. 314, Entries 19(i), (ii)]. If β has degree 7 over α, then

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1, \tag{9.4.42}$$

$$m = \frac{1 - 4\left(\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}}{\{(1-\alpha)(1-\beta)\}^{1/8} - (\alpha\beta)^{1/8}}, \tag{9.4.43}$$

and

$$\frac{7}{m} = \frac{1 - 4\left(\dfrac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}}{(\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}}. \tag{9.4.44}$$

Set

$$t := (\alpha\beta)^{1/8}, \tag{9.4.45}$$

where β has degree 7 over α. Following Son [320], we introduce the notation

$$p_1 := 4\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}, \tag{9.4.46}$$

$$p_2 := 4\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}, \tag{9.4.47}$$

$$p_3 := p_1 - 1. \tag{9.4.48}$$

Lastly, we need the following results from the lost notebook, first proved by Son [319], [15, p. 180], [16, pp. 180, 194].

Theorem 9.4.1. *Let p_1 be given by* (9.4.46), *and let*

$$u := 2q^{1/7}\frac{f(q^5,q^9)}{\varphi(q^7)}, \qquad v := 2q^{4/7}\frac{f(q^3,q^{11})}{\varphi(q^7)}, \qquad w := 2q^{9/7}\frac{f(q,q^{13})}{\varphi(q^7)}.$$

Then

$$\frac{\varphi^8(q)}{\varphi^8(q^7)} - (2+5p_1)\frac{\varphi^4(q)}{\varphi^4(q^7)} + (1-p_1)^3 = 0 \tag{9.4.49}$$

and

$$u^7+v^7+w^7 = \frac{\varphi^8(q)}{\varphi^8(q^7)} - 7(p_1-2)\frac{\varphi^4(q)}{\varphi^4(q^7)} + 7p_1^2 - 49p_1 - 15. \tag{9.4.50}$$

After this preliminary preparation, we are now ready to present Son's proof of Entry 9.4.5 [320]. We first reformulate Ramanujan's Entry 9.1.1 for $n=7$.

Theorem 9.4.2. *Let*

$$\begin{aligned}
&P := f(a^4b^3,a^3b^4), && Q := a^{1/7}f(a^5b^4,a^2b^3),\\
&R := b^{1/7}f(a^4b^5,a^3b^2), && S := a^{3/7}b^{1/7}f(a^6b^5,ab^2),\\
&T := a^{1/7}b^{3/7}f(a^5b^6,a^2b), && U := a^{6/7}b^{3/7}f(a^7b^6,b),\\
&V := a^{3/7}b^{6/7}f(a,a^6b^7).
\end{aligned}$$

Then, for $|ab|<1$,

$$\begin{aligned}
&P^7+Q^7+R^7+S^7+T^7+U^7+V^7\\
&\quad = f(a,b)\left\{\frac{f^7(-ab)}{f(-a^7b^7)} + 7abf^3(-ab)f^3(-a^7b^7) + 7a^2b^2\frac{f^7(-a^7b^7)}{f(-ab)}\right\}.
\end{aligned}$$

To prove Theorem 9.4.2, we see from Entry 9.1.1 that if we set $a=b=q$, then it suffices to prove the following theorem.

Theorem 9.4.3. *For $|q|<1$,*

$$\begin{aligned}
&\varphi^7(q^7) + 2qf^7(q^5,q^9) + 2q^4f^7(q^3,q^{11}) + 2q^9f^7(q,q^{13})\\
&\qquad = \varphi(q)\left\{\frac{f^7(-q^2)}{f(-q^{14})} + 7q^2f^3(-q^2)f^3(-q^{14}) + 7q^4\frac{f^7(-q^{14})}{f(-q^2)}\right\}.
\end{aligned}$$

Lemma 9.4.1. *For p_3 defined by* (9.4.48),

$$m^4 - 7(p_3-1)m^2 - 49p_3 = (m^2-7p_3)^2 + 7\left(m - \frac{p_3^2}{m}\right)^2. \tag{9.4.51}$$

Proof. Let the left side of (9.4.51) be given by

$$L_1 := m^4 - 14p_3m^2 + 7p_3m^2 + 7m^2 - 49p_3.$$

By (9.4.49) and (9.4.46),

$$m^2 = 7 + 5p_3 + \frac{p_3^3}{m^2} = 7p_3 + \frac{p_3^3}{m^2} - 2p_3 + 7.$$

Thus,

$$\begin{aligned} L_1 &= m^4 - 14p_3m^2 + 7p_3\left(7p_3 + \frac{p_3^3}{m^2} - 2p_3 + 7\right) + 7m^2 - 49p_3 \\ &= \left(m^4 - 14p_3m^2 + (7p_3)^2\right) + 7\left(\frac{p_3^4}{m^2} - 2p_3^2 + m^2\right) \\ &= (m^2 - 7p_3)^2 + 7\left(\frac{p_3^2}{m} - m\right)^2, \end{aligned}$$

which completes the proof. □

In the next sequence of lemmas, we require the following representations from Ramanujan's notebooks [282], [55, p. 124, Entry 12(iii)].

Proposition 9.4.1. *We have*

$$f(-q^2) = \sqrt{z_1}2^{-1/3}(\alpha(1-\alpha)/q)^{1/12}, \tag{9.4.52}$$

$$f(-q^{14}) = \sqrt{z_7}2^{-1/3}(\beta(1-\beta)/q^7)^{1/12}. \tag{9.4.53}$$

Lemma 9.4.2. *If p_3 is defined by* (9.4.48), *then*

$$(m^2 - 7p_3)^2 = 64\frac{z_1^{1/2}}{z_7^{7/2}}\frac{f^7(-q^2)}{f(-q^{14})}.$$

Proof. Recall that t and p_1 are defined by (9.4.45) and (9.4.46), respectively. Then, by (9.4.43), (9.4.42), and (9.4.48),

$$1 - 2t = \frac{1-p_1}{m} = -\frac{p_3}{m}, \tag{9.4.54}$$

and by (9.4.47), (9.4.44), (9.4.42), and (9.4.54),

$$p_2 = 1 + \frac{7}{m}(1-2t) = 1 - \frac{7}{m}\frac{p_3}{m} = 1 - \frac{7p_3}{m^2}. \tag{9.4.55}$$

Thus,

$$(m^2 - 7p_3)^2 = m^4\left(1 - \frac{7p_3}{m^2}\right)^2 = m^4p_2^2. \tag{9.4.56}$$

On the other hand, by (9.4.52), (9.4.53), and (9.4.47),

$$
\begin{aligned}
64\frac{z_1^{1/2}}{z_7^{7/2}}\frac{f^7(-q^2)}{f(-q^{14})} &= 64\frac{z_1^{1/2}}{z_7^{7/2}}\frac{z_1^{7/2}2^{-7/3}\big(\alpha(1-\alpha)/q\big)^{7/12}}{z_7^{1/2}2^{-1/3}\big(\beta(1-\beta)/q^7\big)^{1/12}} \\
&= \frac{z_1^4}{z_7^4}\cdot 4^2\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/12} \\
&= m^4 p_2^2. \qquad (9.4.57)
\end{aligned}
$$

Combining (9.4.56) and (9.4.57), we complete the proof. □

Lemma 9.4.3. *We have*

$$
\left(m-\frac{p_3^2}{m}\right)^2 = 64\frac{z_1^{1/2}}{z_7^{7/2}}q^2 f^3(-q^2)f^3(-q^{14}).
$$

Proof. For $t=(\alpha\beta)^{1/8}$, we rewrite (9.4.42) in the form

$$
t(1-t) = \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \qquad (9.4.58)
$$

By (9.4.54),

$$
2t = 1+\frac{p_3}{m} \qquad \text{and} \qquad 2(1-t) = 1-\frac{p_3}{m}.
$$

Thus,

$$
\begin{aligned}
\left(m-\frac{p_3^2}{m}\right)^2 &= m^2\left(1-\frac{p_3^2}{m^2}\right)^2 \\
&= m^2\left(1+\frac{p_3}{m}\right)^2\left(1-\frac{p_3}{m}\right)^2 \\
&= m^2(2t)^2\big(2(1-t)\big)^2 \\
&= 16m^2\big(t(1-t)\big)^2. \qquad (9.4.59)
\end{aligned}
$$

On the other hand, by Proposition 9.4.1,

$$
\begin{aligned}
&64\frac{z_1^{1/2}}{z_7^{7/2}}q^2 f^3(-q^2)f^3(-q^{14}) \\
&\quad = 64\frac{z_1^{1/2}}{z_7^{7/2}}q^2 z_1^{3/2}2^{-1}\big(\alpha(1-\alpha)/q\big)^{1/4}z_7^{3/2}2^{-1}\big(\beta(1-\beta)/q^7\big)^{1/4} \\
&\quad = 16\frac{z_1^2}{z_7^2}\big(\alpha\beta(1-\alpha)(1-\beta)\big)^{1/4} \\
&\quad = 16m^2\big(t(1-t)\big)^2, \qquad (9.4.60)
\end{aligned}
$$

by (9.4.58). Therefore, taking (9.4.59) and (9.4.60) together, we complete the proof of Lemma 9.4.3. □

Lemma 9.4.4. *We have*

$$p_1^2 = 64\frac{z_1^{1/2}}{z_7^{7/2}}q^4\frac{f^7(-q^{14})}{f(-q^2)}. \tag{9.4.61}$$

Proof. By (9.4.52), (9.4.53), and (9.4.46),

$$\begin{aligned}64\frac{z_1^{1/2}}{z_7^{7/2}}q^4\frac{f^7(-q^{14})}{f(-q^2)} &= 64\frac{z_1^{1/2}}{z_7^{7/2}}q^4\frac{z_7^{7/2}2^{-7/3}\big(\beta(1-\beta)/q^7\big)^{7/12}}{z_1^{1/2}2^{-1/3}\big(\alpha(1-\alpha)/q\big)^{1/12}}\\ &= 4^2\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/12}\\ &= p_1^2,\end{aligned}$$

which completes the proof. □

Proof of Theorem 9.4.3. Recall that u, v, and w are defined in Theorem 9.4.1. Then, by (9.4.50),

$$\begin{aligned}L_2 &:= \varphi^7(q^7) + 2qf^7(q^5,q^9) + 2q^4f^7(q^3,q^{11}) + 2q^9f^7(q,q^{13})\\ &= \varphi^7(q^7)\left(1 + 2\left(\frac{u}{2}\right)^7 + 2\left(\frac{v}{2}\right)^7 + 2\left(\frac{w}{2}\right)^7\right)\\ &= \frac{\varphi^7(q^7)}{2^6}(64 + u^7 + v^7 + w^7)\\ &= \frac{\varphi^7(q^7)}{64}\big(m^4 - 7(p_1-2)m^2 + 7p_1^2 - 49(p_1-1)\big).\end{aligned} \tag{9.4.62}$$

Using the definitions (9.4.48), (9.4.40), and (9.4.41), we deduce that

$$\begin{aligned}L_2 &= \frac{z_7^{7/2}}{64}\big(m^4 - 7(p_3-1)m^2 + 7p_1^2 - 49p_3\big)\\ &= \frac{z_7^{7/2}}{64}\left((m^2-7p_3)^2 + 7\left(m - \frac{p_3^2}{m}\right)^2 + 7p_1^2\right),\end{aligned}$$

by Lemma 9.4.1. By Lemmas 9.4.2–9.4.4 and (9.4.40), we find that

$$L_2 = \sqrt{z_1}\left\{\frac{f^7(-q^2)}{f(-q^{14})} + 7q^2f^3(-q^2)f^3(-q^{14}) + 7q^4\frac{f^7(-q^{14})}{f(-q^2)}\right\}. \tag{9.4.63}$$

If we combine (9.4.62) with (9.4.63), we complete the proof of Theorem 9.4.3. □

The foregoing identities for $F_n(q)$, $n = 2, 3, 4, 5, 7$, were established by Rangachari [286] and Son [320]. Several authors have determined the identification of $F_n(q)$ in further special cases. S. Ahlgren [4] considered the cases

$n = 6, 8, 9$, and 10. K. Ono [257] established $F_{11}(q)$, while Chua [115] derived the corresponding result for $F_{13}(q)$. A summary of all known identifications of $F_n(q)$ can be found in Son's paper [322].

One may ask whether other results like Theorem 9.1.1 are known. Indeed, H.H. Chan, Liu, and Ng [101], S.H. Chan and Liu [108], X.-F. Zeng [350], T. Dai and X. Ma [119], and J.-M. Zhu [352] have established further theorems of this sort.

10

Highly Composite Numbers

In 1915, the London Mathematical Society published in its *Proceedings* a paper by Ramanujan entitled *Highly Composite Numbers* [274]. A number N is said to be highly composite if for every integer $M < N$, it happens that $d(M) < d(N)$, where $d(n)$ is the number of divisors of n. In the notes of Ramanujan's *Collected Papers* [281, p. 339], the editors relate, "The paper, long as it is, is not complete. The London Mathematical Society was in some financial difficulty at the time and Ramanujan suppressed part of what he had written in order to save expenses." This suppressed part had been known to G.H. Hardy, who mentioned it in a letter to G.N. Watson in 1930 [283, p. 391], [68, p. 286]. Most of the unpublished portion was published in 1988 with Ramanujan's lost notebook [283, pp. 280–312]. In his analysis of [283], R.A. Rankin devoted a paragraph to the description of this unpublished manuscript [292, p. 361], [69, p. 138]. The manuscript was discussed by J.-L. Nicolas in [245, pp. 238–239] and [246]. Shortly thereafter, G. Robin [301] gave detailed proofs of some of the results therein pertaining to complex analysis and the Riemann zeta function, since Ramanujan gave almost no details. Ramanujan's unpublished manuscript was first set into print by Nicolas and Robin [284] in the first volume of the *Ramanujan Journal* and was accompanied by illuminating comments by them. This chapter contains Ramanujan's original unpublished manuscript with a few minor corrections incorporated, some missing passages added by Nicolas and Robin, a mild revision of Nicolas and Robin's original commentary, and some additional comments by the present authors.

In the unpublished portion of his paper, Ramanujan extends the notion of highly composite numbers to other arithmetic functions, mainly to $Q_{2k}(N)$, $1 \leq k \leq 4$, where $Q_{2k}(N)$ denotes the number of representations of N as the sum of $2k$ squares, and to $\sigma_{-s}(N)$, where $\sigma_{-s}(N)$ denotes the sum of the $(-s)$th powers of the divisors of N. Moreover, the maximal orders of these functions are given.

In reproducing Ramanujan's unpublished manuscript on highly composite numbers, since it was in its final form intended for publication, we have ad-

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*, 359
DOI 10.1007/978-1-4614-3810-6_10,

hered to Ramanujan's manuscript as closely as possible. In particular, we have written sums and products as Ramanujan wrote them, instead of employing more compact notations. For convenience, we have kept the numbering both of paragraphs (which start with number 52 and end with number 75) and formulas (ranging from (10.52.268) to (10.75.408)), so that references to preceding paragraphs or formulas can easily be located in [274]. There is just a small overlap; the last paragraph of [274] is numbered 52 and contains formulas (10.52.268) and (10.52.269). This last paragraph was probably added by Ramanujan to the first part after he had decided to suppress the second part. However, this overlap of two paragraphs numbered 52 should not cause any confusion.

There are two gaps in the manuscript of Ramanujan in [283]. The first one is at the beginning, where the definition of $Q_2(n)$ is missing. Perhaps this definition was sent to the London Mathematical Society in 1915 with the original manuscript, but later deleted because it was irrelevant to the published part. The function $\overline{Q}_2(n)$ is analogously defined in Section 10.55. The second gap is more difficult to explain: Section 10.57 is complete and appears on pages 289 and 290 of [283]. But the lower half of page 290 is empty, and page 291 starts with the end of Section 10.58. Nicolas and Robin completed Section 10.58 by giving the definition of $\sigma_s(N)$ and a proof of formula (10.58.301). All these completions are placed within square brackets in the text below so that readers will be clear about what portions are due to Ramanujan and what portions are not. It should be noted that in [283], pages 295–299 are not in Ramanujan's handwriting, and, as observed by Rankin [292, p. 361], were probably copied by Watson, but this does not create any gap in the text. Pages 282 and 283 of [283] do not belong to the manuscript on highly composite numbers, and one may wonder why these two pages, each containing one or two fragmentary sentences, which are apparently disconnected with anything else in the lost notebook, were chosen for publication. Clearly the text of page 284 follows from that on page 281. On the other hand, pages 309–312 also do not belong to the manuscript on highly composite numbers, but, up to the last formula on page 312, *belong to the subject of highly composite numbers.* With the notation of [274, Section 9], Ramanujan proves on pages 309–310 that

$$\frac{\log\ p_r}{\log(1+1/r)} = \frac{\log\ p_1}{\log\ 2} + O(r),$$

while on pages 311–312 he attempts to extend this formula by replacing p_1 by p_s. More precise results can now be found in [241]. Since pages 309–312 do not belong to the paper *Highly Composite Numbers*, they are not included in the paper below. For completeness, we reproduce these pages without further comment at the conclusion of our comments following the manuscript on highly composite numbers.

In the paper below, Ramanujan studies the maximal order of some classical multiplicative functions, which resemble the number, or the sum, of the divisors of an integer.

In Sections 10.52–10.54, $Q_2(N)$, the number of representations of N as a sum of two squares, is studied, and its maximal order is given under the Riemann Hypothesis, and also without assuming the Riemann Hypothesis. In Sections 10.55–10.56, a similar analysis is provided for $\overline{Q}_2(N)$, the number of representations of N by the form m^2+mn+n^2. In Section 10.57, the number of ways of writing N as a product of $(1+r)$ factors is briefly investigated. Between Section 10.58 and Section 10.71, Ramanujan introduces generalized superior highly composite numbers in providing a deep study of the maximal order of $\sigma_{-s}(N)$ under the Riemann Hypothesis. In Sections 10.72–10.74, $Q_4(N)$, $Q_6(N)$, and $Q_8(N)$, the numbers of representations of N as a sum of 4, 6, and 8 squares, respectively, are studied, as well as their maximal orders. In the concluding Section 10.75, the number of representations of N by some other quadratic forms is considered, but their maximal orders are not studied.

The table of largely composite numbers at the end of the article appears on page 280 in [283]. A number N is largely composite if for every integer $M \le N$, we have $d(M) \le d(N)$.

Several results obtained by Ramanujan in 1915, but remaining in this unpublished manuscript, have been rediscovered and published by other mathematicians. The references for these works are given in the notes at the end of our reproduction of this paper. However, there remain in the paper of Ramanujan some results that have never been published, for instance, the maximal order of $\overline{Q}_2(N)$ (see Section 10.54) and of $\sigma_{-s}(N)$ (cf. Section 10.71) whenever $s \neq 1$. (The case $s=1$ has been studied by Robin [298].)

A few misprints or mistakes were found in Ramanujan's manuscript. These mistakes have been corrected in the text and are mentioned in the notes.

Hardy did not think highly of highly composite numbers. In the preface to Ramanujan's *Collected Papers* [281, p. XXXIV], he writes, "The long memoir [274] represents work, perhaps, in a backwater of mathematics," but a few lines later, he does recognize that "it shews very clearly Ramanujan's extraordinary mastery over the algebra of inequalities." At the conference marking the centenary of Ramanujan's birth held at the University of Illinois on June 1–5, 1987, Freeman Dyson remarked that when he was a research student of Hardy, he wanted to do research on highly composite numbers, but Hardy dissuaded him, because he thought the subject was not sufficiently interesting or important. However, after Ramanujan, several authors have written about them, as can be seen in the survey paper [245] by Nicolas.

THE REMAINING TEXT OF RAMANUJAN'S PAPER

HIGHLY COMPOSITE NUMBERS

10.52

[Let $Q_2(N)$ denote the number of ways in which N can be expressed as m^2+n^2. Let us agree to consider m^2+n^2 as being represented in two ways if m and

n are unequal and in one way if they are equal or one of] them is zero. Then it can be shown that

$$\begin{aligned}&\left(1+2q+2q^4+2q^9+2q^{16}+\cdots\right)^2\\&=1+4\left(\frac{q}{1-q}-\frac{q^3}{1-q^3}+\frac{q^5}{1-q^{-5}}-\frac{q^7}{1-q^7}+\cdots\right)\\&=1+4\{Q_2(1)q+Q_2(2)q^2+Q_2(3)q^3+\cdots\}.\end{aligned}\tag{10.52.268}$$

From this it easily follows that

$$\zeta(s)\zeta_1(s)=\frac{Q_2(1)}{1^s}+\frac{Q_2(2)}{2^s}+\frac{Q_2(3)}{3^s}+\cdots,\tag{10.52.269}$$

where

$$\zeta_1(s)=1^{-s}-3^{-s}+5^{-s}-7^{-s}+\cdots.$$

Since

$$\frac{q}{1-q}+\frac{q^2}{1-q^2}+\frac{q^3}{1-q^3}+\cdots=d(1)q+d(2)q^2+d(3)q^3+\cdots$$

it follows from (10.52.268) that

$$Q_2(N)\le d(N)\tag{10.52.270}$$

for all values of N. Let

$$N=2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p},$$

where $a_\lambda\ge 0$. Then we see that, if any one of $a_3,a_7,a_{11},\ldots$ be odd, where $3,7,11,\ldots$ are the primes of the form $4n-1$, then

$$Q_2(N)=0.\tag{10.52.271}$$

But, if $a_3,a_7,a_{11},\ldots$ be even or zero, then

$$Q_2(N)=(1+a_5)(1+a_{13})(1+a_{17})\cdots\tag{10.52.272}$$

where $5,13,17,\ldots$ are the primes of the form $4n+1$. It is clear that (10.52.270) is a consequence of (10.52.271) and (10.52.272).

10.53

From (10.52.272) it is easy to see that, in order that $Q_2(N)$ should be of maximum order, N must be of the form

$$5^{a_5}\cdot 13^{a_{13}}\cdot 17^{a_{17}}\cdots p^{a_p},$$

where p is a prime of the form $4n+1$, and

$$a_5 \geq a_{13} \geq a_{17} \geq \cdots \geq a_p.$$

Let $\pi_1(x)$ denote the number of primes of the form $4n+1$ which do not exceed x, and let

$$\vartheta_1(x) = \log 5 + \log 13 + \log 17 + \cdots + \log p,$$

where p is the largest prime of the form $4n+1$, not greater than x. Then by arguments similar to those of Section 33 we can show that

$$Q_2(N) \leq N^{1/x} \frac{2^{\pi_1(2^x)}}{e^{(1/x)\theta_1(2^x)}} \frac{\left(\frac{3}{2}\right)^{\pi_1((3/2)^x)}}{e^{(1/x)\theta_1((3/2)^x)}} \frac{\left(\frac{4}{3}\right)^{\pi_1((4/3)^x)}}{e^{(1/x)\theta_1((4/3)^x)}} \cdots \tag{10.53.273}$$

for all values of N and x. From this we can show by arguments similar to those of Section 38 that, in order that $Q_2(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta_1(2^x)+\vartheta_1((3/2)^x)+\vartheta_1((4/3)^x)+\cdots}$$

and $Q_2(N)$ of the form

$$2^{\pi_1(2^x)} \left(\tfrac{3}{2}\right)^{\pi_1((3/2)^x)} \left(\tfrac{4}{3}\right)^{\pi_1((4/3)^x)} \cdots.$$

Then, without assuming the prime number theorem, we can show that the maximum order of $Q_2(N)$ is

$$2^{\log N\left\{1/(\log\log N)+O(1)/(\log\log N)^2\right\}}. \tag{10.53.274}$$

Assuming the prime number theorem we can show that the maximum order of $Q_2(N)$ is

$$2^{(1/2)\mathrm{Li}(2\log N)+O\left\{\log N e^{-a\sqrt{\log N}}\right\}} \tag{10.53.275}$$

where a is a positive constant.

10.54

We shall now assume the Riemann Hypothesis and its analogue for the function $\zeta_1(s)$. Let ρ_1 be a complex root of $\zeta_1(s)$. Then it can be shown that

$$\sum \frac{1}{\rho_1} = \frac{\gamma - 3\log\pi}{2} + \log 2 + 4\log\Gamma\left(\tfrac{3}{4}\right);$$

so that

$$\sum \frac{1}{\rho} + \sum \frac{1}{\rho_1} = 1 + \gamma - 2\log\pi + 4\log\Gamma\left(\tfrac{3}{4}\right). \tag{10.54.276}$$

It can also be shown that

$$\begin{cases} 2\vartheta_1(x) = x - 2\sqrt{x} - \sum x^{\rho}/\rho - \sum x^{\rho_1}/\rho_1 + O\left(x^{1/3}\right) \\ 2\pi_1(x) = \mathrm{Li}(x) - \mathrm{Li}(\sqrt{x}) - \sum \mathrm{Li}(x^{\rho}) - \sum \mathrm{Li}(x^{\rho_1}) + O\left(x^{1/3}\right) \end{cases} \tag{10.54.277}$$

so that

$$\begin{cases} 2\vartheta_1(x) = x + O\left(\sqrt{x}(\log x)^2\right) \\ 2\pi_1(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\log x\right). \end{cases} \tag{10.54.278}$$

Now

$$\begin{aligned} 2\pi_1(x) = \mathrm{Li}(x) &- \frac{1}{\log x}\left(2\sqrt{x} + \sum \frac{x^{\rho}}{\rho} + \sum \frac{x^{\rho_1}}{\rho_1}\right) \\ &- \frac{1}{(\log x)^2}\left(4\sqrt{x} + \sum \frac{x^{\rho}}{\rho^2} + \sum \frac{x^{\rho_1}}{\rho_1^2}\right) + \frac{O(\sqrt{x})}{(\log x)^3}. \end{aligned}$$

But by Taylor's Theorem we have

$$\mathrm{Li}\{2\vartheta_1(x)\} = \mathrm{Li}(x) - \frac{1}{\log x}\left(2\sqrt{x} + \sum \frac{x^{\rho}}{\rho} + \sum \frac{x^{\rho_1}}{\rho_1}\right) + O\left((\log x)^2\right).$$

Hence

$$2\pi_1(x) = \mathrm{Li}\{2\vartheta_1(x)\} - 2R_1(x) + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\} \tag{10.54.279}$$

where

$$R_1(x) = \frac{1}{(\log x)^2}\left(2\sqrt{x} + \frac{1}{2}\sum \frac{x^{\rho}}{\rho^2} + \frac{1}{2}\sum \frac{x^{\rho_1}}{\rho_1^2}\right).$$

It can easily be shown that

$$\sqrt{x}\left(2 + \sum \frac{1}{\rho} + \sum \frac{1}{\rho_1}\right) \geq R_1(x)(\log x)^2 \geq \sqrt{x}\left(2 - \sum \frac{1}{\rho} - \sum \frac{1}{\rho_1}\right)$$

and so from (10.54.276) we see that

$$\begin{aligned} \{3 + \gamma - 2\log \pi + 4\log \Gamma(\tfrac{3}{4})\}\sqrt{x} &\geq R_1(x)(\log x)^2 \\ &\geq \{1 - \gamma + 2\log \pi - 4\log \Gamma(\tfrac{3}{4})\}\sqrt{x}. \end{aligned} \tag{10.54.280}$$

It can easily be verified that

$$\begin{cases} 3 + \gamma - 2\log \pi + 4\log \Gamma(\tfrac{3}{4}) = 2.101, \\ 1 - \gamma + 2\log \pi - 4\log \Gamma(\tfrac{3}{4}) = 1.899, \end{cases} \tag{10.54.281}$$

approximately. Proceeding as in Section 43 we can show that the maximum order of $Q_2(N)$ is

$$2^{(1/2)\mathrm{Li}(2\log N) + \Phi(N)} \tag{10.54.282}$$

where

$$\Phi(N) = \frac{\log \frac{3}{2}}{2\log 2}\mathrm{Li}\left\{\tfrac{3}{2}(\log N)^{\log(3/2)/\log 2}\right\} - \frac{3(\log N)^{\log(3/2)/\log 2}}{4\log(2\log N)}$$
$$-R_1(2\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

10.55

Let $\overline{Q}_2(N)$ denote the number of ways in which N can be expressed as $m^2 + mn + n^2$. Let us agree to consider $m^2 + mn + n^2$ as two ways if m and n are unequal, and as one way if they are equal or one of them is zero. Then it can be shown that

$$\begin{aligned}
&\tfrac{1}{2}\left(1 + 2q^{1/4} + 2q^{4/4} + 2q^{9/4} + \cdots\right)\left(1 + 2q^{3/4} + 2q^{13/4} + 2q^{27/4} + \cdots\right)\\
&+ \tfrac{1}{2}\left(1 - 2q^{1/4} + 2q^{4/4} - 2q^{9/4} + \cdots\right)\left(1 - 2q^{3/4} + 2q^{13/4} - 2q^{27/4} + \cdots\right)\\
&= 1 + 6\left(\frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \cdots\right)\\
&= 1 + 6\left\{\overline{Q}_2(1)q + \overline{Q}_2(2)q^2 + \overline{Q}_2(3)q^3 + \cdots\right\}
\end{aligned} \tag{10.55.283}$$

where $1, 2, 4, 5, \ldots$ are the natural numbers without the multiples of 3. From this it follows that

$$\zeta(s)\zeta_2(s) = 1^{-s}\overline{Q}_2(1) + 2^{-s}\overline{Q}_2(2) + 3^{-s}\overline{Q}_2(3) + \cdots \tag{10.55.284}$$

where

$$\zeta_2(s) = 1^{-s} - 2^{-s} + 4^{-s} - 5^{-s} + \cdots .$$

It also follows that

$$\overline{Q}_2(N) \le d(N) \tag{10.55.285}$$

for all values of N. Let

$$N = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p},$$

where $a_\lambda \ge 0$. Then, if any one of $a_2, a_5, a_{11}, \ldots$ be odd, where $2, 5, 11, \ldots$ are the primes of the form $3n - 1$, then

$$\overline{Q}_2(N) = 0. \tag{10.55.286}$$

But, if a_2, a_5, a_{11} be even or zero, then

$$\overline{Q}_2(N) = (1 + a_7)(1 + a_{13})(1 + a_{19})(1 + a_{31}) \cdots \tag{10.55.287}$$

where $7, 13, 19, \ldots$ are the primes of the form $6n+1$. Let $\pi_2(x)$ be the number of primes of the form $6n + 1$ which do not exceed x, and let

$$\vartheta_2(x) = \log 7 + \log 13 + \log 19 + \cdots + \log p,$$

where p is the largest prime of the form $6n+1$ not greater than x. Then we can show that, in order that $\overline{Q}_2(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta_2(2^x)+\vartheta_2((3/2)^x)+\vartheta_2((4/3)^x)+\cdots}$$

and $\overline{Q}_2(N)$ of the form

$$2^{\pi_2(3^x)}\left(\tfrac{3}{2}\right)^{\pi_2((3/2)^x)}\left(\tfrac{4}{3}\right)^{\pi_2((4/3)^x)}\cdots.$$

Without assuming the prime number theorem we can show that the maximum order of $\overline{Q}_2(N)$ is

$$2^{\log N\left\{1/(\log\log N)+O(1)/(\log\log N)^2\right\}}. \tag{10.55.288}$$

Assuming the prime number theorem we can show that the maximum order of $\overline{Q}_2(N)$ is

$$2^{(1/2)\mathrm{Li}(2\log N)+O\left\{\log N e^{-a\sqrt{\log N}}\right\}}. \tag{10.55.289}$$

10.56

We shall now assume the Riemann hypothesis and its analogue for the function $\zeta_2(s)$. Then we can show that

$$2\pi_2(x) = \mathrm{Li}\left\{2\vartheta_2(x)\right\} - 2R_2(x) + O\{\sqrt{x}/(\log x)^3\} \tag{10.56.290}$$

where

$$R_2(x) = \frac{1}{(\log x)^2}\left\{2\sqrt{x} + \frac{1}{2}\sum\frac{x^\rho}{\rho^2} + \frac{1}{2}\sum\frac{x^{\rho_2}}{\rho_2^2}\right\}$$

where ρ_2 is a complex root of $\zeta_2(s)$. It can also be shown that

$$\sum\frac{1}{\rho} + \sum\frac{1}{\rho_2} = 1 + \gamma + \tfrac{1}{2}\log 3 + 3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \tag{10.56.291}$$

and so

$$\begin{aligned}&\left\{3 + \gamma + \tfrac{1}{2}\log 3 + 3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right\}\sqrt{x}\\ &\geq R_2(x)(\log x)^2\\ &\geq \left\{1 - \gamma - \tfrac{1}{2}\log 3 - 3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right\}\sqrt{x}.\end{aligned} \tag{10.56.292}$$

It can easily be verified that

$$\begin{cases} 3+\gamma+\frac{1}{2}\log 3+3\log\dfrac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}=2.080, \\ 1-\gamma-\frac{1}{2}\log 3-3\log\dfrac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}=1.920, \end{cases} \tag{10.56.293}$$

approximately. Then we can show that the maximum order of $\overline{Q}_2(N)$ is

$$2^{(1/2)\mathrm{Li}(2\log N)+\Phi(N)} \tag{10.56.294}$$

where

$$\begin{aligned} \Phi(N) = {} & \frac{\log\frac{3}{2}}{2\log 2}\mathrm{Li}\left\{\tfrac{3}{2}(\log N)^{\log(3/2)/\log 2}\right\} - \frac{3(\log N)^{\log(3/2)/\log 2}}{4\log(2\log N)} \\ & -R_2(2\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}. \end{aligned}$$

10.57

Let $d_r(N)$ denote the coefficient of N^{-s} in the expansion of $\{\zeta(s)\}^{1+r}$ as a Dirichlet series. Then since

$$\{\zeta(s)\}^{-1} = \left(1-2^{-s}\right)\left(1-3^{-s}\right)\left(1-5^{-s}\right)\cdots\left(1-p^{-s}\right)\cdots,$$

it is easy to see that, if

$$N = p_1^{a_1}\cdot p_2^{a_2}\cdot p_3^{a_3}\cdots p_n^{a_n},$$

where $p_1, p_2, p_3 \ldots$ are any primes, then

$$d_r(N) = \prod_{\nu=1}^{\nu=n}\prod_{\lambda=1}^{\lambda=a_\nu}\left(1+\frac{r}{\lambda}\right) \tag{10.57.295}$$

provided that $r > -1$. It is evident that

$$d_{-1}(N)=0, \quad d_0(N)=1, \quad d_1(N)=d(N);$$

and that, if $-1 \le r \le 0$, then

$$d_r(N) \le 1+r \tag{10.57.296}$$

for all values of N. It is also evident that, if N is a prime then

$$d_r(N) = 1+r$$

for all values of r. It is easy to see from (10.57.295) that, if $r > 0$, then $d_r(N)$ is not bounded when N becomes infinite. Now, if r is positive, it can easily

be shown that, in order that $d_r(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots},$$

and consequently $d_r(N)$ of the form

$$(1+r)^{\pi(x_1)}\left(1+\frac{r}{2}\right)^{\pi(x_2)}\left(1+\frac{r}{3}\right)^{\pi(x_3)}\cdots$$

and proceeding as in Section 46 we can show that N must be of the form

$$e^{\vartheta((1+r)^x)+\vartheta((1+r/2)^x)+\vartheta((1+r/3)^x)+\cdots} \tag{10.57.297}$$

and $d_r(N)$ of the form

$$(1+r)^{\pi((1+r)^x)}\left(1+\frac{r}{2}\right)^{\pi((1+r/2)^x)}\left(1+\frac{r}{3}\right)^{\pi((1+r/3)^x)}\cdots. \tag{10.57.298}$$

From (10.57.297) and (10.57.298) we can easily find the maximum order of $d_r(N)$ as in Section 43. It may be interesting to note that numbers of the form (10.57.297) which may also be written in the form

$$e^{\vartheta\left\{x^{(1/r)\log(1+r)}\right\}+\vartheta\left\{x^{(1/r)\log(1+r/2)}\right\}+\vartheta\left\{x^{(1/r)\log(1+r/3)}\right\}+\cdots}$$

approach the form

$$e^{\vartheta(x)+\vartheta(\sqrt{x})+\vartheta(x^{1/3})+\cdots}$$

as $r\to 0$. That is to say, they approach the form of the least common multiple of the natural numbers as $r\to 0$.

10.58

[Let s be a nonnegative real number, and let $\sigma_{-s}(N)$ denote the sum of the inverses of the sth powers of the divisors of N. If N is defined by

$$N=p_1^{a_1}\cdot p_2^{a_2}\cdot p_3^{a_3}\cdots p_n^{a_n},$$

where $p_1,p_2,p_3,\ldots$ are any primes, then

$$\begin{aligned}\sigma_{-s}(N) = &\left(1+p_1^{-s}+p_1^{-2s}+p_1^{-3s}+\cdots+p_1^{-a_1 s}\right)\\ &\times\left(1+p_2^{-s}+p_2^{-2s}+p_2^{-3s}+\cdots+p_2^{-a_2 s}\right)\\ &\times\cdots\\ &\times\left(1+p_n^{-s}+p_n^{-2s}+p_n^{-3s}+\cdots+p_n^{-a_n s}\right).\end{aligned}$$

For $s=0$, $\sigma_0(N)=d(N)$, the number of divisors of N. For $s>0$,

$$\sigma_{-s}(N) = \left(\frac{1-p_1^{-(a_1+1)s}}{1-p_1^{-s}}\right)\left(\frac{1-p_2^{-(a_2+1)s}}{1-p_2^{-s}}\right)\cdots\left(\frac{1-p_n^{-(a_n+1)s}}{1-p_n^{-s}}\right). \tag{10.58.299}$$

Now, from the concavity of the function $\log(1-e^{-t})$, we see that

$$\begin{aligned}&\frac{1}{n}\left\{\log\left(1-e^{-t_1}\right)+\log\left(1-e^{-t_2}\right)+\cdots+\log\left(1-e^{-t_n}\right)\right\}\\ &\quad\le \log\left\{1-\exp\left(-\frac{t_1+t_2+\cdots+t_n}{n}\right)\right\}.\end{aligned} \tag{10.58.300}$$

Choosing $t_1 = (a_1+1)s\log p_1, t_2 = (a_2+1)s\log p_2, \ldots, t_n = (a_n+1)s\log p_n$ in (10.58.300), we find that (10.58.299) gives]

$$\sigma_{-s}(N) < \frac{\left\{1-(p_1p_2p_3\cdots p_nN)^{-s/n}\right\}^n}{\left(1-p_1^{-s}\right)\left(1-p_2^{-s}\right)\cdots\left(1-p_n^{-s}\right)}. \tag{10.58.301}$$

By arguments similar to those of Section 2 we can show that it is possible to choose the indices $a_1, a_2, a_3, \ldots, a_n$ so that

$$\begin{aligned}&\sigma_{-s}(N)\\ &= \frac{\left\{1-(p_1p_2p_3\cdots p_nN)^{-s/n}\right\}^n}{\left(1-p_1^{-s}\right)\left(1-p_2^{-s}\right)\cdots\left(1-p_n^{-s}\right)}\left\{1-O\left\{N^{-s/n}(\log N)^{-2/(n-1)}\right\}\right\}.\end{aligned} \tag{10.58.302}$$

There are of course results corresponding to (14) and (15) also.

10.59

A number N may be said to be a generalized highly composite number if $\sigma_{-s}(N) > \sigma_{-s}(N')$ for all values of N' less than N. We can easily show that, in order that N should be a generalized highly composite number, N must be of the form

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p} \tag{10.59.303}$$

where

$$a_2 \ge a_3 \ge a_5 \ge \cdots \ge a_p = 1,$$

the exceptional numbers being 36, for the values of s which satisfy the inequality $2^s+4^s+8^s > 3^s+9^s$, and 4 in all cases.

A number N may be said to be a generalized superior highly composite number if there is a positive number ε such that

$$\frac{\sigma_{-s}(N)}{N^{\varepsilon}} \ge \frac{\sigma_{-s}(N')}{(N')^{\varepsilon}} \tag{10.59.304}$$

for all values of N' less than N, and

$$\frac{\sigma_{-s}(N)}{N^{\varepsilon}} > \frac{\sigma_{-s}(N')}{(N')^{\varepsilon}} \tag{10.59.305}$$

for all values of N' greater than N. It is easily seen that all generalized superior highly composite numbers are generalized highly composite numbers. We shall use the expression

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p_1^{a_{p_1}}$$

and the expression

$$\begin{aligned}
&2 \cdot 3 \cdot 5 \cdot 7 \cdots p_1 \\
&\times\ 2 \cdot 3 \cdot 5 \cdots p_2 \\
&\times\ 2 \cdot 3 \cdot 5 \cdots p_3 \\
&\times\ \cdots \\
&\quad\ \cdots \\
&\quad\ \cdots
\end{aligned}$$

as the standard forms of a generalized superior highly composite number.

10.60

Let

$$N' = \frac{N}{\lambda}$$

where $\lambda \leq p_1$. Then from (10.59.304) it follows that

$$1 - \lambda^{-s(1+a_\lambda)} \geq (1 - \lambda^{-sa_\lambda})\lambda^{\varepsilon},$$

or

$$\lambda^{\varepsilon} \leq \frac{1 - \lambda^{-s(1+a_\lambda)}}{1 - \lambda^{-sa_\lambda}}. \tag{10.60.306}$$

Again let $N' = N\lambda$. Then from (10.59.305) we see that

$$1 - \lambda^{-s(1+a_\lambda)} > \left\{1 - \lambda^{-s(2+a_\lambda)}\right\} \lambda^{-\varepsilon}$$

or

$$\lambda^{\varepsilon} > \frac{1 - \lambda^{-s(2+a_\lambda)}}{1 - \lambda^{-s(1+a_\lambda)}}. \tag{10.60.307}$$

Now let us suppose that $\lambda = p_1$ in (10.60.306) and $\lambda = P_1$ in (10.60.307). Then we see that

$$\frac{\log\left(1 + p_1^{-s}\right)}{\log p_1} \geq \varepsilon > \frac{\log\left(1 + P_1^{-s}\right)}{\log P_1}. \tag{10.60.308}$$

From this it follows that, if

$$0 < \varepsilon \leq \frac{\log\left(1 + 2^{-s}\right)}{\log 2},$$

then there is a unique value of p_1 corresponding to each value of ε. It follows from (10.60.306) that

$$a_\lambda \leq \frac{\log\left(\dfrac{\lambda^\varepsilon - \lambda^{-s}}{\lambda^\varepsilon - 1}\right)}{s\log\lambda}, \tag{10.60.309}$$

and from (10.60.307) that

$$1 + a_\lambda > \frac{\log\left(\dfrac{\lambda^\varepsilon - \lambda^{-s}}{\lambda^\varepsilon - 1}\right)}{s\log\lambda}. \tag{10.60.310}$$

From (10.60.309) and (10.60.310) it is clear that

$$a_\lambda = \left[\frac{\log\left(\dfrac{\lambda^\varepsilon - \lambda^{-s}}{\lambda^\varepsilon - 1}\right)}{s\log\lambda}\right]. \tag{10.60.311}$$

Hence N is of the form

$$2^{\left[\log((2^\varepsilon - 2^{-s})/(2^\varepsilon - 1))/(s\log 2)\right]}\ 3^{\left[\log((3^\varepsilon - 3^{-s})/(3^\varepsilon - 1))/(s\log 3)\right]} \cdots p_1 \tag{10.60.312}$$

where p_1 is the prime defined by the inequalities (10.60.308).

10.61

Let us consider the nature of p_r. Putting $\lambda = p_r$ in (10.60.306), and remembering that $a_{p_r} \geq r$, we obtain

$$p_r^\varepsilon \leq \frac{1 - p_r^{-s(1+a_{p_r})}}{1 - p_r^{-sa_{p_r}}} \leq \frac{1 - p_r^{-s(r+1)}}{1 - p_r^{-sr}}. \tag{10.61.313}$$

Again, putting $\lambda = P_r$ in (10.60.307), and remembering that $a_{P_r} \leq r - 1$, we obtain

$$P_r^\varepsilon > \frac{1 - P_r^{-s(2+a_{P_r})}}{1 - P_r^{-s(1+a_{P_r})}} \geq \frac{1 - P_r^{-s(r+1)}}{1 - P_r^{-sr}}. \tag{10.61.314}$$

It follows from (10.61.313) and (10.61.314) that, if x_r be the value of x satisfying the equation

$$x^\varepsilon = \frac{1 - x^{-s(r+1)}}{1 - x^{-sr}} \tag{10.61.315}$$

then p_r is the largest prime not greater than x_r. Hence N is of the form

$$e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots} \tag{10.61.316}$$

where x_r is defined in (10.61.315); and $\sigma_{-s}(N)$ is of the form

$$\Pi_1(x_1)\Pi_2(x_2)\Pi_3(x_3)\cdots\Pi_{a_2}(x_{a_2}) \tag{10.61.317}$$

where

$$\Pi_r(x) = \frac{1-2^{-s(r+1)}}{1-2^{-sr}}\ \frac{1-3^{-s(r+1)}}{1-3^{-sr}}\cdots\frac{1-p^{-s(r+1)}}{1-p^{-sr}}$$

and p is the largest prime not greater than x. It follows from (10.59.304) and (10.59.305) that

$$\sigma_{-s}(N) \leq N^{\varepsilon}\ \frac{\Pi_1(x_1)}{e^{\varepsilon\vartheta(x_1)}}\ \frac{\Pi_2(x_2)}{e^{\varepsilon\vartheta(x_2)}}\ \frac{\Pi_3(x_3)}{e^{\varepsilon\vartheta(x_3)}}\cdots \tag{10.61.318}$$

for all values of N, where $x_1, x_2, x_3, \ldots$ are functions of ε defined by the equation

$$x_r^{\varepsilon} = \frac{1-x_r^{-s(r+1)}}{1-x_r^{-sr}}, \tag{10.61.319}$$

and $\sigma_{-s}(N)$ is equal to the right hand side of (10.61.318) when

$$N = e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots}.$$

10.62

In (16) let us suppose that

$$\Phi(x) = \log\frac{1-x^{-s(r+1)}}{1-x^{-sr}}.$$

Then we see that

$$\begin{aligned}\log\Pi_r(x_r) &= \pi(x_r)\log\frac{1-x_r^{-s(r+1)}}{1-x_r^{-sr}} - \int\pi(x_r)d\left(\log\frac{1-x_r^{-s(r+1)}}{1-x_r^{-sr}}\right)\\ &= \pi(x_r)\log(x_r^{\varepsilon}) - \int\pi(x_r)d(\log x_r^{\varepsilon})\\ &= \varepsilon\pi(x_r)\log x_r - \int\pi(x_r)\log x_r d\varepsilon - \int\frac{\varepsilon\pi(x_r)}{x_r}dx_r\end{aligned}$$

in virtue of (10.61.319). Hence

$$\begin{aligned}&\log\Pi_r(x_r) - \varepsilon\vartheta(x_r)\\ &\quad = \varepsilon\{\pi(x_r)\log x_r - \vartheta(x_r)\} - \int\pi(x_r)\log x_r d\varepsilon - \int\frac{\varepsilon\pi(x_r)}{x_r}dx_r\end{aligned}$$

$$\begin{aligned}
&= \varepsilon \int \frac{\pi(x_r)}{x_r} dx_r - \int \pi(x_r) \log x_r d\varepsilon - \int \frac{\varepsilon\pi(x_r)}{x_r} dx_r \\
&= \int d\varepsilon \int \frac{\pi(x_r)}{x_r} dx_r - \int \pi(x_r) \log x_r d\varepsilon \\
&= \int \left\{ \int \frac{\pi(x_r)}{x_r} dx_r - \pi(x_r) \log x_r \right\} d\varepsilon \\
&= - \int \vartheta(x_r) d\varepsilon. \qquad (10.62.320)
\end{aligned}$$

It follows from (10.61.318) and (10.62.320) that

$$\sigma_{-s}(N) \leq N^{\varepsilon} e^{-\int \{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots\} d\varepsilon} \qquad (10.62.321)$$

for all values of N. By arguments similar to those of Section 38 we can show that the right hand side of (10.62.321) is a minimum when ε is a function of N defined by the equation

$$N = e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots}. \qquad (10.62.322)$$

Now let $\Sigma_{-s}(N)$ be a function of N defined by the equation

$$\Sigma_{-s}(N) = \Pi_1(x_1)\Pi_2(x_2)\Pi_3(x_3)\cdots \qquad (10.62.323)$$

where ϵ is a function of N defined by the equation (10.62.322). Then it follows from (10.61.318) that the order of

$$\sigma_{-s}(N) \leq \Sigma_{-s}(N)$$

for all values of N and $\sigma_{-s}(N) = \Sigma_{-s}(N)$ for all generalized superior highly composite values of N. In other words $\sigma_{-s}(N)$ is of maximum order when N is of the form of a generalized superior highly composite number.

10.63

We shall now consider some important series which are not only useful in finding the maximum order of $\sigma_{-s}(N)$ but also interesting in themselves. Proceeding as in (16) we can easily show that, if $\Phi'(x)$ be continuous, then

$$\begin{aligned}
&\Phi(2)\log 2 + \Phi(3)\log 3 + \Phi(5)\log 5 + \cdots + \Phi(p)\log p \\
&\quad = \Phi(x)\theta(x) - \int_2^x \Phi'(t)\theta(t)dt \qquad (10.63.324)
\end{aligned}$$

where p is the largest prime not exceeding x. Since

$$\int \Phi(x)dx = x\Phi(x) - \int x\Phi'(x)dx,$$

we have

$$\begin{aligned}
&\Phi(x)\vartheta(x) - \int \Phi'(x)\vartheta(x)dx \\
&\quad = \int \Phi(x)dx - \{x-\vartheta(x)\}\Phi(x) + \int \Phi'(x)\{x-\vartheta(x)\}dx. \qquad (10.63.325)
\end{aligned}$$

Remembering that $x-\vartheta(x) = O\left\{\sqrt{x}(\log x)^2\right\}$, we have by Taylor's theorem

$$\begin{aligned}
\int^{\theta(x)} \Phi(t)dt &= \int \Phi(x)dx - \{x-\vartheta(x)\}\Phi(x) \\
&\quad + \tfrac{1}{2}\{x-\vartheta(x)\}^2\Phi'\left\{x + O\left(\sqrt{x}(\log x)^2\right)\right\}. \qquad (10.63.326)
\end{aligned}$$

It follows from (10.63.324)–(10.63.326) that

$$\begin{aligned}
&\Phi(2)\log 2 + \Phi(3)\log 3 + \Phi(5)\log 5 + \cdots + \Phi(p)\log p \\
&\quad = C + \int^{\theta(x)} \Phi(t)dt + \int \Phi'(x)\{x-\vartheta(x)\}dx \\
&\qquad - \tfrac{1}{2}\{x-\vartheta(x)\}^2\Phi'\left\{x + O\left(\sqrt{x}(\log x)^2\right)\right\} \qquad (10.63.327)
\end{aligned}$$

where C is a constant and p is the largest prime not exceeding x.

10.64

Now let us assume that $\Phi(x) = 1/(x^s - 1)$ where $s > 0$. Then from (10.63.327) we see that, if p is the largest prime not greater than x, then

$$\begin{aligned}
&\frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1} \qquad (10.64.328) \\
&= C + \int^{\theta(x)} \frac{dx}{x^s-1} - s\int \frac{x-\vartheta(x)}{x^{1-s}\left(x^s-1\right)^2}dx + O\left\{x^{-s}(\log x)^4\right\}.
\end{aligned}$$

But it is known that

$$x - \theta(x) = x^{1/2} + x^{1/3} + \sum \frac{x^\rho}{\rho} - \sum \frac{x^{\rho/2}}{\rho} + O\left(x^{1/5}\right) \qquad (10.64.329)$$

where ρ is a complex root of $\zeta(s)$. By arguments similar to those of Section 42 we can show that

$$\sum \frac{x^{\rho/2-s}}{\rho(\rho/2-s)} = \int x^{-1-s}\sum \frac{x^{\rho/2}}{\rho}dx.$$

Hence

$$\int \frac{\sum \frac{x^{\rho/2}}{\rho}}{x^{1-s}\left(x^s-1\right)^2}dx = \int O\left\{x^{-1-s}\sum \frac{x^{\rho/2}}{\rho}\right\}dx$$
$$= O\left\{\sum \frac{x^{\rho/2-s}}{\rho(\rho/2-s)}\right\}$$
$$= O\left(x^{1/2-s}\right).$$

Similarly

$$\int \frac{\sum \frac{x^{\rho}}{\rho}}{x^{1-s}(x^s-1)^2}dx = \sum \frac{x^{\rho-s}}{\rho(\rho-s)} + O\left(\sum \frac{x^{\rho-2s}}{\rho(\rho-2s)}\right)$$
$$= \sum \frac{x^{\rho-s}}{\rho(\rho-s)} + O\left(x^{1/2-2s}\right).$$

Hence (10.64.328) may be replaced by

$$\begin{aligned}&\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1}\\ &= C+\int^{\theta(x)}\frac{dt}{t^s-1}-s\int \frac{x^{1/2}+x^{1/3}}{x^{1-s}\left(x^s-1\right)^2}dx\\ &\quad - s\sum \frac{x^{\rho-s}}{\rho(\rho-s)}+O\left(x^{1/2-2s}+x^{1/4-s}\right).\end{aligned} \tag{10.64.330}$$

It can easily be shown that

$$C = -\frac{\zeta'(s)}{\zeta(s)} \tag{10.64.331}$$

when the error term is $o(1)$.

10.65

Let

$$S_s(x) = -s\sum \frac{x^{\rho-s}}{\rho(\rho-s)}.$$

Then

$$\begin{aligned}|S_s(x)| &\le s\sum \left|\frac{x^{\rho-s}}{\rho(\rho-s)}\right|\\ &= s\ x^{1/2-s}\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}}.\end{aligned} \tag{10.65.332}$$

If m and n are any two positive numbers, then it is evident that $1/\sqrt{mn}$ lies between $1/m$ and $1/n$.

Hence

$$\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}}$$

lies between $\chi(1)$ and $\chi(s)$ where

$$\begin{aligned}\chi(s) &= \sum \frac{1}{(\rho-s)(1-\rho-s)} \\ &= \sum \frac{1}{\rho(1-\rho)+s^2-s} \\ &= \frac{1}{1-2s}\left(\sum \frac{1}{\rho-s} + \sum \frac{1}{1-\rho-s}\right) \\ &= \sum \frac{\frac{1}{s-\rho}}{s-\frac{1}{2}}. \end{aligned} \tag{10.65.333}$$

We can show as in Section 41 that

$$\sum \frac{1}{s-\rho} = \frac{2s-1}{s^2-s} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{\zeta'(s)}{\zeta(s)}. \tag{10.65.334}$$

Hence

$$\chi(s) = \frac{2}{s^2-s} + \frac{1}{2s-1}\left\{\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + 2\frac{\zeta'(s)}{\zeta(s)} - \log\pi\right\} \tag{10.65.335}$$

so that

$$\chi(0) = \chi(1) = 2 + \gamma - \log 4\pi. \tag{10.65.336}$$

By elementary algebra, it can easily be shown that if m_r and n_r be not negative and G_r be the geometric mean between m_r and n_r then

$$G_1 + G_2 + G_3 + \cdots < \sqrt{\{m_1+m_2+m_3+\cdots\}\{n_1+n_2+\cdots\}} \tag{10.65.337}$$

unless $\frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m_3}{n_3} = \cdots$.

From this it follows that

$$\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}} < \sqrt{\{\chi(1)\chi(s)\}}. \tag{10.65.338}$$

The following method leads to still closer approximation. It is easy to see that if m and n are positive, then $1/\sqrt{mn}$ is the geometric mean between

$$\frac{1}{3m} + \frac{8}{3(m+3n)} \qquad \text{and} \qquad \frac{1}{3n} + \frac{8}{3(3m+n)} \tag{10.65.339}$$

and so $\dfrac{1}{\sqrt{mn}}$ lies between both. Hence

$$\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}} \tag{10.65.340}$$

lies between

$$\frac{1}{3}\sum \frac{1}{\rho(1-\rho)} + \frac{2}{3}\sum \frac{1}{\rho(1-\rho)+\frac{3}{4}(s^2-s)}$$

and

$$\frac{1}{3}\sum \frac{1}{(\rho-s)(1-\rho-s)} + \frac{2}{3}\sum \frac{1}{\rho(1-\rho)+\frac{1}{4}(s^2-s)}$$

and is also less than the geometric mean[1] between these two in virtue of (10.65.337).

$$\sum \frac{1}{\rho(1-\rho)+\frac{1}{4}(s^2-s)} = \chi\left\{\frac{1+\sqrt{(1-s+s^2)}}{2}\right\}$$

and

$$\sum \frac{1}{\rho(1-\rho)+\frac{3}{4}(s^2-s)} = \chi\left\{\frac{1+\sqrt{(1-3s+3s^2)}}{2}\right\}.$$

Hence

$$\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}} \tag{10.65.341}$$

lies between

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[1]

$$\begin{aligned}
&\frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}}\\
&= \frac{1}{\rho(1-\rho)} - \frac{1}{2}\frac{s^2-s}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^2 - \frac{10}{32}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^3 + \cdots;\\
&\frac{1}{3}\frac{1}{\rho(1-\rho)} + \frac{2}{3}\frac{1}{\rho(1-\rho)+\frac{3}{4}(s^2-s)}\\
&= \frac{1}{\rho(1-\rho)} - \frac{1}{2}\frac{s^2-s}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^2 - \frac{9}{32}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^3 + \cdots;\\
&\frac{1}{3}\frac{1}{\rho(1-\rho)+s^2-s} + \frac{2}{3}\frac{1}{\rho(1-\rho)+\frac{1}{4}(s^2-s)}\\
&= \frac{1}{\rho(1-\rho)} - \frac{1}{2}\frac{s^2-s}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^2 - \frac{11}{32}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^3 + \cdots.
\end{aligned}$$

Since the first value of $\rho(1-\rho)$ is about 200 we see that the geometric mean is a much closer approximation than either.

$$\tfrac{1}{3}\chi(1)+\tfrac{2}{3}\chi\left\{\frac{1+\sqrt{(1-3s+3s^2)}}{2}\right\}$$

and

$$\tfrac{1}{3}\chi(s)+\tfrac{2}{3}\chi\left\{\frac{1+\sqrt{(1-s+s^2)}}{2}\right\}$$

and is also less than the geometric mean between these two.

10.66

In this and the following few sections it is always understood that p is the largest prime not greater than x. It can easily be shown that

$$\begin{aligned}&\int^{\rho(x)}\frac{dt}{t^s-1}-s\int\frac{x^{1/2}+x^{1/3}}{x^{1-s}\,(x^s-1)^2}dx\\&=\frac{\{\theta(x)\}^{1-s}}{1-s}+\frac{\{\theta(x)\}^{1-2s}}{1-2s}+\frac{x^{1-3s}}{1-3s}+\frac{x^{1-4s}}{1-4s}+\cdots+\frac{x^{1-ns}}{1-ns}\\&\quad-\frac{2sx^{1/2-s}}{1-2s}-\frac{3sx^{1/3-s}}{1-3s}-\frac{4sx^{1/2-2s}}{1-4s}+O\left(x^{1/2-2s}\right)\end{aligned}\tag{10.66.342}$$

where $n=\left[2+\frac{1}{2s}\right]$.

It follows from (10.64.330) and (10.66.342) that if $s>0$, then

$$\begin{aligned}&\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1}\\&=-\frac{\zeta'(s)}{\zeta(s)}+\frac{\{\vartheta(x)\}^{1-s}}{1-s}+\frac{\{\vartheta(x)\}^{1-2s}}{1-2s}+\frac{x^{1-3s}}{1-3s}+\frac{x^{1-4s}}{1-4s}+\cdots+\frac{x^{1-ns}}{1-ns}\\&\quad-\frac{2sx^{1/2-s}}{1-2s}-\frac{3sx^{1/3-s}}{1-3s}-\frac{4sx^{1/2-2s}}{1-4s}+S_s(x)+O\left(x^{1/2-2s}+x^{1/4-s}\right)\end{aligned}\tag{10.66.343}$$

where $n=\left[2+\frac{1}{2s}\right]$.

When $s=1,\frac{1}{2},\frac{1}{3}$ or $\frac{1}{4}$ we must take the limit of the right hand side when s approaches $1,\frac{1}{2},\frac{1}{3}$ or $\frac{1}{4}$. We shall consider the following cases:

Case I. $0<s<\frac{1}{4}$

$$\begin{aligned}&\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1}\\&=\frac{\{\vartheta(x)\}^{1-s}}{1-s}+\frac{\{\vartheta(x)\}^{1-2s}}{1-2s}+\frac{x^{1-3s}}{1-3s}+\frac{x^{1-4s}}{1-4s}+\cdots+\frac{x^{1-ns}}{1-ns}\\&\quad-\frac{2sx^{1/2-s}}{1-2s}-\frac{3sx^{1/3-s}}{1-3s}+S_s(x)+O\left(x^{1/2-2s}+x^{1/4-s}\right)\end{aligned}\tag{10.66.344}$$

where $n = \left[2 + \frac{1}{2s}\right]$.

Case II. $s = \frac{1}{4}$

$$\begin{aligned}
&\frac{\log 2}{2^{1/4-1}} + \frac{\log 3}{3^{1/4-1}} + \frac{\log 5}{5^{1/4-1}} + \cdots + \frac{\log p}{p^{1/4-1}} \\
&= \frac{4}{3}\{\vartheta(x)\}^{3/4} + 2\sqrt{\{\vartheta(x)\}} + 3x^{1/4} - 3x^{1/12} \\
&\quad + \frac{1}{2}\log x + S_{1/4}(x) + O(1). \qquad (10.66.345)
\end{aligned}$$

Case III. $s > \frac{1}{4}$

$$\begin{aligned}
&\frac{\log 2}{2^s - 1} + \frac{\log 3}{3^s - 1} + \frac{\log 5}{5^s - 1} + \cdots + \frac{\log p}{p^s - 1} \\
&= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{x^{1-2s} - 2s\ x^{1/2-s}}{1-2s} + \frac{x^{1-3s} - 3s\ x^{1/3-s}}{1-3s} \\
&\quad + S_s(x) + O\left(x^{1/4-s}\right). \qquad (10.66.346)
\end{aligned}$$

10.67

Making $s \to 1$ in (10.66.346), and remembering that

$$\lim_{s\to 1}\left\{\frac{v^{1-s}}{1-s} - \frac{\zeta'(s)}{\zeta(s)}\right\} = \log v - \gamma$$

where γ is the Eulerian constant, we have

$$\begin{aligned}
&\frac{\log 2}{2-1} + \frac{\log 3}{3-1} + \frac{\log 5}{5-1} + \cdots + \frac{\log p}{p-1} \qquad (10.67.347) \\
&= \log\vartheta(x) - \gamma + 2x^{-1/2} + \frac{3}{2}x^{-2/3} + S_1(x) + O\left(x^{-3/4}\right).
\end{aligned}$$

From (10.65.332) we know that

$$\sqrt{x} \mid S_1(x) \mid \leq 2 + \gamma - \log(4\pi) = .046\ldots \qquad (10.67.348)$$

approximately, for all positive values of x.

When $s > 1$, (10.66.346) reduces to

$$\begin{aligned}
&\frac{\log 2}{2^s - 1} + \frac{\log 3}{3^s - 1} + \frac{\log 5}{5^s - 1} + \cdots + \frac{\log p}{p^s - 1} \\
&= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{2s\ x^{1/2-s}}{2s-1} + \frac{3s\ x^{1/3-s}}{3s-1} \\
&\quad + S_s(x) + O\left(x^{1/4-s}\right). \qquad (10.67.349)
\end{aligned}$$

Writing $O\left(x^{1/2-s}\right)$ for $S_s(x)$ in (10.66.343), we see that, if $s > 0$, then

$$\begin{aligned}
&\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1} \\
&=-\frac{\zeta'(s)}{\zeta(s)}+\frac{\{\vartheta(x)\}^{1-s}}{1-s}+\frac{x^{1-2s}}{1-2s}+\frac{x^{1-3s}}{1-3s}+\cdots+\frac{x^{1-ns}}{1-ns} \\
&\quad -\frac{2s\ x^{1/2-s}}{1-2s}+O\left(x^{1/2-s}\right) \qquad (10.67.350)
\end{aligned}$$

when $n = \left[1+\frac{1}{2s}\right]$.

Now the following three cases arise:

Case I. $0 < s < \frac{1}{2}$

$$\begin{aligned}
&\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1} \\
&=\frac{\{\vartheta(x)\}^{1-s}}{1-s}+\frac{x^{1-2s}}{1-2s}+\frac{x^{1-3s}}{1-3s}+\cdots+\frac{x^{1-ns}}{1-ns}+O\left(x^{1/2-s}\right) \quad (10.67.351)
\end{aligned}$$

where $n = \left[1+\frac{1}{2s}\right]$.

Case II. $s = \frac{1}{2}$

$$\begin{aligned}
&\frac{\log 2}{\sqrt{2}-1}+\frac{\log 3}{\sqrt{3}-1}+\frac{\log 5}{\sqrt{5}-1}+\cdots+\frac{\log p}{\sqrt{p}-1} \\
&=2\sqrt{\{\vartheta(x)\}}+\frac{1}{2}\log x+O(1). \qquad (10.67.352)
\end{aligned}$$

Case III. $s > \frac{1}{2}$

$$\begin{aligned}
&\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1} \\
&=-\frac{\zeta'(s)}{\zeta(s)}+\frac{\{\vartheta(x)\}^{1-s}}{1-s}+O\left(x^{1/2-s}\right). \qquad (10.67.353)
\end{aligned}$$

10.68

We shall now consider the product

$$\left(1-2^{-s}\right)\left(1-3^{-s}\right)\left(1-5^{-s}\right)\cdots\left(1-p^{-s}\right).$$

It can easily be shown that

$$\int \frac{x^{a+bs}}{a+bs}ds = \frac{1}{b}\mathrm{Li}\left(x^{a+bs}\right) \qquad (10.68.354)$$

where $\mathrm{Li}(x)$ is the principal value of $\int_0^x \frac{dt}{\log t}$; and that

$$\int S_s(x)ds = -\frac{S_s(x)}{\log x} + O\left\{\frac{x^{1/2-s}}{(\log x)^2}\right\}. \tag{10.68.355}$$

Now remembering (10.68.354) and (10.68.355) and integrating (10.66.343) with respect to s, we see that if $s > 0$, then

$$\begin{aligned}&\log\left\{\left(1-2^{-s}\right)\left(1-3^{-s}\right)\left(1-5^{-s}\right)\cdots\left(1-p^{-s}\right)\right\} \qquad (10.68.356)\\ &= -\log|\zeta(s)| - \mathrm{Li}\{\vartheta(x)\}^{1-s} - \frac{1}{2}\mathrm{Li}\left(x^{1-2s}\right) - \frac{1}{3}\mathrm{Li}\left(x^{1-3s}\right) - \cdots\\ &\quad - \frac{1}{n}\mathrm{Li}\left(x^{1-ns}\right) + \frac{1}{2}\mathrm{Li}\left(x^{1/2-s}\right) - \frac{x^{1/2-s}+S_s(x)}{\log x} + O\left\{\frac{x^{1/2-s}}{(\log x)^2}\right\}\end{aligned}$$

where $n = \left[1 + \frac{1}{2s}\right]$.

Now the following three cases arise.

Case I. $0 < s < \frac{1}{2}$

$$\begin{aligned}&\log\left\{\left(1-2^{-s}\right)\left(1-3^{-s}\right)\left(1-5^{-s}\right)\cdots\left(1-p^{s}\right)\right\} \qquad (10.68.357)\\ &= -\mathrm{Li}\{\vartheta(x)\}^{1-s} - \frac{1}{2}\mathrm{Li}\left(x^{1-2s}\right) - \frac{1}{3}\mathrm{Li}\left(x^{1-3s}\right) - \cdots\\ &\quad - \frac{1}{n}\mathrm{Li}\left(x^{1-ns}\right) + \frac{2sx^{1/2-s}}{(1-2s)\log x} - \frac{S_s(x)}{\log x} + O\left\{x^{1/2-s}(\log x)^2\right\}\end{aligned}$$

where $n = \left[1 + \frac{1}{2s}\right]$. Making $s \to \frac{1}{2}$ in (10.68.356) and remembering that

$$\lim_{h\to 0}\{\mathrm{Li}(1+h) - \log|h|\} = \gamma \tag{10.68.358}$$

where γ is the Eulerian constant, we have

Case II. $s = \frac{1}{2}$

$$\begin{aligned}&\frac{1}{\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{5}}\right)\cdots\left(1-\frac{1}{\sqrt{p}}\right)} \qquad (10.68.359)\\ &= -\sqrt{2}\zeta\left(\frac{1}{2}\right)\exp\left\{\mathrm{Li}\sqrt{\theta(x)} + \frac{1+S_{1/2}(x)}{\log x} + O\left(\frac{1}{(\log x)^2}\right)\right\}.\end{aligned}$$

It may be observed that

$$-(\sqrt{2}-1)\zeta\left(\frac{1}{2}\right) = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots. \tag{10.68.360}$$

Case III. $s > \frac{1}{2}$

$$\begin{aligned}&\frac{1}{\left(1-2^{-s}\right)\left(1-3^{-s}\right)\left(1-5^{-s}\right)\cdots\left(1-p^{-s}\right)} \\ &= |\zeta(s)|\exp\left[\operatorname{Li}\{\theta(x)\}^{1-s}+\frac{2s\ x^{1/2-s}}{(2s-1)\log x}+\frac{S_s(x)}{\log x}+O\left\{\frac{x^{1/2-s}}{(\log x)^2}\right\}\right].\end{aligned} \tag{10.68.361}$$

Remembering (10.68.358) and making $s\to 1$ in (10.68.361) we obtain

$$\begin{aligned}&\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{p}\right)} \\ &= e^{\gamma}\left\{\log\vartheta(x)+\frac{2}{\sqrt{x}}+S_1(x)+O\left(\frac{1}{\sqrt{x}\log x}\right)\right\}.\end{aligned} \tag{10.68.362}$$

It follows from this and (10.67.347) that

$$\begin{aligned}&\frac{e^{-\gamma}}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{p}\right)} \\ &= \gamma+\frac{\log 2}{2-1}+\frac{\log 3}{3-1}+\cdots+\frac{\log p}{p-1}+O\left(\frac{1}{\sqrt{p}\log p}\right).\end{aligned} \tag{10.68.363}$$

10.69

We shall consider the order of x_r. Putting $r=1$ in (10.61.319) we have

$$\varepsilon=\frac{\log\left(1+x_1^{-s}\right)}{\log x_1};$$

and so

$$x_r^{\log(1+x_1^{-s})/\log x_1}=\frac{1-x_r^{-s(r+1)}}{1-x_r^{-sr}}. \tag{10.69.364}$$

Let

$$x_r=x_1^{t_r/r}.$$

Then we have

$$\left(1+x_1^{-s}\right)^{t_r/r}=\frac{1-x_1^{-st_r(1+1/r)}}{1-x_1^{-st_r}}.$$

From this we can easily deduce that

$$t_r=1+\frac{\log r}{s\log x_1}+O\left\{\frac{1}{(\log x_1)^2}\right\}.$$

Hence

$$x_r = x_1^{1/r} \left\{ r^{1/(rs)} + O\left(\frac{1}{\log x_1}\right) \right\} \tag{10.69.365}$$

and so

$$x_r \sim \left(r^{1/s} x_1\right)^{1/r}. \tag{10.69.366}$$

Putting $\lambda = 2$ in (10.60.311) we see that the greatest possible value of r is

$$a_2 = \frac{\log(1/\varepsilon)}{s \log 2} + O(1) = \frac{\log x_1}{\log 2} + \frac{\log\log x_1}{s \log 2} + O(1). \tag{10.69.367}$$

Again

$$\log N = \vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots = \vartheta(x_1) + x_2 + O\left(x_1^{1/3}\right) \tag{10.69.368}$$

in virtue of (10.69.366). It follows from Section 10.68 and the definition of $\Pi_r(x)$, that, if sr and $s(r+1)$ are not equal to 1, then

$$\Pi_r(x) = \left|\frac{\zeta(sr)}{\zeta\{s(r+1)\}}\right| \exp\left(O\left(x^{1-sr}\right)\right);$$

and consequently

$$\Pi_r(x_r) = \left|\frac{\zeta(sr)}{\zeta\{s(r+1)\}}\right| \exp\left(O\left(x_1^{1/r-s}\right)\right) \tag{10.69.369}$$

in virtue of (10.69.366). But if sr or $s(r+1)$ is unity, it can easily be shown that

$$\Pi_{r-1}(x_{r-1})\Pi_r(x_r)\Pi_{r+1}(x_{r+1}) = \left|\frac{\zeta\{s(r-1)\}}{\zeta\{s(r+2)\}}\right| \exp\left(O\left(x_1^{1/(r-1)-s}\right)\right). \tag{10.69.370}$$

10.70

We shall now consider the order of $\Sigma_{-s}(N)$ i.e. the maximum order of $\sigma_{-s}(N)$. It follows from (10.61.317) that if $3s \neq 1$, then

$$\Sigma_{-s}(N) = \Pi_1(x_1)\Pi_2(x_2)|\zeta(3s)| \exp\left(O\left(x_1^{1/3-s}\right)\right) \tag{10.70.371}$$

in virtue of (10.69.367), (10.69.369) and (10.69.370). But if $3s = 1$, we can easily show, by using (10.68.362), that

$$\Sigma_{-s}(N) = \Pi_1(x_1)\Pi_2(x_2) \exp\left(O\left(\log\log x_1\right)\right). \tag{10.70.372}$$

It follows from Section 10.68 that

$$\begin{aligned}\log \Pi_1(x_1) = \log\left|\frac{\zeta(s)}{\zeta(2s)}\right| &+ \mathrm{Li}\{\theta(x_1)\}^{1-s} - \frac{1}{2}\mathrm{Li}\{\vartheta(x_1)\}^{1-2s} \\ &+\frac{1}{3}\mathrm{Li}\{\vartheta(x_1)\}^{1-3s} - \cdots - \frac{(-1)^n}{n}\mathrm{Li}\{\vartheta(x_1)\}^{1-ns} \\ &-\frac{1}{2}\mathrm{Li}\left(x_1^{1/2-s}\right) + \frac{x_1^{1/2-s}+S_s(x_1)}{\log x_1} + O\left\{\frac{x_1^{1/2-s}}{(\log x_1)^2}\right\}\end{aligned} \tag{10.70.373}$$

where $n = \left[1+\frac{1}{2s}\right]$; and also that, if $3s \neq 1$, then,

$$\log \Pi_2(x_2) = \log\left|\frac{\zeta(2s)}{\zeta(3s)}\right| + \mathrm{Li}\left(x_2^{1-2s}\right) + O\left\{\frac{x_1^{1/2-s}}{(\log x_1)^2}\right\}; \tag{10.70.374}$$

and when $3s = 1$

$$\log \Pi_2(x_2) = \mathrm{Li}\left(x_2^{1-2s}\right) + O\left\{\frac{x_1^{1/2-s}}{(\log x_1)^2}\right\}. \tag{10.70.375}$$

It follows from (10.70.371)–(10.70.375) that

$$\begin{aligned}\log \Sigma_{-s}(N) = \log|\zeta(s)| &+ \mathrm{Li}\{\vartheta(x_1)\}^{1-s} - \frac{1}{2}\mathrm{Li}\{\vartheta(x_1)\}^{1-2s} \\ &+\frac{1}{3}\mathrm{Li}\{\vartheta(x_1)\}^{1-3s} - \cdots - \frac{(-1)^n}{n}\mathrm{Li}\{\vartheta(x_1)\}^{1-ns} \\ &-\frac{1}{2}\mathrm{Li}\left(x_1^{1/2-s}\right) + \mathrm{Li}\left(x_2^{1-2s}\right) \\ &+\frac{x_1^{1/2-s}+S_s(x_1)}{\log x_1} + O\left\{\frac{x_1^{1/2-s}}{(\log x_1)^2}\right\}\end{aligned} \tag{10.70.376}$$

where $n = \left[1+\frac{1}{2s}\right]$. But from (10.69.368) it is clear that, if $m > 0$ then

$$\begin{aligned}&\mathrm{Li}\{\vartheta(x_1)\}^{1-ms} \\ &= \mathrm{Li}\left\{\log N - x_2 + O\left(x_1^{1/3}\right)\right\}^{1-ms} \\ &= \mathrm{Li}\left\{(\log N)^{1-ms} - (1-ms)x_2(\log N)^{-ms} + O\left(x_1^{1/3-ms}\right)\right\} \\ &= \mathrm{Li}(\log N)^{1-ms} - \frac{x_2(\log N)^{-ms}}{\log\log N} + O\left(x_1^{1/3-ms}\right).\end{aligned}$$

By arguments similar to those of Section 42 we can show that

$$S_s(x_1) = S_s\left\{\log N + O\left(\sqrt{x_1}(\log x_1)^2\right)\right\} = S_s(\log N) + O\left\{x_1^{-s}(\log x_1)^4\right\}.$$

Hence

$$\log \Sigma_{-s}(N) = \log|\zeta(s)| + \mathrm{Li}(\log N)^{1-s} - \frac{1}{2}\mathrm{Li}(\log N)^{1-2s} \tag{10.70.377}$$

$$+\frac{1}{3}\mathrm{Li}(\log N)^{1-3s} - \cdots - \frac{(-1)^n}{n}\mathrm{Li}(\log N)^{1-ns}$$
$$-\frac{1}{2}\mathrm{Li}(\log N)^{1/2-s} + \frac{(\log N)^{1/2-s} + S_s(\log N)}{\log\log N}$$
$$+\mathrm{Li}\left(x_2^{1-2s}\right) - \frac{x_2(\log N)^{-s}}{\log\log N} + O\left\{\frac{(\log N)^{1/2-s}}{(\log\log N)^2}\right\}$$

where $n = \left[1 + \frac{1}{2s}\right]$ and

$$\begin{aligned} x_2 &= 2^{1/(2s)}\sqrt{x_1} + O\left(\frac{\sqrt{x_1}}{\log x_1}\right) \\ &= 2^{1/(2s)}\sqrt{(\log N)} + O\left\{\frac{\sqrt{(\log N)}}{\log\log N}\right\} \end{aligned} \tag{10.70.378}$$

in virtue of (10.69.365).

10.71

Let us consider the order of $\Sigma_{-s}(N)$ in the following three cases.

Case I. $0 < s < \frac{1}{2}$

Here we have

$$\mathrm{Li}(\log N)^{1/2-s} = \frac{(\log N)^{1/2-s}}{(\frac{1}{2} - s)\log\log N} + O\left\{\frac{(\log N)^{1/2-s}}{(\log\log N)^2}\right\}.$$

$$\begin{aligned} \mathrm{Li}\left(x_2^{1-2s}\right) &= \frac{x_2^{1-2s}}{(1-2s)\log x_2} + O\left\{\frac{x_2^{1-2s}}{(\log x_2)^2}\right\} \\ &= \frac{2^{1/(2s)}(\log N)^{1/2-s}}{(1-2s)\log\log N} + O\left\{\frac{(\log N)^{1/2-s}}{(\log\log N)^2}\right\}. \end{aligned}$$

$$\frac{x_2(\log N)^{-s}}{\log\log N} = \frac{2^{1/(2s)}(\log N)^{1/2-s}}{\log\log N} + O\left\{\frac{(\log N)^{1/2-s}}{(\log\log N)^2}\right\}.$$

It follows from these and (10.70.377) that

$$\begin{aligned} \log \Sigma_{-s}(N) = {} & \mathrm{Li}(\log N)^{1-s} - \frac{1}{2}\mathrm{Li}(\log N)^{1-2s} \\ & + \frac{1}{3}\mathrm{Li}(\log N)^{1-3s} - \cdots - \frac{(-1)^n}{n}\mathrm{Li}(\log N)^{1-ns} \\ & + \frac{2s\left(2^{1/(2s)} - 1\right)(\log N)^{1/2-s}}{(1-2s)\log\log N} \\ & + \frac{S_s(\log N)}{\log\log N} + O\left\{\frac{(\log N)^{1/2-s}}{(\log\log N)^2}\right\} \end{aligned} \tag{10.71.379}$$

where $n = \left[1 + \frac{1}{2s}\right]$. Remembering (10.68.358) and (10.70.378) and making $s \to \frac{1}{2}$ in (10.70.377) we have

Case II. $s = \frac{1}{2}$

$$\Sigma_{-1/2}(N) = -\frac{\sqrt{2}}{2}\zeta\left(\frac{1}{2}\right)\exp\left\{\mathrm{Li}\sqrt{(\log N)} + \frac{2\log 2 - 1 + S_{1/2}(\log N)}{\log\log N} + \frac{O(1)}{(\log\log N)^2}\right\}. \quad (10.71.380)$$

Case III. $s > \frac{1}{2}$

$$\Sigma_{-s}(N) = |\zeta(s)|\exp\left\{\mathrm{Li}(\log N)^{1-s} - \frac{2s\left(2^{1/(2s)} - 1\right)}{2s-1}\frac{(\log N)^{1/2-s}}{\log\log N} + \frac{S_s(\log N)}{\log\log N} + O\left\{\frac{(\log N)^{1/2-s}}{(\log\log N)^2}\right\}\right\}. \quad (10.71.381)$$

Now making $s \to 1$ in this we have

$$\Sigma_{-1}(N) = e^{\gamma}\left\{\log\log N - \frac{2(\sqrt{2}-1)}{\sqrt{(\log N)}} + S_1(\log N) + \frac{O(1)}{\sqrt{(\log N)}\log\log N}\right\}. \quad (10.71.382)$$

Hence

$$\underline{\mathrm{Lim}}\{\Sigma_{-1}(N) - e^{\gamma}\log\log N\}\sqrt{(\log N)} \geq -e^{\gamma}(2\sqrt{2} + \gamma - \log 4\pi) = -1.558$$

approximately and

$$\overline{\mathrm{Lim}}\{\Sigma_{-1}(N) - e^{\gamma}\log\log N\}\sqrt{(\log N)} \leq -e^{\gamma}(2\sqrt{2} - 4 - \gamma + \log 4\pi) = -1.393$$

approximately.

The maximum order of $\sigma_s(N)$ is easily obtained by multiplying the values of $\Sigma_{-s}(N)$ by N^s. It may be interesting to see that $x_r \to x_1^{1/r}$ as $s \to \infty$, and ultimately N assumes the form

$$e^{\vartheta(x_1) + \vartheta(x_1^{1/2}) + \vartheta(x_1^{1/3}) + \cdots}$$

that is to say the form of a generalized superior highly composite number approaches that of the least common multiple of the natural numbers when s becomes infinitely large.

The maximum order of $\sigma_{-s}(N)$ without assuming the prime number theorem is obtained by changing $\log N$ to $\log Ne^{O(1)}$ in all the preceding results. In particular

$$\Sigma_{-1}(N) = e^{\gamma}\{\log\log N + O(1)\}. \quad (10.71.383)$$

10.72

Let

$$\left(1+2q+2q^4+2q^9+\cdots\right)^4 = 1+8\left\{Q_4(1)q+Q_4(2)q^2+Q_4(3)q^3+\cdots\right\}.$$

Then, by means of elliptic functions, we can show that

$$\begin{aligned} &Q_4(1)q+Q_4(2)q^2+Q_4(3)q^3+\cdots \\ &\quad = \frac{q}{1-q}+\frac{2q^2}{1+q^2}+\frac{3q^3}{1-q^3}+\frac{4q^4}{1+q^4}+\cdots \\ &\quad = \frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\frac{4q^4}{1-q^4}+\cdots \\ &\qquad -\left(\frac{4q^4}{1-q^4}+\frac{8q^8}{1-q^8}+\frac{12q^{12}}{1-q^{12}}+\cdots\right). \end{aligned} \tag{10.72.384}$$

But

$$\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\cdots = \sigma_1(1)q+\sigma_1(2)q^2+\sigma_1(3)q^3+\cdots.$$

It follows that

$$Q_4(N) \le \sigma_1(N) \tag{10.72.385}$$

for all values of N. It also follows from (10.72.384) that

$$\left(1-4^{1-s}\right)\zeta(s)\zeta(s-1) = 1^{-s}Q_4(1)+2^{-s}Q_4(2)+3^{-s}Q_4(3)+\cdots. \tag{10.72.386}$$

Let

$$N = 2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p}$$

where $a_\lambda \ge 0$. Then, the coefficient of q^N in

$$\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\cdots$$

is

$$N\frac{1-2^{-a_2-1}}{1-2^{-1}}\,\frac{1-3^{-a_3-1}}{1-3^{-1}}\,\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}};$$

and that in

$$\frac{4q^4}{1-q^4}+\frac{8q^8}{1-q^8}+\frac{12q^{12}}{1-q^{12}}+\cdots$$

is 0 when N is not a multiple of 4 and

$$N\frac{1-2^{-a_2-1}}{1-2^{-1}}\,\frac{1-3^{-a_3-1}}{1-3^{-1}}\,\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}$$

when N is a multiple of 4. From this and (10.72.384) it follows that, if N is not a multiple of 4, then

$$Q_4(N) = N\frac{1-2^{-a_2-1}}{1-2^{-1}}\frac{1-3^{-a_3-1}}{1-3^{-1}}\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}; \quad (10.72.387)$$

and if N is a multiple of 4, then

$$Q_4(N) = 3N\frac{1-2^{-a_2-1}}{1-2^{-1}}\frac{1-3^{a_3-1}}{1-3^{-1}}\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}. \quad (10.72.388)$$

It is easy to see from (10.72.387) and (10.72.388) that, in order that $Q_4(N)$ should be of maximum order, a_2 must be 1. From (10.71.382) we see that the maximum order of $Q_4(N)$ is

$$\begin{aligned}&\frac{3}{4}e^{\gamma}\left\{\log\log N - \frac{2(\sqrt{2}-1)}{\sqrt{(\log N)}} + S_1(\log N) + O\left(\frac{1}{\sqrt{(\log N)}\log\log N}\right)\right\}\\ &= \frac{3}{4}e^{\gamma}\left\{\log\log N + O\left(\frac{1}{\sqrt{(\log N)}}\right)\right\}. \qquad (10.72.389)\end{aligned}$$

It may be observed that, if N is not a multiple of 4, then

$$Q_4(N) = \sigma_1(N);$$

and if N is a multiple of 4, then

$$Q_4(N) = \frac{3\sigma_1(N)}{2^{a_2+1}-1}.$$

10.73

Let

$$\left(1+2q+2q^4+2q^9+\cdots\right)^6 = 1+12\left\{Q_6(1)q+Q_6(2)q^2+Q_6(3)q^3+\cdots\right\}.$$

Then, by means of elliptic functions, we can show that

$$\begin{aligned}&Q_6(1)q+Q_6(2)q^2+Q_6(3)q^3+\cdots\\ &= \frac{4}{3}\left(\frac{1^2q}{1+q^2}+\frac{2^2q^2}{1+q^4}+\frac{3^2q^3}{1+q^6}+\cdots\right)\\ &\quad -\frac{1}{3}\left(\frac{1^2q}{1-q}-\frac{3^2q^3}{1-q^3}+\frac{5^2q^5}{1-q^5}-\cdots\right). \qquad (10.73.390)\end{aligned}$$

But

$$\frac{5}{3}\left\{\sigma_2(1)q+\sigma_2(2)q^2+\sigma_2(3)q^3+\cdots\right\}$$
$$=\frac{4}{3}\left\{\frac{1^2q}{1-q}+\frac{2^2q^2}{1-q^2}+\frac{3^2q^3}{1-q^3}+\cdots\right\}$$
$$+\frac{1}{3}\left\{\frac{1^2q}{1-q}+\frac{2^2q^2}{1-q^2}+\frac{3^2q^3}{1-q^3}+\cdots\right\}.$$

It follows that

$$Q_6(N)\leq\frac{5\sigma_2(N)-2}{3} \tag{10.73.391}$$

for all values of N. It also follows from (10.73.390) that

$$\frac{4}{3}\zeta(s-2)\zeta_1(s)-\frac{1}{3}\zeta(s)\zeta_1(s-2)$$
$$=1^{-s}Q_6(1)+2^{-s}Q_6(2)+3^{-s}Q_6(3)+\cdots. \tag{10.73.392}$$

Let

$$N=2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p},$$

where $a_\lambda\geq 0$. Then from (10.73.390) we can show, as in the previous section, that if $2^{-a_2}N$ be of the form $4n+1$, then

$$Q_6(N)=N^2\frac{1-\left(2^2\right)^{-a_2-1}}{1-2^{-2}}\frac{1-\left(-3^2\right)^{-a_3-1}}{1+3^{-2}}$$
$$\times\frac{1-\left(5^2\right)^{-a_5-1}}{1-5^{-2}}\cdots\frac{1-\left\{(-1)^{(p-1)/2}p^2\right\}^{-a_p-1}}{1-(-1)^{(p-1)/2}p^{-2}}; \tag{10.73.393}$$

and if $2^{-a_2}N$ be of the form $4n-1$, then

$$Q_6(N)=N^2\frac{1+\left(2^2\right)^{-a_2-1}}{1-2^{-2}}\frac{1-\left(-3^2\right)^{-a_3-1}}{1+3^{-2}}$$
$$\times\frac{1-\left(5^2\right)^{-a_5-1}}{1-5^{-2}}\cdots\frac{1-\left\{(-1)^{(p-1)/2}p^2\right\}^{-a_p-1}}{1-(-1)^{(p-1)/2}p^{-2}}. \tag{10.73.394}$$

It follows from (10.73.393) and (10.73.394) that, in order that $Q_6(N)$ should be of maximum order, $2^{-a_2}N$ must be of the form $4n-1$ and $a_2,a_3,a_7,a_{11},\ldots$ must be 0; $3,7,11,\ldots$ being primes of the form $4n-1$. But all these cannot be satisfied at the same time since $2^{-a_2}N$ cannot be of the form $4n-1$, when $a_3,a_7,a_{11},\ldots$ are all zeros. So let us retain a single prime of the form $4n-1$ in the end, that is to say, the largest prime of the form $4n-1$ not exceeding p. Thus we see that, in order that $Q_6(N)$ should be of maximum order, N must be of the form

$$5^{a_5}\cdot 13^{a_{13}}\cdot 17^{a_{17}}\cdots p^{a_p}\cdot p'$$

where p is a prime of the form $4n+1$ and p' is the prime of the form $4n-1$ next above or below p; and consequently

$$Q_6(N) = \frac{5}{3}N^2 \frac{1-5^{-2(a_5+1)}}{1-5^{-2}} \frac{1-13^{-2(a_{13}+1)}}{1-13^{-2}} \cdots \frac{1-p^{-2(a_p+1)}}{1-p^{-2}} \left\{1-(p')^{-2}\right\}.$$

From this we can show that the maximum order of $Q_6(N)$ is

$$\begin{aligned} &\frac{5N^2 e^{1/2\ \mathrm{Li}(1/(2\log N))+O\left\{(\log\log N)/(\log N\sqrt{(\log N)})\right\}}}{3\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{17^2}\right)\left(1-\frac{1}{29^2}\right)\cdots} \\ &= \frac{5N^2\left\{1+\frac{1}{2}\mathrm{Li}\left(\frac{1}{2\log N}\right)+O\left(\frac{\log\log N}{\log N\sqrt{(\log N)}}\right)\right\}}{3\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{17^2}\right)\left(1-\frac{1}{29^2}\right)\cdots} \end{aligned} \quad (10.73.395)$$

where $5, 13, 17, \ldots$ are the primes of the form $4n+1$.

10.74

Let

$$\left(1+2q+2q^4+2q^9+\cdots\right)^8 = 1+16\left\{Q_8(1)q+Q_8(2)q^2+Q_8(3)q^3+\cdots\right\}.$$

Then, by means of elliptic functions, we can show that

$$\begin{aligned} &Q_8(1)q+Q_8(2)q^2+Q_8(3)q^3+\cdots \\ &\quad = \frac{1^3q}{1+q}+\frac{2^3q^2}{1-q^2}+\frac{3^3q^3}{1+q^3}+\frac{4^3q^4}{1-q^4}+\cdots. \end{aligned} \quad (10.74.396)$$

But

$$\begin{aligned} &\sigma_3(1)q+\sigma_3(2)q^2+\sigma_3(3)q^3+\cdots \\ &\quad = \frac{1^3q}{1-q}+\frac{2^3q^2}{1-q^2}+\frac{3^3q^3}{1-q^3}+\cdots. \end{aligned}$$

It follows that

$$Q_8(N) \le \sigma_3(N) \quad (10.74.397)$$

for all values of N. It can also be shown from (10.74.396) that

$$\left(1-2^{1-s}+4^{2-s}\right)\zeta(s)\zeta(s-3) = Q_8(1)1^{-s}+Q_8(2)2^{-s}+Q_8(3)3^{-s}+\cdots. \quad (10.74.398)$$

Let

$$N = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p},$$

where $a_\lambda \geq 0$. Then from (10.74.396) we can easily show that, if N is odd, then

$$Q_8(N) = N^3 \frac{1-2^{-3(a_2+1)}}{1-2^{-3}} \frac{1-3^{-3(a_3+1)}}{1-3^{-3}} \cdots \frac{1-p^{-3(a_p+1)}}{1-p^{-3}}; \qquad (10.74.399)$$

and if N is even then

$$Q_8(N) = N^3 \frac{1-15\cdot 2^{-3(a_2+1)}}{1-2^{-3}} \frac{1-3^{-3(a_3+1)}}{1-3^{-3}} \cdots \frac{1-p^{-3(a_p+1)}}{1-p^{-3}}. \qquad (10.74.400)$$

Hence the maximum order of $Q_8(N)$ is

$$\begin{aligned} &\zeta(3)N^3 e^{\mathrm{Li}(\log N)^{-2}+O\left(((\log N)^{-5/2})(\log\log N)\right)} \\ &= \zeta(3)N^3\left\{1+\mathrm{Li}(\log N)^{-2}+O\left(\frac{(\log N)^{-5/2}}{\log\log N}\right)\right\} \end{aligned}$$

or more precisely

$$\begin{aligned} \zeta(3)N^3 \Bigg\{ 1+\mathrm{Li}(\log N)^{-2} &- \frac{6\left(2^{1/6}-1\right)(\log N)^{-5/2}}{5\log\log N} \\ &+\frac{S_3(\log N)}{\log\log N}+O\left(\frac{(\log N)^{-5/2}}{(\log\log N)^2}\right)\Bigg\}. \end{aligned} \qquad (10.74.401)$$

10.75

There are, of course, results corresponding to those of Sections 10.72–10.74 for the various powers of $\overline{Q}$ where

$$\overline{Q} = 1+6\left(\frac{q}{1-q}-\frac{q^2}{1-q^2}+\frac{q^4}{1-q^4}-\frac{q^5}{1-q^5}+\cdots\right).$$

Thus for example

$$(\overline{Q})^2 = 1+12\left(\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{4q^4}{1-q^4}+\frac{5q^5}{1-q^5}+\cdots\right), \qquad (10.75.402)$$

$$\begin{aligned} (\overline{Q})^3 &= 1-9\left(\frac{1^2q}{1-q}-\frac{2^2q^2}{1-q^2}+\frac{4^2q^4}{1-q^4}-\frac{5^2q^5}{1-q^5}+\cdots\right) \\ &\quad +27\left(\frac{1^2q}{1+q+q^2}+\frac{2^2q^2}{1+q^2+q^4}+\frac{3^2q^3}{1+q^3+q^6}+\cdots\right), \end{aligned} \qquad (10.75.403)$$

$$(\overline{Q})^4 = 1+24\left(\frac{1^3q}{1-q}+\frac{2^3q^2}{1-q^2}+\frac{3^2q^3}{1-q^3}+\cdots\right)$$

$$+ 8\left(\frac{3^3q^3}{1-q^3}+\frac{6^3q^6}{1-q^6}+\frac{9^3q^9}{1-q^9}+\cdots\right). \tag{10.75.404}$$

The number of ways in which a number can be expressed in the forms m^2+2n^2, $k^2+l^2+2m^2+2n^2$, m^2+3n^2, and $k^2+l^2+3m^2+3n^2$ can be found from the following formulae.

$$\left(1+2q+2q^4+2q^9+\cdots\right)\left(1+2q^2+2q^8+2q^{18}+\cdots\right)$$
$$= 1+2\left(\frac{q}{1-q}+\frac{q^3}{1-q^3}-\frac{q^5}{1-q^5}-\frac{q^7}{1-q^7}+\cdots\right), \tag{10.75.405}$$
$$\left(1+2q+2q^4+2q^9+\cdots\right)^2\left(1+2q^2+2q^8+2q^{18}+\cdots\right)^2$$
$$= 1+4\left(\frac{q}{1-q^2}+\frac{2q^2}{1-q^4}+\frac{3q^3}{1-q^6}+\frac{4q^4}{1-q^8}+\cdots\right), \tag{10.75.406}$$
$$\left(1+2q+2q^4+2q^9+\cdots\right)\left(1+2q^3+2q^{12}+2q^{27}+\cdots\right)$$
$$= 1+2\left(\frac{q}{1-q}-\frac{q^2}{1+q^2}+\frac{q^4}{1+q^4}-\frac{q^5}{1-q^5}+\frac{q^7}{1-q^7}-\cdots\right), \tag{10.75.407}$$
$$\left(1+2q+2q^4+2q^9+\cdots\right)^2\left(1+2q^3+2q^{12}+2q^{27}+\cdots\right)^2$$
$$= 1+4\left(\frac{q}{1+q}+\frac{2q^2}{1-q^2}+\frac{4q^4}{1-q^4}+\frac{5q^5}{1+q^5}+\frac{7q^7}{1+q^7}+\cdots\right) \tag{10.75.408}$$

where $1,2,4,5\ldots$ are the natural numbers without the multiples of 3.

NOTES

10.52

The definition of $Q_2(N)$ given in square brackets is missing in [283]. It has been formulated in the same terms as the definition of $\overline{Q}_2(N)$ given in Section 10.55. For $N \neq 0$, $4Q_2(N)$ is the number of pairs $(x, y) \in \mathbf{Z}^2$ such that $x^2+y^2 = N$. The content of this section is well known.

Formula (10.52.269) links Dirichlet's series with Lambert's series [168, p. 258].

10.53

Effective upper bounds for $Q_2(N)$ can be found in [296, p. 50]; for instance

$$\log Q_2(N) \le \frac{(\log 2)(\log N)}{\log\log N}\left(1+\frac{1-\log 2}{\log\log N}+\frac{2.40104}{(\log\log N)^2}\right).$$

The maximal order of $Q_2(N)$ is studied in [242], but not so deeply as here. See also [245, pp. 218–219].

10.54

For a proof of (10.54.276), see [301, p. 22]. In (10.54.276), we remind the reader that ρ is a zero of the Riemann zeta function. Formula (10.54.279) has been rediscovered and extended to all arithmetic progressions [299].

10.56

For a proof of (10.56.291), see [301, p. 22]. In the definition of $R_2(x)$, between formulas (10.56.290) and (10.56.291), and in the definition of $\Phi(N)$, after formula (10.56.294), three misprints in [283] have been corrected, namely $\sum \frac{x^\rho}{\rho^2}$ and $\sum \frac{x^{\rho_2}}{\rho_2^2}$ have been written instead of $\sum \frac{x^\rho}{\rho}$ and $\sum \frac{x^{\rho_2}}{\rho_2}$, respectively, and $R_2(2\log N)$ instead of $R_2(\log N)$.

10.57

Effective upper bounds for $d_2(N)$ can be found in [297, p. 51]; for instance

$$\log d_2(N) \leq \frac{(\log 3)(\log N)}{\log\log N}\left(1 + \frac{1}{\log\log N} + \frac{5.5546}{(\log\log N)^2}\right).$$

For a more general study of $d_k(n)$, when k and n tend to infinity, see [126] and [253].

In the final (short) section of Ramanujan's published paper [274], [281, p. 128] on highly composite numbers, he proves that

$$m_2(x) := \max_{n\leq x} \log d(d(n)) \geq (\sqrt{2}\log 4 + o(1))\frac{\sqrt{\log x}}{\log\log x}.$$

As this volume was nearing completion, Y. Buttkewitz, C. Elsholtz, K. Ford, and J.-C. Schlage-Puchta [95] proved an asymptotic formula for $m_2(x)$, namely,

$$m_2(x) = \frac{\sqrt{\log x}}{\log\log x}\left(c + O\left(\frac{\log\log\log x}{\log\log x}\right)\right),$$

where

$$c = \left(8\sum_{j=1}^{\infty} \log^2\left(1 + \frac{1}{j}\right)\right)^{1/2} = 2.7959802335\ldots.$$

10.58

The words in square brackets do not occur in [283], where the definition of $\sigma_{-s}(N)$ and the proof of (10.58.301) are missing. It is not clear why Ramanujan considered $\sigma_{-s}(N)$ only with $s \geq 0$. Of course, he knew that

$$\sigma_s(N) = N^s \sigma_{-s}(N)$$

(see, for instance, Section 10.71, after formula (10.71.382)), but for $s > 0$, the generalized highly composite numbers for $\sigma_s(N)$ are quite different, and for instance, property (10.59.303) does not hold for them.

10.59

It would be better to call these numbers s-generalized highly composite numbers, because their definition depends on s. For $s = 1$, these numbers have been called superabundant by L. Alaoglu and P. Erdős [9], [135], and the generalized superior highly composite numbers have been called colossally abundant. The real solution of $2^s + 4^s + 8^s = 3^s + 9^s$ is approximately 1.6741.

10.60

For $s = 1$, the results of this and the following section are in [9] and [135].

10.62

The references given here, formula (16), and Section 38 are from [274]. For a geometrical interpretation of $\Sigma_{-s}(N)$, see [245, p. 230]. Consider the piecewise linear function $u \mapsto f(u)$ such that for all generalized superior highly composite numbers N, $f(\log N) = \log \sigma_{-s}(N)$. Then for all N,

$$\Sigma_{-s}(N) = \exp\left(f(\log N)\right).$$

Indefinite integrals mean, in fact, definite integrals. For instance, in formula (10.62.320),

$$\int \frac{\varepsilon\pi(x_r)}{x_r} dx_r \quad \text{should be read as} \quad \int_2^{x_r} \frac{\varepsilon\pi(t)}{t} dt.$$

10.64

Formula (10.64.329) is proved in [301, p. 29] from a classical explicit formula in prime number theory.

10.65

There is a misprint in the last term of formula (10.65.340) in [283], but it may be only a mistake of copying, since the next formula is correct. This section belongs to that part of the manuscript that is not in Ramanujan's handwriting in [283].

The approximations given for $1/\sqrt{mn}$ arise from the Padé approximant of $\sqrt{t}$ in the neighborhood of $t = 1$, namely,

$$\sqrt{t} = 1 + \frac{t-1}{2 + \frac{t-1}{2}} = \frac{3t+1}{t+3} = \frac{1}{\frac{1}{3} + \frac{8}{3(3t+1)}}.$$

10.68

There are two formulas (10.68.362) in [283, p. 299]. Formula (10.68.362) can be found in [244]. As observed by Birch [75, p. 74], there is some similarity between the calculations of Sections 10.63–10.68, and those appearing in [283, pp. 228–232].

10.71

There is an incorrect sign in formula (10.71.379) of [283], and also in formulas (10.71.381) and (10.71.382). The two inequalities following formula (10.71.382) are also incorrect. In formula (10.71.380), the right coefficient in the right-hand side is $-\frac{\sqrt{2}}{2}\zeta(1/2)$ instead of $-\sqrt{2}\zeta(1/2)$ in [283]. It follows from (10.71.382) that under the Riemann hypothesis, and for n_0 large enough,

$$n > n_0 \Rightarrow \sigma(n)/n \leq e^{\gamma} \log\log n.$$

It has been shown in [298] that the relation above with $n_0 = 5040$ is equivalent to the Riemann hypothesis.

10.72

Formula (10.72.384) is due to Jacobi, but Ramanujan also discovered it; see Entry 8(ii) in Chapter 17 of his second notebook [282], [55, p. 114] and formula (3.12) in his lost notebook [283, p. 356], [15, Entry 18.2.1]. See also Hardy and Wright's text [168, p. 311] and Hardy's book [166, pp. 132–160]. In [323], B.K Spearman and K.S. Williams give a purely arithmetic proof. Further proofs and references can be found in Berndt's book [60, pp. 59–63, 79]. In formula (10.72.389) of [283], the sign of the second term in the curly brackets is wrong.

10.73

Formula (10.73.390) is proved in [265, p. 198, Equation (90.3)]. Two particularly simple proofs have been given by S.H. Chan [103] and A. Alaca, S. Alaca, and Williams [11]. Chan's proof is reproduced in [60, pp. 63–67]. Further proofs can be found in the paper [223] by E. McAfee and Williams and M. Nathanson's book [237, p. 436]. It is true that if

$$N = 5^{a_5} \cdot 13^{a_{13}} \cdot 17^{a_{17}} \cdots p^{a_p} p',$$

with $p' \sim p$, then $Q_6(N)$ will have the maximal order (10.73.395). But if we define a superior champion for Q_6, that is to say an N that maximizes $Q(N)N^{-2-\varepsilon}$ for an $\varepsilon > 0$, it will be of the form above, with $p' \sim p\sqrt{\frac{\log p}{2}}$. In (10.73.395), the error term was written $O\left(\frac{1}{(\log N)^{3/2} \log\log N}\right)$ in [283]; see [301].

10.74

Formula (10.74.396) is proved in [265, p. 198, Equation (90.4)]. This famous formula of Jacobi was also given by Ramanujan in a fragment published with the lost notebook [283, p. 353, Formula (1.14); p. 356, Formula (3.14)]. See also Entry 18.2.3 of [15], where references to further proofs can be found. A simple arithmetic proof has been given by Spearman and Williams [324]. Further proofs and references can be found in Berndt's book [60, pp. 67–71, 80]. In formula (10.74.401), the sign of the third term in the curly brackets is wrong in [283]. In [283], the right-hand side of (10.74.398) is written as the left-hand side of (10.74.396).

10.75

The formulas (10.75.402), (10.75.403), and (10.75.404) are given in a fragment on Lambert series published with the lost notebook [283, p. 355]. See Entries 18.2.9–18.2.11 in Chapter 18 of [15, pp. 402–403] for these formulas, a proof of (10.75.403), and references to proofs of (10.75.402) and (10.75.404).

One can find (10.75.405) as Entry 8(iii) of Chapter 17 in Ramanujan's second notebook [282], [55, p. 114]. The formulas (10.75.407) and (10.75.408) can also be found in the aforementioned fragment [283, p. 354]. See Entries 18.2.24 and 18.2.25 in [15, p. 407], where references to proofs of the former formula can be found and where a proof of the latter formula can be read.

The formula (10.75.406) is incorrect as it stands. The correct formula is given by

$$\varphi^2(q)\varphi^2(q^2) = 1 + 4\sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} - 8\sum_{n=1}^{\infty} \frac{n(-1)^n q^{4n}}{1-q^{4n}}. \qquad (10.75.409)$$

A different, but equivalent, Lambert series representation for $\varphi^2(q)\varphi^2(q^2)$ can be found in Ramanujan's second notebook [282, p. 266], [58, p. 373, Entry 31]. Equating coefficients of q^n on both sides of (10.75.409), we find that the number $R_2(n)$ of integral solutions of the equation $n = x^2 + y^2 + 2z^2 + 2t^2$, $n \geq 1$, is given by

$$R_2(n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8),$$

where it is understood that if $n/2^j$ is not an integer, then $\sigma(n/2^j) = 0$. This result was first proved by J. Liouville [217], with later proofs provided by T. Pepin [262], [263], P. Bachmann [30, p. 414], J.I. Deutsch [123], S.H. Chan [104, p. 68, Corollary 3.6.4], and Alaca, Alaca, M.F. Lemire, and Williams [10].

TWO FRAGMENTS ON PAGES 309–312

The fragment on pages 309–310

$$N \text{ is highly composite}$$

$$N = e^{\vartheta(p_1)+\vartheta(p_2)+\vartheta(p_3)+\cdots}.$$

$$d(N) = 2^{\pi(p_1)} \left(\frac{3}{2}\right)^{\pi(p_2)} \left(\frac{4}{3}\right)^{\pi(p_3)} \cdots$$

and

$$N' = Ne^{\vartheta(q_1)+\vartheta(q_r)-\vartheta(p_1)-\vartheta(p_r)}.$$

Then

$$d(N') \geq d(N) \cdot 2^{\pi(q_1)-\pi(p_1)} \left(1 + \frac{1}{r}\right)^{\pi(q_r)-\pi(p_r)}.$$

First let us suppose that $q_1 > p_1$, $q_r < p_r$ and

$$\pi(q_1) = \pi(p_1) + n,$$
$$\pi(q_r) = \pi(p_r) - \left[\frac{n \log 2}{\log(1 + 1/r)}\right].$$

Then we have $d(N') \geq d(N)$ and so $N' \geq N$. That is to say

$$\vartheta(q_1) + \vartheta(q_r) \geq \vartheta(p_1) + \vartheta(p_r).$$

But if $\alpha > \beta$, it is easy to see that

$$\{\pi(\alpha) - \pi(\beta)\} \log \alpha \geq \vartheta(\alpha) - \vartheta(\beta) \geq \{\pi(\alpha) - \pi(\beta)\} \log \beta.$$

Hence,

$$\{\pi(q_1) - \pi(p_1)\} \log q_1 \geq \{\pi(p_r) - \pi(q_r)\} \log q_r,$$

or

$$n \log q_1 \geq \left[\frac{n \log 2}{\log(1 + 1/r)}\right] \log q_r.$$

But it is easy to show that, if $\pi(\alpha) = \pi(\beta) + x$ and $x = o\left(\frac{\beta}{\log N}\right)$, then

$$\log \alpha = \log \beta + o(1).$$

Hence,

$$\log q_1 = \log p_1 + o(1), \qquad \log q_r = \log p_r + o(1),$$

and so

$$n \{\log p_1 + o(1)\} \geq \left[\frac{n \log 2}{\log(1 + 1/r)}\right] \{\log p_r + o(1)\},$$

or

$$\log p_1 + o(1) \geq \left\{\frac{\log 2}{\log(1 + 1/r)} + O\left(\frac{1}{n}\right)\right\} \{\log p_r + o(1)\}.$$

In other words,

$$\frac{\log p_1}{\log 2} - \frac{\log p_r}{\log(1 + 1/r)} \geq O\left(r + \frac{\log p_r}{n}\right).$$

Now we can choose n so that $\log p_r = O(nr)$. Hence,

$$\frac{\log p_1}{\log 2} - \frac{\log p_r}{\log(1 + 1/r)} > O(r).$$

Again supposing that $q_1 < p_1$, $q_r > p_r$, and

$$\begin{aligned} \pi(q_1) &= \pi(p_1) - n - 1, \\ \pi(q_r) &= \pi(p_r) + \left[\frac{n \log 2}{\log(1 + 1/r)}\right] \end{aligned}$$

and proceeding as before we can show that

$$\frac{\log p_1}{\log 2} - \frac{\log p_r}{\log(1 + 1/r)} < O(r).$$

It follows that

$$\frac{\log p_r}{\log(1 + 1/r)} = \frac{\log p_1}{\log 2} + O(r). \tag{1}$$

The second fragment is almost identical to approximately the first half of the first fragment.

The fragment on pages 311–312

If

$$N = e^{\vartheta(\alpha)+\vartheta(\beta)+\vartheta(\gamma)+\cdots},$$

then

$$d(N) \geq 2^{\pi(\alpha)} \left(\frac{3}{2}\right)^{\pi(\beta)} \left(\frac{4}{3}\right)^{\pi(\gamma)} \cdots .$$

If $\alpha \geq \beta \geq \gamma \geq \cdots$, then only

$$N = e^{\vartheta(\alpha)+\vartheta(\beta)+\vartheta(\gamma)+\cdots}$$

and

$$d(N) = 2^{\pi(\alpha)} \left(\frac{3}{2}\right)^{\pi(\beta)} \left(\frac{4}{3}\right)^{\pi(\gamma)} \cdots .$$

Let

$$N' = N e^{\vartheta(q_r)+\vartheta(q_s)-\vartheta(p_r)-\vartheta(p_s)}.$$

Then

$$d(N') \geq d(N) \left(1+\frac{1}{r}\right)^{\pi(q_r)-\pi(p_r)} \left(1+\frac{1}{s}\right)^{\pi(q_s)-\pi(p_s)} .$$

Let us suppose that

$$q_r > p_r, \qquad q_s < p_s$$

and

$$\begin{cases} \pi(q_r) &= \pi(p_r) + n, \\ \pi(q_s) &= \pi(p_s) - \left[\dfrac{n \log(1+1/r)}{\log(1+1/s)}\right]. \end{cases} \tag{1}$$

We have then

$$d(N') \geq d(N)$$

and so

$$N' \geq N.$$

That is to say

$$\vartheta(q_r) + \vartheta(q_s) \geq \vartheta(p_r) + \vartheta(p_s).$$

But if $\alpha > \beta$, then

$$\{\pi(\alpha) - \pi(\beta)\} \log \alpha \geq \vartheta(\alpha) - \vartheta(\beta) \geq \{\pi(\alpha) - \pi(\beta)\} \log \beta.$$

Hence,

$$\{\pi(q_r) - \pi(p_r)\} \log q_r \geq \{\pi(p_s) - \pi(q_s)\} \log q_s,$$

or

$$n \log q_r \geq \left[\frac{n \log(1+1/r)}{\log(1+1/s)}\right] \log q_s.$$

Now we have to express q_r and q_s in terms of p_r and p_s respectively.

TABLE OF LARGELY COMPOSITE NUMBERS

The table appearing on page 280 in [283] is a table of largely composite numbers. An integer N is said to be largely composite if whenever $M \leq N$, then $d(M) \leq d(N)$. There are a few errors in the table. The entry 150840 is not a largely composite number; in particular,

$$150840 = 2^3 \cdot 3^2 \cdot 5 \cdot 419 \quad \text{and} \quad d(150840) = 48,$$

while the four numbers $4200, 151200, 415800, 491400$ are largely composite and do not appear in the table of Ramanujan. Largely composite numbers are studied in [243]. The table below is a corrected version of Ramanujan's table [283, p. 280]. The numbers marked with one asterisk are superior highly composite numbers.

n	d		n	d		n	d	
1	1		7560	64	$2^3.3^3.5.7$	942480	240	$2^4.3^2.5.7.11.17$
*2	2	2	9240	64	$2^3.3.5.7.11$	982800	240	$2^4.3^3.5^2.7.13$
3	2	3	10080	72	$2^5.3^2.5.7$	997920	240	$2^5.3^4.5.7.11$
4	3	2^2	12600	72	$2^3.3^2.5^2.7$	1053360	240	$2^4.3^2.5.7.11.19$
*6	4	2.3	13860	72	$2^2.3^2.5.7.11$	1081080	256	$2^3.3^3.5.7.11.13$
8	4	2^3	15120	80	$2^4.3^3.5.7$	1330560	256	$2^7.3^3.5.7.11$
10	4	2.5	18480	80	$2^4.3.5.7.11$	1413720	256	$2^3.3^3.5.7.11.17$
*12	6	$2^2.3$	20160	84	$2^6.3^2.5.7$	*1441440	288	$2^5.3^2.5.7.11.13$
18	6	2.3^2	25200	90	$2^4.3^2.5^2.7$	1663200	288	$2^5.3^3.5^2.7.11$
20	6	$2^2.5$	27720	96	$2^3.3^2.5.7.11$	1801800	288	$2^3.3^2.5^2.7.11.13$
24	8	$2^3.3$	30240	96	$2^5.3^3.5.7$	1884960	288	$2^5.3^2.5.7.11.17$
30	8	$2.3.5$	32760	96	$2^3.3^2.5.7.13$	1965600	288	$2^5.3^3.5^2.7.13$
36	9	$2^2.3^2$	36960	96	$2^5.3.5.7.11$	2106720	288	$2^5.3^2.5.7.11.19$
48	10	$2^4.3$	37800	96	$2^3.3^3.5^2.7$	2162160	320	$2^4.3^3.5.7.11.13$
*60	12	$2^2.3.5$	40320	96	$2^7.3^2.5.7$	2827440	320	$2^4.3^3.5.7.11.17$
72	12	$2^3.3^2$	41580	96	$2^2.3^3.5.7.11$	2882880	336	$2^6.3^2.5.7.11.13$
84	12	$2^2.3.7$	42840	96	$2^3.3^2.5.7.17$	3326400	336	$2^6.3^3.5^2.7.11$
90	12	$2.3^2.5$	43680	96	$2^5.3.5.7.13$	3603600	360	$2^4.3^2.5^2.7.11.13$
96	12	$2^5.3$	45360	100	$2^4.3^4.5.7$	*4324320	384	$2^5.3^3.5.7.11.13$
108	12	$2^2.3^3$	50400	108	$2^5.3^2.5^2.7$	5405400	384	$2^3.3^3.5^2.7.11.13$
*120	16	$2^3.3.5$	*55440	120	$2^4.3^2.5.7.11$	5654880	384	$2^5.3^3.5.7.11.17$
168	16	$2^3.3.7$	65520	120	$2^4.3^2.5.7.13$	5765760	384	$2^7.3^2.5.7.11.13$
180	18	$2^2.3^2.5$	75600	120	$2^4.3^3.5^2.7$	6126120	384	$2^3.3^2.5.7.11.13.17$
240	20	$2^4.3.5$	83160	128	$2^3.3^3.5.7.11$	6320160	384	$2^5.3^3.5.7.11.19$
336	20	$2^4.3.7$	98280	128	$2^3.3^3.5.7.13$	6486480	400	$2^4.3^4.5.7.11.13$
*360	24	$2^3.3^2.5$	110880	144	$2^5.3^2.5.7.11$	7207200	432	$2^5.3^2.5^2.7.11.13$
420	24	$2^2.3.5.7$	131040	144	$2^5.3^2.5.7.13$	8648640	448	$2^6.3^3.5.7.11.13$
480	24	$2^5.3.5$	138600	144	$2^3.3^2.5^2.7.11$	10810800	480	$2^4.3^3.5^2.7.11.13$
504	24	$2^3.3^2.7$	151200	144	$2^5.3^3.5^2.7$	12252240	480	$2^4.3^2.5.7.11.13.17$
540	24	$2^2.3^3.5$	163800	144	$2^3.3^2.5^2.7.13$	12972960	480	$2^5.3^4.5.7.11.13$
600	24	$2^3.3.5^2$	166320	160	$2^4.3^3.5.7.11$	13693680	480	$2^4.3^2.5.7.11.13.19$
630	24	$2.3^2.5.7$	196560	160	$2^4.3^3.5.7.13$	14137200	480	$2^4.3^3.5^2.7.11.17$
660	24	$2^2.3.5.11$	221760	168	$2^6.3^2.5.7.11$	14414400	504	$2^6.3^2.5^2.7.11.13$
672	24	$2^5.3.7$	262080	168	$2^6.3^2.5.7.13$	17297280	512	$2^7.3^3.5.7.11.13$
720	30	$2^4.3^2.5$	277200	180	$2^4.3^2.5^2.7.11$	18378360	512	$2^3.3^3.5.7.11.13.17$
840	32	$2^3.3.5.7$	327600	180	$2^4.3^2.5^2.7.13$	20540520	512	$2^3.3^3.5.7.11.13.19$
1080	32	$2^3.3^3.5$	332640	192	$2^5.3^3.5.7.11$	*21621600	576	$2^5.3^3.5^2.7.11.13$
1260	36	$2^2.3^2.5.7$	360360	192	$2^3.3^2.5.7.11.13$	24504480	576	$2^5.3^2.5.7.11.13.17$
1440	36	$2^5.3^2.5$	393120	192	$2^5.3^3.5.7.13$	27387360	576	$2^5.3^2.5.7.11.13.19$
1680	40	$2^4.3.5.7$	415800	192	$2^3.3^3.5^2.7.11$	28274400	576	$2^5.3^3.5^2.7.11.17$
2160	40	$2^4.3^3.5$	443520	192	$2^7.3^2.5.7.11$	28828800	576	$2^7.3^2.5^2.7.11.13$
*2520	48	$2^3.3^2.5.7$	471240	192	$2^3.3^2.5.7.11.17$	30270240	576	$2^5.3^3.5.7^2.11.13$
3360	48	$2^5.3.5.7$	480480	192	$2^5.3.5.7.11.13$	30630600	576	$2^3.3^2.5^2.7.11.13.17$
3780	48	$2^2.3^3.5.7$	491400	192	$2^3.3^3.5^2.7.13$	31600800	576	$2^5.3^3.5^2.7.11.19$
3960	48	$2^3.3^2.5.11$	498960	200	$2^4.3^4.5.7.11$	32432400	600	$2^4.3^4.5^2.7.11.13$
4200	48	$2^3.3.5^2.7$	554400	216	$2^5.3^2.5^2.7.11$	36756720	640	$2^4.3^3.5.7.11.13.17$
4320	48	$2^5.3^3.5$	655200	216	$2^5.3^2.5^2.7.13$	41081040	640	$2^4.3^3.5.7.11.13.19$
4620	48	$2^2.3.5.7.11$	665280	224	$2^6.3^3.5.7.11$	43243200	672	$2^6.3^3.5^2.7.11.13$
4680	48	$2^3.3^2.5.13$	*720720	240	$2^4.3^2.5.7.11.13$	49008960	672	$2^6.3^2.5.7.11.13.17$
*5040	60	$2^4.3^2.5.7$	831600	240	$2^4.3^3.5^2.7.11$	54774720	672	$2^6.3^2.5.7.11.13.19$

n	d		n	d	
56548800	672	$2^6.3^3.5^2.7.11.17$	232792560	960	$2^4.3^2.5.7.11.13.17.19$
60540480	672	$2^6.3^3.5.7^2.11.13$	245044800	1008	$2^6.3^2.5^2.7.11.13.17$
61261200	720	$2^4.3^2.5^2.7.11.13.17$	273873600	1008	$2^6.3^2.5^2.7.11.13.19$
64864800	720	$2^5.3^4.5^2.7.11.13$	294053760	1024	$2^7.3^3.5.7.11.13.17$
68468400	720	$2^4.3^2.5^2.7.11.13.19$	328648320	1024	$2^7.3^3.5.7.11.13.19$
73513440	768	$2^5.3^3.5.7.11.13.17$	349188840	1024	$2^3.3^3.5.7.11.13.17.19$
82162080	768	$2^5.3^3.5.7.11.13.19$	$*$367567200	1152	$2^5.3^3.5^2.7.11.13.17$
86486400	768	$2^7.3^3.5^2.7.11.13$	410810400	1152	$2^5.3^3.5^2.7.11.13.19$
91891800	768	$2^3.3^3.5^2.7.11.13.17$	465585120	1152	$2^5.3^2.5.7.11.13.17.19$
98017920	768	$2^7.3^2.5.7.11.13.17$	490089600	1152	$2^7.3^2.5^2.7.11.13.17$
99459360	768	$2^5.3^3.5.7.11.13.23$	497296800	1152	$2^5.3^3.5^2.7.11.13.23$
102702600	768	$2^3.3^3.5^2.7.11.13.19$	514594080	1152	$2^5.3^3.5.7^2.11.13.17$
107442720	768	$2^5.3^3.5.7.11.17.19$	537213600	1152	$2^5.3^3.5^2.7.11.17.19$
108108000	768	$2^5.3^3.5^3.7.11.13$	547747200	1152	$2^7.3^2.5^2.7.11.13.19$
109549440	768	$2^7.3^2.5.7.11.13.19$	551350800	1200	$2^4.3^4.5^2.7.11.13.17$
110270160	800	$2^4.3^4.5.7.11.13.17$	616215600	1200	$2^4.3^4.5^2.7.11.13.19$
122522400	864	$2^5.3^2.5^2.7.11.13.17$	698377680	1280	$2^4.3^3.5.7.11.13.17.19$
136936800	864	$2^5.3^2.5^2.7.11.13.19$	735134400	1344	$2^6.3^3.5^2.7.11.13.17$
147026880	896	$2^6.3^3.5.7.11.13.17$	821620800	1344	$2^6.3^3.5^2.7.11.13.19$
164324160	896	$2^6.3^3.5.7.11.13.19$	931170240	1344	$2^6.3^2.5.7.11.13.17.19$
183783600	960	$2^4.3^3.5^2.7.11.13.17$	994593600	1344	$2^6.3^3.5^2.7.11.13.23$
205405200	960	$2^4.3^3.5^2.7.11.13.19$	1029188160	1344	$2^6.3^3.5.7^2.11.13.17$
220540320	960	$2^5.3^4.5.7.11.13.17$	1074427200	1344	$2^6.3^3.5^2.7.11.17.19$

Scratch Work

Introduction

In the introduction, partially on the basis of several pages of scratch work, we tendered the conjecture that Ramanujan had devoted his last hours to cranks before his suffering became too intense to work on mathematics in the last four days of his brief life. In this short appendix we briefly examine some of the pages of scratch work. Most of the scratch work that we can identify pertains to calculations involving theta functions, cranks, or the partition function. We emphasize that these pages contain no exciting discoveries. The scratch work gives us glimpses of some of Ramanujan's thoughts, but perhaps more importantly, the scratch work demonstrates the importance of calculations for Ramanujan.

We discuss pages in the order in which they appear in [283].

Page 61

As discussed in Chapter 4, the first five tables are preliminary versions of the tables given on page 179. These tables are followed by seven lists of numbers, three belonging to residue classes 1 modulo 3, three belonging to residue classes -1 modulo 3, and the last belonging to multiples of 3. However, generally, neither the numbers nor the values of $p(n)$ for these n belong to the requisite residue classes. The seven classes contain a total of 71 numbers, with the largest being 130.

Below the tables is the quotient of (apparently) infinite products

$$\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}.$$

The scratch work at the bottom of the page contains several theta functions depicted by the first several terms of their infinite series representations, in particular,

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*, DOI 10.1007/978-1-4614-3810-6,

$$\begin{aligned}1-q-q^2+q^5+q^7-q^{12}-q^{15}-\cdots &= f(-q),\\ 1-q^2-q^7+q^{13}+q^{23}-q^{33}-q^{48}+\cdots &= f(-q^2,-q^7),\\ 1-q^4-q^5+q^{17}+q^{19}-q^{39}-q^{42}+\cdots &= f(-q^4,-q^5).\end{aligned}$$

We also see

$$1+q^2+q^4+q^6-q^7+q^8-q^9+q^{10}+\cdots=\frac{1-q^7}{1-q^2}.$$

Page 65=73, 66

Inexplicably, the publisher photocopied page 65 twice. Calculations on these pages appear to be related to cranks.

Page 72

Some of the calculations appear to be related to cranks.

Pages 74–77

These pages may be related to the generating functions in Chapters 2 and 3. In the upper left-hand corner of page 76, we find the expressions

$$\begin{aligned}&\frac{1-q^2-q^3+\cdots}{(1-q)^2(1-q^4)^2}, &&\frac{(1-q^5)(1-q^{10})}{(1-q)(1-q^4)},\\ &\frac{(1-q^5)(1-q^{10})}{(1-q^2)(1-q^5)}, &&\frac{1-q-q^4+\cdots}{(1-q^2)^2(1-q^3)^2}.\end{aligned}$$

Page 78

Printed upside down, we find

$$\begin{aligned}(1-q^7-q^8+q^{29}+q^{31}-\cdots)&(1-q^5-q^{10}+q^{25}+q^{35}-\cdots)\\ &=f(-q^7,-q^8)f(-q^5,-q^{10}).\end{aligned}$$

Immediately following, we find

$$1-q-q^4+q^7+q^{13}-q^{18}-q^{27}+q^{34}+q^{46}-\cdots=f(-q,-q^4).$$

We remark that we have inserted $-\cdots$ in each instance above.

Page 79

If the series

$$1 - q + q^2 + q^6 + q^8 - q^9 + q^{10} - q^{11} + 2q^{12}$$

were extended to infinity, it would equal

$$\frac{(q;q^{10})_\infty(q^9;q^{10})_\infty(q^{10};q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{f(-q^5)f(-q,-q^9)}{f(-q^2,-q^3)}.$$

In the middle of the page, we see

$$1 + q + q^2 + q^3 + 2q^4 + q^6 + q^7 + 2q^8 + q^9,$$

which, if the summands were extended to infinity, would be equal to

$$\frac{(q^5;q^5)_\infty^2}{(q;q^5)_\infty(q^4;q^5)_\infty} = \frac{f^3(-q^5)}{f(-q,-q^4)}.$$

Page 80

This page appears to be related to calculations related to cranks. Reading sideways, we again see many functions identical to or similar to others that we have seen before, including

$$\frac{f(-q^2,-q^3)}{(q;q^5)_\infty^2(q^4;q^5)_\infty^2}, \qquad \frac{(q^5;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}, \qquad \frac{(q^5;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Page 81

This page may be related to Chapters 2 and 3.

Page 82

At the top of the page we see

$$\begin{aligned} &1 + q - q^2 - q^5 + q^7 + q^{12} - q^{15} - q^{22} + q^{26} + q^{35} - q^{40} - \dots \\ &= \sum_{n=-\infty}^{0} (-1)^n q^{n(3n-1)/2} - \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2}, \end{aligned}$$

which is a false theta function.

Pages 83–85

These pages are likely related to calculations involving cranks.

Pages 86–89

These pages contain a table of the residue classes of $p(n)$ for n from 1 to nearly 200. The first column is n, the second is $p(n) \pmod 2$, the third is $p(n) \pmod 3$, the fourth is $p(n) \pmod 5$, the fifth is $p(n) \pmod 7$, and the sixth is $p(n) \pmod{11}$. For the first 17 values of n, the residue of $p(n)$ modulo 13, 17, 19, and 23 are also given. On the right side of page 87, Ramanujan also tabulates the number of values in each residue class modulo 2, 3, 5, and 7 for the first and second 50 values of n. On page 86, Ramanujan appears to have calculated residues modulo various n for some arithmetic function that we cannot identify.

Page 213

This page contains a small amount of scratch work on Eisenstein series, but no theorem is offered.

Location Guide

For each page of Ramanujan's lost notebook on which we have discussed or proved entries in this book, we provide below a list of those chapters or entries in which these pages are discussed.

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*,
DOI 10.1007/978-1-4614-3810-6,

Provenance

Chapter 2

A.O.L. Atkin and H.P.F. Swinnerton-Dyer, [28]
F.G. Garvan, [144], [146]

Chapter 3

B.C. Berndt, H.H. Chan, S.H. Chan, and W.-C. Liaw, [62]

Chapter 4

B.C. Berndt, H.H. Chan, S.H. Chan, and W.-C. Liaw, [63], [64]
W.-C. Liaw, [216]

Chapter 5

B.C. Berndt and K. Ono, [67]

Chapter 6

B.C. Berndt, A.J. Yee, and J. Yi, [70]
B.C. Berndt, C. Gugg, and S. Kim, [66]

Chapter 7

J.M. Rushforth, [305], [306]

Chapter 8

B.C. Berndt, G. Choi, Y.-S. Choi, H. Hahn, B. P. Yeap, A. J. Yee, H. Yesilyurt, and J. Yi, [65]
B.C. Berndt and H. Yesilyurt, [73]
H. Yesilyurt, [347], [348]

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DOI 10.1007/978-1-4614-3810-6,

Chapter 9

A. Berkovich, F.G. Garvan, and H. Yesilyurt, [53]
H.H. Chan, Z.-G. Liu, and S.T. Ng, [101]
S.H. Son, [320]
P. Xu, [345]

Chapter 10

J.-L. Nicolas and G. Robin, [250]

References

1. S. Ahlgren, *Distribution of parity of the partition function in arithmetic progressions*, Indag. Math. **10** (1999), 173–181.
2. S. Ahlgren, *Non-vanishing of the partition function modulo odd primes ℓ*, Mathematika **46** (1999), 185–192.
3. S. Ahlgren, *The partition function modulo composite integers M*, Math. Ann. **318** (2000), 795–803.
4. S. Ahlgren, *The sixth, eighth, ninth and tenth powers of Ramanujan's theta function*, Proc. Amer. Math. Soc. **128** (2000), 1333–1338.
5. S. Ahlgren and M. Boylan, *Arithmetic properties of the partition function*, Invent. Math. **153** (2003), 487–502.
6. S. Ahlgren and M. Boylan, *Coefficients of half-integral weight modular forms modulo ℓ*, Math. Ann. **331** (2005), 219–241.
7. S. Ahlgren and K. Ono, *Addition and counting: The arithmetic of partitions*, Notices Amer. Math. Soc. **48** (2001), 978–984.
8. S. Ahlgren and S. Treneer, *Rank generating functions as weakly holomorphic modular forms*, Acta Arith. **133** (2008), 267–279.
9. L. Alaoglu and P. Erdős, *On highly composite and similar numbers*, Trans. Amer. Math. Soc. **56** (1944), 448–469.
10. A. Alaca, S. Alaca, M.F. Lemire, and K.S. Williams, *Nineteen quatenary quadratic forms*, Acta Arith. **130** (2007), 277–310.
11. A. Alaca, S. Alaca, and K.S. Williams, *The simplest proof of Jacobi's six squares theorem*, Far East J. Math. Sci. **27** (2007), 187–192.
12. G.E. Andrews, *The Theory of Partitions*, Addison–Wesley, Reading, MA, 1976; reissued: Cambridge University Press, Cambridge, 1998.
13. G.E. Andrews, *Partitions: Yesterday and Today*, The New Zealand Mathematical Society, Wellington, 1979.
14. G.E. Andrews, *Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks*, Invent. Math. **169** (2007), 37–73.
15. G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook, Part* I, Springer, New York, 2005.
16. G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook, Part* II, Springer, New York, 2009.

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*,
DOI 10.1007/978-1-4614-3810-6,

17. G.E. Andrews and F.G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. **18** (1988), 167–171.
18. G.E. Andrews and D. Hickerson, *Ramanujan's "Lost" Notebook VII: The sixth order mock theta functions*, Adv. Math. **89** (1991), 60–105.
19. G.E. Andrews and R. Lewis, *The ranks and cranks of partitions moduli* 2, 3, *and* 4, J. Number Theory **85** (2000), 74–84.
20. G.E. Andrews and K. Ono, *Ramanujan's congruences and Dyson's crank* (commentary), Proc. Natl. Acad. Sci. USA **102** (2005), 15277.
21. G.E. Andrews and R. Roy, *Ramanujan's method in q-series congruences*, Elec. J. Combin. **4**(2) (1997), R2, 7pp.
22. T. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, New York, 1990.
23. A.O.L. Atkin, *Proof of a conjecture of Ramanujan*, Glasgow Math. J. **8** (1967), 14–32.
24. A.O.L. Atkin, *Multiplicative congruence properties and density problems for* $p(n)$, Proc. London Math. Soc. **18** (1968), 563–576.
25. A.O.L. Atkin, *Ramanujan congruences for* $p_k(n)$, Canad. J. Math. **20** (1968), 67–78.
26. A.O.L. Atkin and F.G. Garvan, *Relations between the ranks and cranks of partitions*, Ramanujan J. **7** (2003), 343–366.
27. A.O.L. Atkin and J.N. O'Brien, *Some properties of* $p(n)$ *and* $c(n)$ *modulo powers of* 13, Trans. Amer. Math. Soc. **126** (1967), 442–459.
28. A.O.L. Atkin and H.P.F. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. (3) **4** (1954), 84–106.
29. F.C. Auluck, S. Chowla, and H. Gupta, *On the maximum value of the number of partitions of* n *into* k *parts*, J. Indian Math. Soc. (N.S.) **6** (1942), 105–112.
30. P. Bachmann, *Niedere Zahlentheorie*, Chelsea, New York, 1968.
31. W.N. Bailey, *A note on two of Ramanujan's formulae*, Quart. J. Math. (Oxford) **3** (1952), 29–31.
32. W.N. Bailey *A further note on two of Ramanujan's formulae*, Quart. J. Math. (Oxford) **3** (1952), 158–160.
33. R.P. Bambah, *Two congruence properties of Ramanujan's function* $\tau(n)$, J. London Math. Soc. **21** (1946), 91–93.
34. R.P. Bambah, *Ramanujan's function* $\tau(n)$*—a congruence property*, Bull. Amer. Math. Soc. **53** (1947), 764–765.
35. R.P. Bambah and S. Chowla, *A congruence property of Ramanujan's function* $\tau(n)$, Proc. Nat. Inst. Sci. India **12** (1946), 431–432.
36. R.P. Bambah and S. Chowla, *Some new congruence properties of Ramanujan's function* $\tau(n)$, Math. Student **14** (1946), 24–26.
37. R.P. Bambah and S. Chowla, *A new congruence property of Ramanujan's function* $\tau(n)$, Bull. Amer. Math. Soc. **53** (1947), 768–769.
38. R.P. Bambah and S. Chowla, *Congruence properties of Ramanujan's function* $\tau(n)$, Bull. Amer. Math. Soc. **53** (1947), 950–955.
39. R.P. Bambah and S. Chowla, *The residue of Ramanujan's function* $\tau(n)$ *to the modulus* 2^8, J. London Math. Soc. **22** (1947), 140–147.
40. R.P. Bambah, S. Chowla, and H. Gupta, *A congruence property of Ramanujan's function* $\tau(n)$, Bull. Amer. Math. Soc. **53** (1947), 766–767.
41. R.P. Bambah, S. Chowla, H. Gupta, and D.B. Lahiri, *Congruence properties of Ramanujan's function* $\tau(n)$, Quart. J. Math. **18** (1947), 143–146.

42. D.P. Banerji, *Congruence properties of Ramanujan's function* $\tau(n)$, J. London Math. Soc. **17** (1942), 144–145.
43. N.D. Baruah and J. Bora, *New proofs of Ramanujan's modular equations of degree* 9, Indian J. Math. **47** (2005), 99–122.
44. N.D. Baruah and J. Bora, *Further analogues of the Rogers–Ramanujan functions with applications to partitions*, Elec. J. Combin. Number Thy. **7**(2) (2007), #A05, 22 pp.
45. N.D. Baruah and J. Bora, *Modular relations for the nonic analogues of the Rogers–Ramanujan functions with applications to partitions*, J. Number Thy. **128** (2008), 175–206.
46. N.D. Baruah, J. Bora, and N. Saikia, *Some new proofs of modular identities for Göllnitz–Gordon functions*, Ramanujan J. **15** (2008), 281–301.
47. N.D. Baruah and K.K. Ojah, *Some congruences deducible from Ramanujan's cubic continued fraction*, Internat. J. Number Thy. **7** (2011), 1331–1343.
48. A. Berkovich and F.G. Garvan, *Some observations on Dyson's new symmetries of partitions*, J. Combin. Thy., Ser. A **100** (2002), 61–93.
49. A. Berkovich and F.G. Garvan, *On the Andrews-Stanley refinement of Ramanujan's partition congruence modulo* 5 *and generalizations*, Trans. Amer. Math. Soc. **358** (2006), 703–726.
50. A. Berkovich and F.G. Garvan, *The BG-rank of a partition and its applications*, Adv. Appl. Math. **40** (2008), 377–400.
51. A. Berkovich and F.G. Garvan, *K. Saito's conjecture for nonnegative eta products and analogous results for other infinite products*, J. Number Thy. **128** (2008), 1731–1748.
52. A. Berkovich and H. Yesilyurt, *Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms*, Ramanujan J. **20** (2009), 375–408.
53. A. Berkovich, F.G. Garvan, and H. Yesilyurt, manuscript in preparation.
54. B.C. Berndt, *Ramanujan's Notebooks, Part* I, Springer-Verlag, New York, 1985.
55. B.C. Berndt, *Ramanujan's Notebooks, Part* III, Springer-Verlag, New York, 1991.
56. B.C. Berndt, *On a certain theta-function in a letter of Ramanujan from Fitzroy House*, Ganita **43** (1992), 33–43.
57. B. C. Berndt, *Ramanujan's Notebooks, Part* IV, Springer-Verlag, New York, 1994.
58. B.C. Berndt, *Ramanujan's Notebooks, Part* V, Springer-Verlag, New York, 1998.
59. B.C. Berndt, *Fragments by Ramanujan on Lambert series*, in *Number Theory and Its Applications*, S. Kanemitsu and K. Győry, eds., Kluwer, Dordrecht, 1999, pp. 35–49.
60. B.C. Berndt, *Number Theory in the Spirit of Ramanujan*, American Mathematical Society, Providence, RI, 2006.
61. B.C. Berndt, S. Bhargava, and F.G. Garvan, *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc. **347** (1995), 4163–4244.
62. B.C. Berndt, H.H. Chan, S.H. Chan, and W.-C. Liaw, *Cranks and dissections in Ramanujan's lost notebook*, J. Combin. Thy., Ser. A **109** (2005), 91–120.
63. B.C. Berndt, H.H. Chan, S.H. Chan, and W.-C. Liaw, *Ramanujan and cranks*, in *Theory and Applications of Special Functions. A Volume Dedicated to Mizan*

Rahman, M.E.H. Ismail and E. Koelink, eds., Springer, New York, 2005, pp. 77–98.

64. B.C. Berndt, H.H. Chan, S.H. Chan, and W.-C. Liaw, *Cranks—really the final problem*, Ramanujan J. **23** (2010), 3–15.
65. B.C. Berndt, G. Choi, Y.-S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi, *Ramanujan's forty identities for the Rogers–Ramanujan functions*, Memoir No. 880, **188** American Mathematical Society, Providence, RI, 2007.
66. B.C. Berndt, C. Gugg, and S. Kim, *Ramanujan's elementary method in partition congruences*, in *Partitions, q-Series and Modular Forms*, K. Alladi and F. Garvan, eds., Develop. in Math. **23**, 2011, Springer, New York, pp. 13–22.
67. B.C. Berndt and K. Ono, *Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary*, Sém. Lotharingien de Combinatoire **42** (1999), 63 pp.; in *The Andrews Festschrift*, D. Foata and G.-N. Han, eds., Springer-Verlag, Berlin, 2001, pp. 39–110.
68. B.C. Berndt and R.A. Rankin, *Ramanujan: Letters and Commentary*, American Mathematical Society, Providence, RI, 1995; London Mathematical Society, London, 1995.
69. B.C. Berndt and R.A. Rankin, *Ramanujan: Essays and Surveys*, American Mathematical Society, Providence, 2001; London Mathematical Society, London, 2001.
70. B.C. Berndt, A.J. Yee, and J. Yi, *Theorems on partitions from a page in Ramanujan's lost notebook*, J. Comp. Appl. Math. **160** (2003), 53–68.
71. B.C. Berndt, A.J. Yee, and A. Zaharescu, *On the parity of partition functions*, Internat. J. Math. **14** (2003), 437–459.
72. B.C. Berndt, A.J. Yee, and A. Zaharescu, *New theorems on the parity of partition functions*, J. Reine Angew. Math. **566** (2004), 91–109.
73. B.C. Berndt and H. Yesilyurt, *New identities for the Rogers–Ramanujan functions*, Acta Arith. **120** (2005), 395–413.
74. A.J.F. Biagioli, *A proof of some identities of Ramanujan using modular forms*, Glasgow Math. J. **31** (1989), 271–295.
75. B.J. Birch, *A look back at Ramanujan's notebooks*, Proc. Cambridge Philos. Soc. **78** (1975), 73–79.
76. R. Blecksmith, J. Brillhart, and I. Gerst, *Some infinite product identities*, Math. Comp. **51** (1988), 301–314.
77. R. Blecksmith, J. Brillhart, and I. Gerst, *A fundamental modular identity and some applications*, Math. Comp. **61** (1993), 83–95.
78. F. van der Blij, *The function $\tau(n)$ of S. Ramanujan* (*an expository lecture*), Math. Student **18** (1950), 83–99.
79. J.M. Borwein and P.B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc. **323** (1991), 691–701.
80. J.M. Borwein, P.B. Borwein, and F.G. Garvan, *Some cubic modular identities of Ramanujan*, Trans. Amer. Math. Soc. **343** (1994), 35–47.
81. D. Bressoud, *Proof and Generalization of Certain Identities Conjectured by Ramanujan*, Ph.D. Thesis, Temple University, 1977.
82. D. Bressoud, *Some identities involving Rogers–Ramanujan-type functions*, J. London Math. Soc. (2) **16** (1977), 9–18.
83. K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J. **144** (2008), 195–233.

84. K. Bringmann, *Congruences for Dyson's ranks*, Internat. J. Number Thy. **5** (2009), 573–584.
85. K. Bringmann, *Asymptotics for rank partition functions*, Trans. Amer. Math. Soc. **361** (2009), 3483–3500.
86. K. Bringmann, F. Garvan, and K. Mahlburg, *Partition statistics and quasiharmonic Maass forms*, Internat. Math. Res. Notices (IMRN) (2009), 63–97.
87. K. Bringmann and B. Kane, *Inequalities for differences of Dyson's rank for all odd moduli*, Math. Res. Lett. **17** (2010), 927–942.
88. K. Bringmann and J. Lovejoy, *Rank and congruences for overpartition pairs*, Internat. J. Number Thy. **4** (2008), 303–322.
89. K. Bringmann, J. Lovejoy, and R. Osburn, *Automorphic properties of generating functions for generalized rank moments and Durfee symbols*, Internat. Math. Res. Notices (IMRN) (2010), 238–260.
90. K. Bringmann and K. Mahlburg, *Inequalities between ranks and cranks*, Proc. Amer. Math. Soc. **137** (2009), 2567–2574.
91. K. Bringmann, K. Mahlburg, and R.C. Rhoades, *Asymptotics for rank and crank moments*, Bull. London Math. Soc. **43** (2011), 661–672.
92. K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. Math. (2) **171** (2010), 419–449.
93. K. Bringmann and H. Swisher, *On a conjecture of Koike on identities between Thompson series and Rogers–Ramanujan functions*, Proc. Amer. Math. Soc. **135** (2007), 2317–2326.
94. K. Bringmann and S. Zwegers, *Rank-Crank type PDE's and non-holomorphic Jacobi forms*, Math. Res. Lett. **17** (2010), 589–600.
95. Y. Buttkewitz, C. Elsholtz, K. Ford, and J.-C. Schlage-Puchta, *A problem of Ramanujan, Erdős and Kátai on the iterated divisor function*, Internat. Math. Res. Notices (IMRN) (2012), doi:10.1093/imrn/rnr175.
96. Z. Cao, *Integer matrix exact covering systems and product identities for theta functions*, Internat. Math. Res. Notices (IMRN) (2011), 4471–4514.
97. H.H. Chan, *New proofs of Ramanujan's partition identities for moduli* 5 *and* 7, J. Number Thy. **53** (1995), 144–159.
98. H.H. Chan, *Email correspondence with B.C. Berndt*, February, 2002.
99. H.H. Chan, S. Cooper, and W.-C. Liaw, *On* $\eta^3(a\tau)\eta^3(b\tau)$ *with* $a+b=8$, J. Austral. Math. Soc. **84** (2008), 301–313.
100. H.H. Chan, H. Hahn, R.P. Lewis, and S.L. Tan, *New Ramanujan–Kolberg type partition identities*, Math. Res. Lett. **9** (2002), 801–811.
101. H.H. Chan, Z.-G. Liu, and S.T. Ng, *Circular summation of theta functions in Ramanujan's lost notebook*, J. Math. Anal. Appl. **316** (2006), 628–641.
102. O.-Y. Chan, *Some asymptotics for cranks*, Acta Arith. **120** (2005), 107–143.
103. S.H. Chan, *An elementary proof of Jacobi's six squares theorem*, Amer. Math. Monthly **111** (2004), 806–811.
104. S.H. Chan, *On Cranks of Partitions, Generalized Lambert Series, and Basic Hypergeometric Series*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, Urbana, 2005.
105. S.H. Chan, *Generalized Lambert series*, Proc. London Math. Soc. (3) **91** (2005), 598–622.
106. S.H. Chan, *Some congruences for Andrews–Paule's broken* 2*-diamond partitions*, Discrete Math. **308** (2008), 5735–5741.

107. S.H. Chan, A. Dixit, and F.G. Garvan, *Rank–crank type PDEs and generalized Lambert series identities*, Ramanujan J., to appear.
108. S.H. Chan and Z.-G. Liu, *On a new circular summation of theta functions*, J. Number Thy. **130** (2010), 1190–1196.
109. S.-L. Chen and S.-S. Huang, *New modular relations for the Göllnitz–Gordon functions*, J. Number Thy. **93** (2002), 58–75.
110. D. Choi, S.-Y. Kang, and J. Lovejoy, *Partitions weighted by the parity of the crank*, J. Combin. Thy., Ser. A **116** (2009), 1034–1046.
111. S. Chowla, *Outline of a new method for proving results of elliptic function theory* (*such as identities of the Ramanujan–Rademacher–Zuckerman type*), Proc. Lahore Philos. Soc. **7** (1945), 45–55.
112. S. Chowla, *On a theorem of Walfisz*, J. London Math. Soc. **22** (1947), 136–140.
113. S. Chowla, *Collected Papers of Sarvadaman Chowla*, Vol. II, J.G. Huard and K.S. Williams, eds., Les Publications CRM, Montreal, 1999.
114. W. Chu, *Common source of numerous theta function identities*, Glasgow Math. J. **49** (2007), 61–79.
115. K.S. Chua, *Circular summation of the* 13*th powers of Ramanujan's theta function*, Ramanujan J. **5** (2001), 353–354.
116. K.S. Chua, *The root lattice* A_n^* *and Ramanujan's circular summation of theta functions*, Proc. Amer. Math. Soc. **130** (2002), 1–8.
117. K.S. Chua, *Explicit congruences of the partition function modulo every prime*, Arch. Math. **81** (2003), 11–21.
118. H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, in *Modular Functions of One Variable* VI, Lecture Notes in Math. No. 627, Springer-Verlag, Berlin, 1977, pp. 69–78.
119. T. Dai and X. Ma, *A bivariate circular summation formula for cubic theta functions and its implications*, J. Ramanujan Math. Soc. **26** (2011), 237–259.
120. H.B.C. Darling, *Proofs of certain identities and congruences enunciated by S. Ramanujan*, Proc. London Math. Soc. (2) **19** (1921), 350–372.
121. H. Delange, *Généralisation du théorème de Ikehara*, Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 213–242.
122. P. Deligne, *Formes modulaires et représentations* ℓ*-adiques*, in *Séminaire Bourbaki* (1968/1969), Exp. No. 355, Lecture Notes in Math. No. 179, Springer-Verlag, Berlin, 1971, pp. 139–172.
123. J.I. Deutsch, *A quaternionic proof of the representation formula of a quatenary quadratic form*, J. Number Thy. **113** (2005), 149–179.
124. M. Dewar, *The nonholomorphic parts of certain weak Maass forms*, J. Number Thy. **130** (2010), 559–573.
125. J.M. Dobbie, *A simple proof of some partition formulae of Ramanujan's*, Quart. J. Math. (Oxford) **6** (1955), 193–196.
126. J.-L. Duras, J.-L. Nicolas, and G. Robin, *Grandes valeurs de la fonction* d_k, in *Number Theory in Progress*, Vol. 2 (Zakopane, Poland), J. Urbanowicz, K. Győry, and H. Iwaniec, eds., Walter de Gruyter, Berlin, 1999, pp. 743–770.
127. F.J. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8** (1944), 10–15.
128. F.J. Dyson, *A walk through Ramanujan's garden*, in *Ramanujan Revisited*, G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, and R.A. Rankin, eds., Academic Press, Boston, 1988, pp. 7–28.

129. F.J. Dyson, *Mappings and symmetries of partitions*, J. Combin. Thy., Ser. A **51** (1989), 169–180.
130. D.E. Eichhorn, *Comments on p.* 178 *of Ramanujan's lost notebook: Some congruences for tau functions*, unpublished report on research assistantship under B.C. Berndt, April 23, May 5, 1999.
131. A.B. Ekin, *The rank and the crank modulo* 5, Turkish J. Math. **21** (1997), 169–178.
132. A.B. Ekin, *Inequalities for the crank*, J. Combin. Thy., Ser. A **83** (1998), 283–289.
133. A.B. Ekin, *Some properties of partitions in terms of crank*, Trans. Amer. Math. Soc. **352** (2000), 2145–2156.
134. P. Erdős and J. Lehner, *The distribution of the number of summands in the partitions of a positive integer*, Duke Math. J. **8** (1941), 335–345.
135. P. Erdős and J.-L. Nicolas, *Répartition des nombres superabondants*, Bull. Soc. Math. France **103** (1975), 113–122.
136. R.J. Evans, *Generalized Lambert series*, in *Analytic Number Theory* (Allerton Park, IL, 1995), Vol. 1, B.C. Berndt, H.G. Diamond, and A.J. Hildebrand, eds., Birkhäuser, Boston, 1996, pp. 357–370.
137. N.J. Fine, *On a system of modular functions connected with the Ramanujan identities*, Tôhoku J. Math. **8** (1956), 149–164.
138. A. Folsom, Z. Kent, and K. Ono, *ℓ-adic properties of the partition function*, Adv. Math. **229** (2012), 1586–1609.
139. R. Fricke, *Lehrbuch der Algebra*, Bd. 3, F. Vieweg, Braunschweig, 1928.
140. J.M. Gandhi, *Congruences for $p_r(n)$ and Ramanujan's τ function*, Amer. Math. Monthly **70** (1963), 265–274.
141. S.A. Garthwaite, *Convolution congruences for the partition function*, Proc. Amer. Math. Soc. **135** (2007), 13–20.
142. F.G. Garvan, *Some congruence properties of the partition function*, M.S. Thesis, University of New South Wales, Sydney, Australia, 1982.
143. F.G. Garvan, *A simple proof of Watson's partition congruences for powers of* 7, J. Austral. Math. Soc. **36** (1984), 316–334.
144. F.G. Garvan, *Generalizations of Dyson's Rank*, Ph.D. Thesis, Pennsylvania State University, University Park, PA, 1986.
145. F.G. Garvan, *Combinatorial interpretations of Ramanujan's partition congruences*, in *Ramanujan Revisited*, G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, and R.A. Rankin, eds., Academic Press, Boston, 1988, pp. 29–45.
146. F.G. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod* 5, 7 *and* 11, Trans. Amer. Math. Soc. **305** (1988), 47–77.
147. F.G. Garvan, *The crank of partitions mod* $8, 9$ *and* 10, Trans. Amer. Math. Soc. **322** (1990), 79–94.
148. F.G. Garvan, *Generalizations of Dyson's rank and non-Rogers-Ramanujan partitions*, Manuscripta Math. **84** (1994), 343–359.
149. F.G. Garvan, *More cranks and t-cores*, Bull. Austral. Math. Soc. **63** (2001), 379–391.
150. F.G. Garvan, *Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank*, Internat. J. Number Thy. **6** (2010), 281–309.

151. F.G. Garvan, *Biranks for partitions into* 2 *colors*, in *Ramanujan Rediscovered: Proceedings of a Conference on Elliptic Functions, Partitions, and q-Series in memory of K. Venkatachaliengar: Bangalore,* 1–5 *June,* 2009, N.D. Baruah, B.C. Berndt, S. Cooper, T. Huber, and M.J. Schlosser, eds., Ramanujan Mathematical Society, Mysore, 2010, pp. 89–111.
152. F.G. Garvan, *Higher order spt-functions*, Adv. Math. **228** (2011), 241–265.
153. F.G. Garvan, D. Kim, and D. Stanton, *Cranks and t–cores*, Invent. Math. **101** (1990), 1–17.
154. F. Garvan and D. Stanton, *Sieved partition functions and q-binomial coefficients*, Math. Comp. **55** (1990), 299–311.
155. B. Gordon, *Ramanujan congruences for p_{-k} mod* 11^r, Glasgow Math. J. **24** (1983), 107–123.
156. B. Gordon and R.J. McIntosh, *Modular transformations of Ramanujan's fifth and seventh order mock theta functions*, Ramanujan J. **7** (2003), 193–222.
157. B. Gordon and K. Ono, *Divisibility of certain partition functions by powers of primes*, Ramanujan J. **1** (1997), 25–34.
158. B. Gordon and D. Sinor, *Multiplicative properties of η-products*, in *Number Theory, Madras* 1987, K. Alladi, ed., Lecture Notes in Math. No. 1395, Springer-Verlag, New York, 1988, pp. 173–200.
159. C. Gugg, *Two modular equations for squares of the Rogers–Ramanujan functions with applications*, Ramanujan J. **18** (2009), 183–207.
160. C. Gugg, *A new proof of Ramanujan's modular equation relating $R(q)$ with $R(q^5)$*, Ramanujan J. **20** (2009), 163–177.
161. C. Gugg, *Modular Identities for the Rogers–Ramanujan Functions and Analogues*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, Urbana, 2010.
162. C. Gugg, *Modular equations for cubes of the Rogers–Ramanujan and Ramanujan–Göllnitz–Gordon functions and their associated continued fractions*, J. Number Thy. **132** (2012), 1519–1553.
163. H. Gupta, *A congruence relation between $\tau(n)$ and $\sigma(n)$*, J. Indian Math. Soc. **9** (1945), 59–60.
164. H. Gupta, *A congruence property of $\tau(n)$*, Proc. Indian Acad. Sci. A **24** (1946), 441–442.
165. H. Hahn, *Septic analogues of the Rogers–Ramanujan functions*, Acta Arith. **110** (2003), 381–399.
166. G.H. Hardy, *Ramanujan*, Cambridge University Press, Cambridge, 1940; reprinted by Chelsea, New York, 1960; reprinted by the American Mathematical Society, Providence, RI, 1999.
167. G.H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115.
168. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Clarendon Press, Oxford, 1979.
169. C.B. Haselgrove and H.N.V. Temperley, *Asymptotic formulae in the theory of partitions*, Proc. Cambridge Philos. Soc. **50** (1954), 225–241.
170. E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung* I, Math. Ann. **114** (1937), 1–28.
171. D. Hickerson, *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), 639–660.

172. M.D. Hirschhorn, *A simple proof of an identity of Ramanujan*, J. Austral. Math. Soc. Ser. A **34** (1983), 31–35.
173. M.D. Hirschhorn, *A birthday present for Ramanujan*, Amer. Math. Monthly **97** (1990), 398–400.
174. M.D. Hirschhorn, *On the parity of* $p(n)$ II, J. Combin. Thy., Ser. A **62** (1993), 128–138.
175. M.D. Hirschhorn, *Ramanujan's partition congruences*, Discrete Math. **131** (1994), 351–355.
176. M.D. Hirschhorn, *Another short proof of Ramanujan's mod* 5 *partition congruence, and more*, Amer. Math. Monthly **106** (1999), 580–583.
177. M.D. Hirschhorn, *An identity of Ramanujan, and applications*, in *q-Series from a Contemporary Perspective*, M.E.H. Ismail and D. Stanton, eds., American Mathematical Society, Providence, RI, 2000, pp. 229–234.
178. M.D. Hirschhorn, *Winquist and the Atkin–Swinnerton-Dyer partition congruences for modulus* 11, Australasian J. Comb. **22** (2000), 101–104.
179. M.D. Hirschhorn, *On the* 2- *and* 4-*dissections of Ramanujan's continued fraction and its reciprocal*, Ramanujan J. **24** (2011), 85–92.
180. M.D. Hirschhorn, *Personal communication*, October 24, 2011.
181. M.D. Hirschhorn and D.C. Hunt, *A simple proof of the Ramanujan conjecture for powers of* 5, J. Reine Angew. Math. **336** (1981), 1–17.
182. M.D. Hirschhorn and M.V. Subbarao, *On the parity of* $p(n)$, Acta Arith. **50** (1988), 355–356.
183. S.-S. Huang, *On modular relations for the Göllnitz–Gordon functions with applications to partitions*, J. Number Thy. **68** (1998), 178–216.
184. K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, 1982.
185. M. Jackson, *On some formulae in partition theory, and bilateral basic hypergeometric series*, J. London Math. Soc. **24** (1949), 233–237.
186. S.J. Kaavya, *Crank* 0 *partitions and the parity of the partition function*, Internat. J. Number Thy. **7** (2011), 793–801.
187. V.G. Kač and D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. Math. **53** (1984), 125–264.
188. V.G. Kač and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, in *Lie Theory and Geometry*, Birkhäuser, Boston, 1994, pp. 415–456.
189. D.M. Kane, *Resolution of a conjecture of Andrews and Lewis involving cranks of partitions*, Proc. Amer. Math. Soc. **132** (2004), 2247–2256.
190. S.-Y. Kang, *Some theorems on the Rogers–Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook*, Ramanujan J. **3** (1999), 91–111.
191. W.J. Keith, *Distribution of the full rank in residue classes for odd moduli*, Discrete Math. **309** (2009), 4960–4968.
192. B. Kim, *An analog of crank for a certain kind of partition function arising from the cubic continued fraction*, Acta Arith. **148** (2011), 1–19.
193. I. Kiming and J. Olsson, *Congruences like Ramanujan's for powers of the partition function*, Arch. Math. **59** (1992), 348–360.
194. M.I. Knopp, *Modular Functions in Analytic Number Theory*, Chelsea, New York, 1993.

195. N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, New York, 1984.
196. M. Koike, *Thompson series and Ramanujan's identities*, in *Galois Theory and Modular Forms*, K. Hashimoto, K. Miyake, and H. Nakamura, eds., Kluwer, Dordrecht, 2003, pp. 367–374.
197. O. Kolberg, *Some identities involving the partition function*, Math. Scand. **5** (1957), 77–92.
198. O. Kolberg, *Note on the parity of the partition function*, Math. Scand. **7** (1959), 377–378.
199. O. Kolberg, *Note on Ramanujan's function* $\tau(n)$, Math. Scand. **10** (1962), 171–172.
200. L.W. Kolitsch, *Generalized Frobenius partitions, k-cores, k-quotients, and cranks*, Acta Arith. **62** (1992), 97–102.
201. D. Kruyswijk, *On some well-known properties of the partition function* $p(n)$ *and Euler's infinite product*, Nieuw Arch. Wisk. **23** (1950), 97–107.
202. D.B. Lahiri, *On Ramanujan's function* $\tau(n)$ *and the divisor function* $\sigma_k(n)$, II, Bull. Calcutta Math. Soc. **39** (1947), 33–52.
203. E. Landau, *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*, Arch. Math. Phys. **13** (1908), 305–312.
204. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Chelsea, New York, 1953.
205. D.H. Lehmer, *The vanishing of Ramanujan's function* $\tau(n)$, Duke Math. J. **14** (1947), 429–433.
206. D.H. Lehmer, *Some functions of Ramanujan*, Math. Student **27** (1959), 105–116.
207. J. Lehner, *A partition function connected with the modulus five*, Duke Math. J. **8** (1941), 631–655.
208. J. Lehner, *Ramanujan identities involving the partition function for the moduli* 11^α, Amer. J. Math. **65** (1943), 492–520.
209. R. Lewis, *On the rank and the crank modulo* 4, Proc. Amer. Math. Soc. **112** (1991), 925–933.
210. R. Lewis, *On the ranks of partitions modulo* 9, Bull. London Math. Soc. **23** (1991), 417–421.
211. R. Lewis, *On some relations between the rank and the crank*, J. Combin. Thy., Ser. A **59** (1992), 104–110.
212. R. Lewis, *Relations between the rank and the crank modulo* 9, J. London Math. Soc. (2) **45** (1992), 222–231.
213. R. Lewis, *The ranks of partitions modulo* 2, Discrete Math. **167/168** (1997), 445–449.
214. R. Lewis, *The generating functions of the rank and crank modulo* 8, Ramanujan J. **18** (2009), 121–146.
215. W.-C. Liaw, *Contributions to Ramanujan's Theories of Modular Equations, Ramanujan-Type Series for* $1/\pi$*, and Partitions*, Ph.D. Thesis, University of Illinois at Urbana–Champaign, Urbana, 1999.
216. W.-C. Liaw, *Note on the monotonicity of coefficients for some q-series arising from Ramanujan's lost notebook*, Ramanujan J. **3** (1999), 385–388.
217. J. Liouville, *Sur la forme* $x^2 + y^2 + 2(z^2 + t^2)$, J. Math. Pures Appl. **5** (1860), 269–272.

218. J. Lovejoy, *Rank and conjugation for the Frobenius representation of an overpartition*, Ann. Combin. **9** (2005), 321–334.
219. J. Lovejoy and R. Osburn, *M_2-rank differences for partitions without repeated odd parts*, J. Théor. Nombres Bordeaux **21** (2009), 313–334.
220. J. Lovejoy and R. Osburn, *M_2-rank differences for overpartitions*, Acta Arith. **144** (2010), 193–212.
221. K. Mahlburg, *Partition congruences and the Andrews-Garvan-Dyson crank*, Proc. Natl. Acad. Sci. USA **102** (2005), 15373–15376.
222. J. Malenfant, *Generalizing Ramanujan's J functions*, Preprint.
223. E. McAfee and K.S. Williams, *Sums of six squares*, Far East J. Math. Sci. **16** (2005), 17–41.
224. A. Milas, *Ramanujan's "lost notebook" and the Virasoro algebra*, Comm. Math. Phys. **251** (2004), 567–588.
225. M. Monks, *Number theoretic properties of generating functions related to Dyson's rank for partitions into distinct parts*, Proc. Amer. Math. Soc. **138** (2010), 481–494.
226. L.J. Mordell, *On Ramanujan's empirical expansions of modular functions*, Proc. Cambridge Philos. Soc. **19** (1920), 117–124.
227. L.J. Mordell, *Note on certain modular relations considered by Messrs Ramanujan, Darling and Rogers*, Proc. London Math. Soc. **20** (1922), 408–416.
228. P. Moree, *On some claims in Ramanujan's 'unpublished' manuscript on the partition and tau functions*, Ramanujan J. **8** (2004), 317–330.
229. P. Moree, *Counting numbers in multiplicative sets: Landau versus Ramanujan*, Mathematics Newsletter, Ramanujan Math. Soc. **21** (2011), 73–81.
230. P. Moree and J. Cazaran, *On a claim of Ramanujan in his first letter to Hardy*, Expos. Math. **17** (1999), 289–311.
231. P. Moree and H.J.J. te Riele, *The hexagonal versus the square lattice*, Math. Comp. **73** (2004), 451–473.
232. E. Mortenson, *On the broken 1-diamond partition*, Internat. J. Number Thy. **4** (2008), 199–218.
233. M.R. Murty and V.K. Murty, *Prime divisors of Fourier coefficients of modular forms*, Duke Math. J. **51** (1984), 57–76.
234. T. Murayama, *On an identity of theta functions obtained from weight enumerators of linear codes*, Proc. Japan Acad., Ser. A **74** (1998), 98–100.
235. M.R. Murty and V.K. Murty, *Odd values of Fourier coefficients of certain modular forms*, Internat. J. Number Thy. **3** (2007), 455–470.
236. M.R. Murty, V.K. Murty, and T.N. Shorey, *Odd values of the Ramanujan τ-function*, Bull. Soc. Math. France **115** (1987), 391–395.
237. M.B. Nathanson, *Elementary Methods in Number Theory*, Springer, New York, 2000.
238. M. Newman, *Remarks on some modular identities*, Trans. Amer. Math. Soc. **73** (1952), 313–320.
239. M. Newman, *Congruences for the coefficients of modular forms and some new congruences for the partition function*, Canad. J. Math. **9** (1957), 549–552.
240. M. Newman, *Periodicity modulo m and divisibility properties of the partition function*, Trans. Amer. Math. Soc. **97** (1960), 225–236.
241. J.-L. Nicolas, *Répartition des nombres hautement composés de Ramanujan*, Canad. J. Math. **23** (1971), 115–130.

242. J.-L. Nicolas, *Grandes valeurs des fonctions arithmétiques*, Séminaire D. P. P. Paris (16e année, 1974/75), n^o G20, 5p.
243. J.-L. Nicolas, *Répartition des nombres largement composés*, Acta Arith. **34** (1980), 379–390.
244. J.-L. Nicolas, *Petites valeurs de la fonction d'Euler*, J. Number Thy. **17** (1983), 375–388.
245. J.-L. Nicolas, *On highly composite numbers*, in *Ramanujan Revisited*, G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, and R.A. Rankin, eds., Academic Press, Boston, 1988, pp. 216–244.
246. J.-L. Nicolas, *On composite numbers*, in *Number Theory, Madras* 1987, Lecture Notes in Math. No. 1395, K. Alladi, ed., Springer-Verlag, 1989, pp. 18–20.
247. J.-L. Nicolas, *Valeurs impaires de la fonction de partition* $p(n)$, Internat. J. Number Thy. **2** (2006), 469–487.
248. J.-L. Nicolas, *Parité des valeurs de la fonction de partition* $p(n)$ *and anatomy of the integers*, in *Anatomy of Integers*, CRM Proc. Lecture Notes, 46, American Mathematical Society, Providence, RI, 2008, pp. 97–113.
249. J.-L. Nicolas and G. Robin, *Majorations explicites pour le nombre de diviseurs de* N, Canad. Math. Bull. **26** (1983), 485–492.
250. J.-L. Nicolas and G. Robin, Annotators, *Highly composite numbers,* by S. Ramanujan, Ramanujan J. **1** (1997), 119–153.
251. J.-L. Nicolas, I.Z. Ruzsa, and A. Sárközy, *On the parity of additive representation functions, with an appendix by J.-P. Serre*, J. Number Thy. **73** (1998), 292–317.
252. J.-L. Nicolas and A. Sárközy, *On the parity of generalized partition functions*, in *Number Theory for the Millennium*, M.A. Bennett, B.C. Berndt, N. Boston, H.G. Diamond, A.J. Hildebrand, and W. Philipp, eds., AK Peters, Boston, 2002, pp. 55–72.
253. K.K. Norton, *Upper bounds for sums of powers of divisor functions*, J. Number Thy. **40** (1992), 60–85.
254. K. Ono, *On the parity of the partition function*, J. Reine Angew. Math. **472** (1996), 1–15.
255. K. Ono, *The residue of* $p(n)$ *modulo small primes*, Ramanujan J. **2** (1998), 47–54.
256. K. Ono, *The partition function in arithmetic progressions*, Math. Ann. **312** (1998), 251–260.
257. K. Ono, *On the circular summation of the eleventh powers of Ramanujan's theta functions*, J. Number Thy. **76** (1999), 62–65.
258. K. Ono, *Distribution of the partition function modulo* m, Ann. Math. **151** (2000), 293–307.
259. K. Ono, *Mock theta functions, ranks, and Maass forms*, in *Surveys in Number Theory*, Dev. Math., Vol. 17, Springer, New York, 2008, pp. 119–141.
260. K. Ono, *Parity of the partition function*, Adv. Math. **225** (2010), 349–366.
261. A.E. Patkowski, *A note on the rank parity function*, Discrete Math. **310** (2010), 961–965.
262. T. Pepin, *Étude sur quelques formules d'analyse utiles dans la théorie des nombres*, Atti. Accad. Pont. Nuovi Lincei **38** (1884–85), 139–196.
263. T. Pepin, *Sur quelques formes quadratiques quaternaires*, J. Math. Pures Appl. **6** (1890), 5–67.

264. H. Rademacher, *The Ramanujan identities under modular substitutions*, Trans. Amer. Math. Soc. **51** (1942), 609–636.
265. H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, New York, 1973.
266. H. Rademacher, *Collected Papers of Hans Rademacher*, Vol. II, MIT Press, Cambridge, MA, 1974.
267. H. Rademacher and H.S. Zuckerman, *A new proof of two of Ramanujan's identities*, Ann. Math. **40** (1939), 473–489.
268. S. Raghavan, *On Ramanujan and Dirichlet series with Euler products*, Glasgow Math. J. **25** (1984), 203–206.
269. S. Raghavan, *On certain identities due to Ramanujan*, Quart. J. Math. (Oxford) **37** (1986), 221–229.
270. K.G. Ramanathan, *Congruence properties of Ramanujan's function* $\tau(n)$, Proc. Indian Acad. Sci. A **19** (1944), 146–148.
271. K.G. Ramanathan, *Congruence properties of Ramanujan's function* $\tau(n)$, II, J. Indian Math. Soc. **9** (1945), 55–59.
272. K.G. Ramanathan, *Identities and congruences of the Ramanujan type*, Canad. J. Math. **2** (1950), 168–178.
273. K.G. Ramanathan, *Ramanujan and the congruence properties of partitions*, Proc. Indian Acad. Sci. (Math. Sci.) **89** (1980), 133–157.
274. S. Ramanujan, *Highly composite numbers*, Proc. London Math. Soc. (2) **14** (1915), 347–409.
275. S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159–184.
276. S. Ramanujan, *Some properties of* $p(n)$*, the number of partitions of* n, Proc. Cambridge Philos. Soc. **19** (1919), 207–210.
277. S. Ramanujan, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Philos. Soc. **19** (1919), 214–216.
278. S. Ramanujan, *Algebraic relations between certain infinite products*, Proc. London Math. Soc. **2** (1920), xviii.
279. S. Ramanujan, *Congruence properties of partitions*, Proc. London Math. Soc. **18** (1920), xix.
280. S. Ramanujan, *Congruence properties of partitions*, Math. Z. **9** (1921), 147–153.
281. S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.
282. S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
283. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
284. S. Ramanujan, *Highly composite numbers,* J.-L. Nicolas and G. Robin, eds., Ramanujan J. **1** (1997), 119–153.
285. S.S. Rangachari, *Ramanujan and Dirichlet series with Euler products*, Proc. Indian Acad. Sci. (Math. Sci.) **91** (1982), 1–15.
286. S.S. Rangachari, *On a result of Ramanujan on theta functions*, J. Number Thy. **48** (1994), 364–372.
287. S.S. Rangachari, *Euler products, modular identities and elliptic integrals in Ramanujan's manuscripts,* II, in *Ramanujan Revisited*, G.E. Andrews, R.A.

Askey, B.C. Berndt, K.G. Ramanathan, and R.A. Rankin, eds., Academic Press, Boston, 1988, pp. 347–357.

288. R.A. Rankin, *The divisibility of divisor functions*, Glasgow Math. J. **5** (1961), 35–40.
289. R.A. Rankin, *Ramanujan's unpublished work on congruences*, in *Modular Functions of One Variable* V, Lecture Notes in Math. No. 601, Springer-Verlag, Berlin, 1977, pp. 3–15.
290. R.A. Rankin, *Modular Forms and Functions*, Cambridge University Press, Cambridge, 1977.
291. R.A. Rankin, *Ramanujan's manuscripts and notebooks*, Bull. London Math. Soc. **14** (1982), 81–97; reprinted in [69, pp. 117–128].
292. R.A. Rankin, *Ramanujan's manuscripts and notebooks* II, Bull. London Math. Soc. **21** (1989), 351–365; reprinted in [69, pp. 129–142].
293. K. Ribet, *Galois representations attached to eigenforms with Nebentypus*, in *Modular Functions of One Variable* V, Lecture Notes in Math. No. 601, Springer-Verlag, Berlin, 1977, pp. 17–51.
294. L.B. Richmond, *Some general problems on the number of parts in partitions*, Acta Arith. **66** (1994), 297–313.
295. L.B. Richmond, *Personal communication*, November 29, 2011.
296. G. Robin, *Sur l'ordre maximum de la fonction somme des diviseurs*, Séminaire Delange-Pisot-Poitou. Paris, 1981–1982, Birkhäuser, Boston, 1983, pp. 223–244.
297. G. Robin, *Grandes valeurs de fonctions arithmétiques et problèmes d'optimisation en nombres entiers*, Thèse d'Etat, Université de Limoges, France, 1983.
298. G. Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*, J. Math. Pures Appl. **63** (1984), 187–213.
299. G. Robin, *Sur la différence* $\mathrm{Li}(\theta(x)) - \pi(x)$, Ann. Fac. Sc. Toulouse **6** (1984), 257–268.
300. G. Robin, *Grandes valeurs de la fonction somme des diviseurs dans les progressions arithmétiques*, J. Math. Pures Appl. **66** (1987), 337–349.
301. G. Robin, *Sur des travaux non publiés de S. Ramanujan sur les nombres hautement composés*, Publications du département de Mathématiques de l'Université de Limoges, France, 1991, pp. 1–60.
302. S. Robins, *Arithmetic Properties of Modular Forms*, Ph.D. Thesis, University of California at Los Angeles, 1991.
303. L.J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.
304. L.J. Rogers, *On a type of modular relation*, Proc. London Math. Soc. **19** (1921), 387–397.
305. J.M. Rushforth, *Congruence Properties of the Partition Function and Associated Functions*, Doctoral Thesis, University of Birmingham, 1950.
306. J.M. Rushforth, *Congruence properties of the partition function and associated functions*, Proc. Cambridge Philos. Soc. **48** (1952), 402–413.
307. N. Santa-Gadea, *On some relations for the rank moduli* 9 *and* 12, J. Number Thy. **40** (1992), 130–145.
308. B.K. Sarmah, *Contributions to Partition Identities and Sums of Polygonal Numbers by Using Ramanujan's Theta Functions*, Ph.D. Thesis, Tezpur University, Napaam, India, 2012.

309. E.J. Scourfield, *On the divisibility of $\sigma_\nu(n)$*, Acta Arith. **10** (1964), 245–285.
310. J.-P. Serre, *Une interprétation des congruences relatives à la fonction τ de Ramanujan*, in *Séminaire Delange–Pisot–Poitou:* 1967/68*: Théorie des Nombres,* Fasc. 1, Exp. 14, Secrétariat Mathématique, Paris, 1969, 17 pp.
311. J.-P. Serre, *Congruences et formes modulaires* [*d'après H.P.F. Swinnerton-Dyer*], in *Séminaire Bourbaki,* 24e année (1971/1972), Exp. No. 416, Lecture Notes in Math. No. 317, Springer-Verlag, Berlin, 1973, pp. 319–338.
312. J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*, in *Modular Functions of One Variable* III, Lecture Notes in Math. No. 350, Springer-Verlag, Berlin, 1973, pp. 191–268.
313. J.-P. Serre, *Divisibilité des coefficients des formes modulaires de poids entier*, C. R. Acad. Sci. (Paris), Sér. A **279** (1974), 670–682.
314. J.-P. Serre, *Divisibilité de certaines fonctions arithmétiques*, L'Enseign. Math. **22** (1976), 227–260.
315. J.-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Publ. Math. I. H. E. S. **54** (1981), 123–201.
316. A.V. Sills, *A Rademacher type formula for partitions and overpartitions*, Internat. J. Math. Math. Sci. **2010** (2010), Article ID 630458, 21 pages.
317. W.H. Simons, *Congruences involving the partition function $p(n)$*, Bull. Amer. Math. Soc. **50** (1944), 883–892.
318. S.H. Son, *Some integrals of theta functions*, in *Number Theory, Fifth Conference of the Canadian Number Theory Association*, R. Gupta and K.S. Williams, eds., American Mathematical Society, Providence, RI, 1999, pp. 329–339.
319. S.H. Son, *Septic theta function identities in Ramanujan's lost notebook*, Acta Arith. **98** (2001), 361–374.
320. S.H. Son, *Circular summations of theta functions in Ramanujan's lost notebook*, Ramanujan J. **8** (2004), 235–272.
321. S.H. Son, *Basic functional equations of the Rogers–Ramanujan functions*, Rocky Mt. J. Math. **37** (2007), 653–662.
322. S.H. Son, *Ramanujan's symmetric theta functions in his lost notebook*, in *Special Functions and Orthogonal Polynomials*, D. Dominici and R.S. Maier, eds., Contemp. Math. 471, American Mathematical Society, Providence, RI, 2008, pp. 187–202.
323. B.K. Spearman and K.S. Williams, *The simplest arithmetic proof of Jacobi's four squares theorem*, Far East J. Math. Sci. **2** (2000), 433–439.
324. B.K. Spearman and K.S. Williams, *An arithmetic proof of Jacobi's eight squares theorem*, Far East J. Math. Sci. **3** (2001), 1001–1005.
325. G.K. Stanley, *Two assertions made by Ramanujan*, J. London Math. Soc. **3** (1928), 232–237.
326. H.P.F. Swinnerton-Dyer, *On ℓ-adic representations and congruences for coefficients of modular forms*, in *Modular Functions of One Variable* III, Lecture Notes in Math. No. 350, Springer-Verlag, Berlin, 1973, pp. 1–55.
327. H.P.F. Swinnerton-Dyer, *Correction to: "On ℓ-adic representations and congruences for coefficients of modular forms"*, in *Modular Functions of One Variable* IV, Lecture Notes in Math. No. 476, Springer-Verlag, Berlin, 1975, p. 149.
328. H.P.F. Swinnerton-Dyer, *On ℓ-adic representations and congruences for coefficients of modular forms* (II), in *Modular Functions of One Variable* V, Lecture Notes in Math. No. 601, Springer-Verlag, Berlin, 1977, pp. 63–90.

329. H.P.F. Swinnerton-Dyer, *Congruence properties of* $\tau(n)$, in *Ramanujan Revisited*, G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, and R.A. Rankin, eds., Academic Press, Boston, 1988, pp. 289–311.
330. G. Szekeres, *An asymptotic formula in the theory of partitions*, Quart. J. Math. Oxford (2) **2** (1951), 85–108.
331. J. Tannery and J. Molk, *Éléments de la Théorie des Fonctions Elliptiques,* Tome III, Gauthier-Villars, Paris, 1898.
332. G.N. Watson, *Theorems stated by Ramanujan* (VII)*: Theorems on continued fractions*, J. London Math. Soc. **4** (1929), 39–48.
333. G.N. Watson, *Proof of certain identities in combinatory analysis*, J. Indian Math. Soc. **20** (1933), 57–69.
334. G.N. Watson, *Über Ramanujansche Kongruenzeigenschaften der Zerfällungsanzahlen,* I, Math. Z. **39** (1935), 712–731.
335. G.N. Watson, *The final problem: An account of the mock theta functions*, J. London Math. Soc. **11** (1936), 55–80.
336. G.N. Watson, *Ramanujans Vermutung über Zerfällungsanzahlen*, J. Reine Angew. Math. **179** (1938), 97–128.
337. G.N. Watson, *A table of Ramanujan's function* $\tau(n)$, Proc. London Math. Soc. **51** (1948), 1–13.
338. R.L. Weaver, *New congruences for the partition function*, Ramanujan J. **5** (2001), 53–63.
339. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1966.
340. J.R. Wilton, *On Ramanujan's arithmetical function* $\sigma_{r,s}(n)$, Proc. Cambridge Philos. Soc. **25** (1929), 255–264.
341. J.R. Wilton, *Congruence properties of Ramanujan's function* $\tau(n)$, Proc. London Math. Soc. **31** (1930), 1–10.
342. L. Winquist, *An elementary proof of* $p(11n+6) \equiv 0 \pmod{11}$, J. Comb. Thy. **6** (1969), 56–59.
343. E.X.W. Xia and X.M. Yao, *Some modular relations for the Göllnitz–Gordon functions by an even–odd method*, J. Math. Anal. Appl. **387** (2012), 126–138.
344. E.X.W. Xia and X.M. Yao, *Some modular relations for the Göllnitz–Gordon functions and Ramanujan's modular equation*, submitted for publication.
345. P. Xu, *An elementary proof of Ramanujan's circular summation formula and its generalizations*, Ramanujan J. **27** (2012), 409–417.
346. Q. Yan, *Several identities for certain products of theta functions*, Ramanujan J. **19** (2009), 79–94.
347. H. Yesilyurt, *A generalization of a modular identity of Rogers*, J. Number Thy. **129** (2009), 1256–1271.
348. H. Yesilyurt, *Elementary proofs of some identities of Ramanujan for the Rogers–Ramanujan functions*, J. Math. Anal. Appl. **388** (2012), 420–434.
349. B. Yuttanan, *Modular Equations and Ramanujan's Cubic and Quartic Theories of Theta Functions*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, Urbana, 2011.
350. X.-F. Zeng, *A generalized circular summation of theta function and its applications*, J. Math. Anal. Appl. **356** (2009), 698–703.
351. L.-C. Zhang, *A note on Ramanujan's* 40 *identities for the Rogers–Ramanujan functions*, South East Asian J. Math. Math. Sci. **8** (2010), 85–101.

352. J.-M. Zhu, *An alternate circular summation formula of theta functions and its applications*, Applic. Anal. Discrete Math. **6** (2012), 114–125.
353. H.S. Zuckerman, *Identities analogous to Ramanujan's identities involving the partition function*, Duke Math. J. **5** (1939), 88–110.
354. H.S. Zuckerman, *On the coefficients of certain modular forms belonging to subgroups of the modular group*, Trans. Amer. Math. Soc. **45** (1939), 298–321.
355. S. Zwegers, *Mock Theta Functions*, Doctoral Dissertation, Universiteit Utrecht, 2002.
356. S. Zwegers, *Rank-crank type PDE's for higher level Appell functions*, Acta Arith. **144** (2010), 263–273.

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